CHAPTER 4

Initial Cases of Transfinite Induction

The goal here is to study the in a sense "most complex" proofs in first-order arithmetic. The main tool for proving theorems in arithmetic is clearly the induction schema

$$A(0) \to \forall_x (A(x) \to A(Sx)) \to \forall_x A(x).$$

An equivalent form of this schema is "course-of-values" or cumulative induction

$$\forall_x (\forall_{y<x} A(y) \to A(x)) \to \forall_x A(x).$$

Both schemes refer to the standard ordering of $\mathbb{N}$. It is tempting to try to strengthen arithmetic by allowing more general induction schemes, e.g., w.r.t. the lexicographical ordering of $\mathbb{N} \times \mathbb{N}$. Even more generally, let $\prec$ be a well-ordering of $\mathbb{N}$ and use *transfinite induction*:

$$\forall_x (\forall_{y \prec x} A(y) \to A(x)) \to \forall_x A(x).$$

It can be understood as

> Suppose the property $A(x)$ is "progressive", i.e., from the validity of $A(y)$ for all $y \prec x$ we can conclude that $A(x)$ holds. Then $A(x)$ holds for all x.

For which well-orderings this schema is derivable in arithmetic? We will discuss a classic result of Gentzen (1943) which in a sense answers this question completely. However, to state the result we have to be more explicit about the well-orderings used.

4.1. Ordinals below ε_0

We need some knowledge and notations for ordinals. This can be done without relying on set theory: we introduce an initial segment of the ordinals (the ones $< \varepsilon_0$) in a formal, combinatorial way, i.e., via ordinal notations based on "Cantor normal form". From now on "ordinal" means "ordinal notation".

Definition. We define

(a) α is an ordinal,
(b) $\alpha < \beta$ for ordinals α, β.

simultaneously by induction, as follows.

(a) If $\alpha_m, \dots, \alpha_0$ are ordinals, $m \geq -1$ and $\alpha_m \geq \dots \geq \alpha_0$ (where $\alpha \geq \beta$ means $\alpha > \beta$ or $\alpha = \beta$), then

$$\omega^{\alpha_m} + \dots + \omega^{\alpha_0}$$

is an ordinal. The empty sum (denoted by 0) is allowed.

(b) If $\omega^{\alpha_m} + \dots + \omega^{\alpha_0}$ and $\omega^{\beta_n} + \dots + \omega^{\beta_0}$ are ordinals, then

$$\omega^{\alpha_m} + \dots + \omega^{\alpha_0} < \omega^{\beta_n} + \dots + \omega^{\beta_0}$$

iff there is an $i \geq 0$ such that $\alpha_{m-i} < \beta_{n-i}$, $\alpha_{m-i+1} = \beta_{n-i+1}$, $\dots$, $\alpha_m = \beta_n$, or else $m < n$ and $\alpha_m = \beta_n$, $\dots$, $\alpha_0 = \beta_{n-m}$.

We shall use the notation:

$$\begin{aligned} 1 &:= \omega^0, \\ k &:= \omega^0 + \dots + \omega^0 \quad \text{with } k \text{ copies of } \omega^0, \\ \omega^\alpha k &:= \omega^\alpha + \dots + \omega^\alpha \quad \text{with } k \text{ copies of } \omega^\alpha. \end{aligned}$$

The *level* of an ordinal is defined by $\text{lev}(0) := 0$, $\text{lev}(\omega^{\alpha_m} + \dots + \omega^{\alpha_0}) := \text{lev}(\alpha_m) + 1$. For ordinals of level $k+1$ we have $\omega_k \leq \alpha < \omega_{k+1}$, where $\omega_0 := 0$, $\omega_1 := \omega^1$, $\omega_{k+1} := \omega^{\omega_k}$.

LEMMA 4.1.1. *$<$ is a linear order with 0 the least element.*

PROOF. By induction on the levels. □

EXAMPLE.

$$\begin{aligned} &0 < 1 < 2 \dots < \omega < \omega + 1 \dots < \omega 2 < \omega 2 + 1 \dots < \omega 3 \dots < \omega^2 \\ &< \omega^2 + 1 \dots < \omega^2 + \omega \dots < \omega^3 \dots < \omega^\omega = \omega_2 \dots < \omega_3 \dots \end{aligned}$$

DEFINITION (Addition of ordinals).

$$(\omega^{\alpha_m} + \dots + \omega^{\alpha_0}) + (\omega^{\beta_n} + \dots + \omega^{\beta_0}) := \omega^{\alpha_m} + \dots + \omega^{\alpha_i} + \omega^{\beta_n} + \dots + \omega^{\beta_0}$$

where i is minimal such that $\alpha_i \geq \beta_n$.

LEMMA 4.1.2. *$+$ is an associative operation which is strictly monotonic in the second argument and weakly monotonic in the first argument.*

PROOF. Exercise. □

REMARK. $+$ is *not* commutative:

$$1 + \omega = \omega \neq \omega + 1.$$

There is also a commutative version on addition, the *natural sum* (or Hessenberg sum). It is defined by

$$(\omega^{\alpha_m} + \cdots + \omega^{\alpha_0}) \# (\omega^{\beta_n} + \cdots + \omega^{\beta_0}) := \omega^{\gamma_{m+n}} + \cdots + \omega^{\gamma_0},$$

where $\gamma_{m+n}, \ldots, \gamma_0$ is a decreasing permutation of $\alpha_m, \ldots, \alpha_0, \beta_n, \ldots, \beta_0$. It is easy to see that # is associative, commutative and strictly monotonic in both arguments.

How ordinals of the form $\beta+\omega^\alpha$ can be approximated from below? First note that

$$\delta < \alpha \to \beta + \omega^\delta k < \beta + \omega^\alpha.$$

For any $\gamma < \beta + \omega^\alpha$ we can find a $\delta < \alpha$ and a k such that

$$\gamma < \beta + \omega^\delta k.$$

It is easy to code ordinals $< \varepsilon_0$ bijectively by natural numbers:

$$\mathrm{o}(\ulcorner \alpha \urcorner) = \alpha \quad \text{and} \quad \ulcorner \mathrm{o}(x) \urcorner = x$$

such that relations and operations on ordinals transfer to elementary relations and operations on $\mathbb{N}$. Abbreviations:

$$\begin{aligned} x \prec y &:= \mathrm{o}(x) < \mathrm{o}(y), \\ \omega^x &:= \ulcorner \omega^{\mathrm{o}(x)} \urcorner, \\ x \oplus y &:= \ulcorner \mathrm{o}(x) + \mathrm{o}(y) \urcorner, \\ xk &:= \ulcorner \mathrm{o}(x) k \urcorner, \\ \omega_k &:= \ulcorner \omega_k \urcorner. \end{aligned}$$

4.2. Provability of initial cases of transfinite induction

We will derive initial cases of transfinite induction in arithmetic:

$$\forall_x (\forall_{y \prec x} Py \to Px) \to \forall_{x \prec a} Px$$

for some number a and a predicate symbol P, where $\prec$ is the standard order of order type ε_0 defined before.

REMARK. Gentzen (1943) proved that this result is optimal in the sense that for the full system of ordinals $< \varepsilon_0$ the principle

$$\forall_x (\forall_{y \prec x} Py \to Px) \to \forall_x Px$$

of transfinite induction is underivable. However, we will not present a proof in these notes.

By an *arithmetical system* **Z** we mean a theory based on minimal logic in the $\forall\to$-language (including equality axioms) such that

(a) The language of **Z** consists of a fixed supply of function and relation constants assumed to denote computable functions and relations on the non-negative integers.
(b) Among the function constants there is a constant S for the successor function and 0 for (the 0-place function) zero.
(c) Among the relation constants we have $=$, P and also $\prec$ for the ordering of type ε_0 of $\mathbb{N}$, as introduced before.
(d) *Terms* are built up from object variables x, y, z by $f(t_1, \dots, t_m)$, where f is a function constant.
(e) We identify closed terms which have the same value; this expresses that each function constant is computable.
(f) Terms of the form $S(S(\dots S0\dots))$ are called *numerals*. Notation: $S^n 0$ or $\underline{n}$ or just n.
(g) *Formulas* are built up from atomic formulas $R(t_1, \dots, t_m)$, with R a relation constant, by $A \to B$ and $\forall_x A$.

The axioms of **Z** are

- Compatibility of equality

$$x = y \to A(x) \to A(y),$$

- the *Peano axioms*, i.e., the universal closures of

$$Sx = Sy \to x = y, \tag{42}$$

$$Sx = 0 \to A, \tag{43}$$

$$A(0) \to \forall_x (A(x) \to A(Sx)) \to \forall_x A(x), \tag{44}$$

  with $A(x)$ an arbitrary formula.
- $R\vec{n}$ whenever $R\vec{n}$ is true (to express that R is computable).
- Irreflexivity and transitivity for $\prec$

$$x \prec x \to A,$$
$$x \prec y \to y \prec z \to x \prec z$$

Further axioms – following Schütte – are the universal closures of

$$x \prec 0 \to A, \tag{45}$$

$$z \prec y \oplus \omega^0 \to (z \prec y \to A) \to (z = y \to A) \to A, \tag{46}$$

$$x \oplus 0 = x, \tag{47}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \tag{48}$$

$$0 \oplus x = x, \tag{49}$$

$$\omega^x 0 = 0, \tag{50}$$

$$\omega^x (Sy) = \omega^x y \oplus \omega^x, \tag{51}$$

(52) $$z \prec y \oplus \omega^{Sx} \to z \prec y \oplus \omega^{e(x,y,z)} m(x,y,z),$$

(53) $$z \prec y \oplus \omega^{Sx} \to e(x,y,z) \prec Sx,$$

where $\oplus$, $\lambda_{x,y}(\omega^x y)$, e and m denote function constants and A is any formula. These axioms are formal counterparts to the properties of the ordinal notations observed above.

THEOREM 4.2.1 (Provable initial cases of transfinite induction in **Z**). *Transfinite induction up to ω_n, i.e., for arbitrary $A(x)$ the formula*

$$\forall_x(\forall_{y \prec x} A(y) \to A(x)) \to \forall_{x \prec \omega_n} A(x),$$

is derivable in **Z**.

PROOF. To every formula $A(x)$ we assign a formula $A^+(x)$ (with respect to a fixed variable x) by

$$A^+(x) := \forall_y(\forall_{z \prec y} A(z) \to \forall_{z \prec y \oplus \omega^x} A(z)).$$

We first show

If $A(x)$ is progressive, then $A^+(x)$ is progressive,

where "$B(x)$ is *progressive*" means $\forall_x(\forall_{y \prec x} B(y) \to B(x))$. Assume that $A(x)$ is progressive and

(54) $$\forall_{y \prec x} A^+(y).$$

Our goal is $A^+(x) := \forall_y(\forall_{z \prec y} A(z) \to \forall_{z \prec y \oplus \omega^x} A(z))$. Assume

(55) $$\forall_{z \prec y} A(z)$$

and $z \prec y \oplus \omega^x$. We have to show $A(z)$.

Case $x = 0$. Then $z \prec y \oplus \omega^0$. By (46):

$$z \prec y \oplus \omega^0 \to (z \prec y \to A) \to (z = y \to A) \to A$$

it suffices to derive $A(z)$ from $z \prec y$ as well as from $z = y$. If $z \prec y$, then $A(z)$ follows from (55), and if $z = y$, then $A(z)$ follows from (55) and the progressiveness of $A(x)$.

Case Sx. From $z \prec y \oplus \omega^{Sx}$ we obtain $z \prec y \oplus \omega^{e(x,y,z)} m(x,y,z)$ by (52) and $e(x,y,z) \prec Sx$ by (53). By (54) we have $A^+(e(x,y,z))$, i.e.

$$\forall_{u \prec y \oplus \omega^{e(x,y,z)} v} A(u) \to \forall_{u \prec (y \oplus \omega^{e(x,y,z)} v) \oplus \omega^{e(x,y,z)}} A(u)$$

and hence, using (48) and (51)

$$\forall_{u \prec y \oplus \omega^{e(x,y,z)} v} A(u) \to \forall_{u \prec y \oplus \omega^{e(x,y,z)}(Sv)} A(u).$$

Also from (55) and (50), (47) we obtain

$$\forall_{u \prec y \oplus \omega^{e(x,y,z)} 0} A(u).$$

By induction:

$$\forall_{u \prec y \oplus \omega^{e(x,y,z)} m(x,y,z)} A(u)$$

and hence $A(z)$.

Next we show, by induction on n, how to derive

$$\forall_x (\forall_{y \prec x} A(y) \to A(x)) \to \forall_{x \prec \omega_n} A(x) \quad \text{for arbitrary } A(x).$$

Assume the left hand side, i.e., that $A(x)$ is progressive.

Case 0. Then $x \prec \omega^0$ and hence $x \prec 0 \oplus \omega^0$ by (49). By (46) it suffices to derive $A(x)$ from $x \prec 0$ as well as from $x = 0$. Now $x \prec 0 \to A(x)$ holds by (45), and $A(0)$ then follows from the progressiveness of $A(x)$.

Case $n+1$. Since $A(x)$ is progressive, also $A^+(x)$ is. By IH: $\forall_{x \prec \omega_n} A^+(x)$, hence $A^+(\omega_n)$ since $A^+(x)$ is progressive. By definition of $A^+(x)$ (with (45): $x \prec 0 \to A$ and (49): $0 \oplus x = x$) we obtain $\forall_{z \prec \omega^{\omega_n}} A(z)$. □

Remark. In the induction step we derived transfinite induction up to ω_{n+1} for $A(x)$ from transfinite induction up to ω_n for $A^+(x)$. Define the *level* of a formula by

$$\begin{aligned}
\operatorname{lev}(R\vec{t}\,) &:= 0,\\
\operatorname{lev}(A \to B) &:= \max(\operatorname{lev}(A) + 1, \operatorname{lev}(B)),\\
\operatorname{lev}(\forall_x A) &:= \max(1, \operatorname{lev}(A)).
\end{aligned}$$

Then $\operatorname{lev}(A^+(x)) = \operatorname{lev}(A(x)) + 1$. Hence to prove transfinite induction up to ω_n, the induction scheme in **Z** is used for formulas of level n.

4.3. Iteration operators of higher types

We have just seen that the strength of the induction scheme increases with the level of the formula proved by induction. A similar phenomenon occurs when one considers types instead of formulas, and iteration (a special case of recursion) instead of induction. Such operators have a similar relation to ordinals $<\varepsilon_0$.

Definition. An ordinal $\omega^{\alpha_n} + \dots + \omega^{\alpha_0}$ is a *successor* if $\alpha_0 = 0$. It is a *limit* if α_0 it is neither 0 nor a successor. For every limit $\lambda = \omega^{\alpha_n} + \dots + \omega^{\alpha_0}$ we define its *fundamental sequence* $\lambda[x]$ by

$$\lambda[x] := \begin{cases} \omega^{\alpha_n} + \ldots + \omega^{\alpha_1} + \omega^{\alpha_0 - 1} \cdot x & \text{if } \alpha_0 \text{ is a successor} \\ \omega^{\alpha_n} + \ldots + \omega^{\alpha_1} + \omega^{\alpha_0[x]} & \text{if } \alpha_0 \text{ is a limit.} \end{cases}$$

EXAMPLES.

$$\omega[x] = x,$$
$$(\omega + \omega)[x] = \omega + x,$$
$$\omega^2[x] = \omega x,$$
$$\omega^3[x] = \omega^2 x,$$
$$\omega^\omega[x] = \omega^x.$$

DEFINITION (Extended Grzegorczyk hierarchy $(F_\alpha)_{\alpha<\varepsilon_0}$).

$$F_0(x) := 2^x,$$
$$F_{\alpha+1}(x) := F_\alpha^{(x)}(x) \quad (F_\alpha^{(x)} \text{ } x\text{-th iterate of } F_\alpha),$$
$$F_\lambda(x) := F_{\lambda[x]}(x).$$

We also define $F_{\varepsilon_0}(x) := F_{\omega_x}(x)$.

REMARK. F_ω is a variant of the Ackermann function (1940), and the F_n for $n < \omega$ were (essentially) defined and studied by Grzegorczyk (1953).

LEMMA 4.3.1. *The function F_1 is not an elementary function, but its graph is an elementary relation.*

PROOF. That F_1 is not elementary was essentially proved as a lemma in Section 2.2.1. The see that the graph of F_1 is elementary observe that

$$F_1(x) = y \leftrightarrow \exists_z\big((z)_0 = 0 \wedge \forall_{i<x}((z)_{i+1} = 2^{(z)_i}) \wedge (z)_x = y\big).$$

Now it suffices to prove that z can be bounded by an elementary function in x and y. But since F_0 is increasing we can bound z by $\langle y, \dots, y\rangle$ with x occurrences of y, and by a lemma in Section 2.2.5 we have

$$\langle \underbrace{y, \dots, y}_{x} \rangle < (y + 1)^{2^x}. \qquad \square$$

Using similar arguments one can prove that all functions F_α for $\alpha < \varepsilon_0$ have elementary graphs.

Let T be a theory in a language containing $0, S$ with the property that every elementary relation is representable in T. We call a function $f : \mathbb{N} \to \mathbb{N}$ *provably recursive* in T if we have a formula A_f representing the graph of f such that

$$T \vdash \forall_x \exists_y A_f(x, y).$$

In standard arithmetical systems like **Z** one can prove that all functions F_α for $\alpha < \varepsilon_0$ are provably recursive, with methods similar to what we used in Section 4.2. Again ε_0 is a sharp bound: F_{ε_0} is not be provably recursive.

We can characterize $(F_\alpha)_{\alpha<\varepsilon_0}$ by higher type iteration. To this end we extend the definition of the functions F_α into higher types.

Types are generated from the base type $\mathbb{N}$ by the formation of *function types* $\tau \to \sigma$. The *level* of a type is defined similar to the level of a formula (in Section 4.2), by

$$\begin{aligned} \mathrm{lev}(\mathbb{N}) &:= 0, \\ \mathrm{lev}(\tau \to \sigma) &:= \max(\mathrm{lev}(\tau) + 1, \mathrm{lev}(\sigma)). \end{aligned}$$

It is convenient here to introduce *integer types* ρ_n:

$$\begin{aligned} \rho_0 &:= \mathbb{N}, \\ \rho_{n+1} &:= \rho_n \to \rho_n. \end{aligned}$$

If $x_0, \dots, x_{n+1}$ are of integer types $\rho_0, \dots, \rho_{n+1}$, then we can form $x_{n+1}(x_n)$ (of type ρ_n) and so on, finally $x_{n+1}(x_n) \dots (x_0)$, or shortly $x_{n+1}(x_n, \dots, x_0)$. Note that $\mathrm{lev}(\rho_n) = n$.

We define F^{n+1}_α of type ρ_{n+1} for $\alpha < \varepsilon_0$:

$$\begin{aligned} F^{n+1}_0(x_n, \dots, x_0) &:= \begin{cases} 2^{x_0} & \text{if } n = 0 \\ x_n^{(x_0)}(x_{n-1}, \dots, x_0) & \text{otherwise.} \end{cases} \\ F^{n+1}_{\alpha+1}(x_n, \dots, x_0) &:= (F^{n+1}_\alpha)^{(x_0)}(x_n, \dots, x_0), \\ F^{n+1}_\lambda(x_n, \dots, x_0) &:= F^{n+1}_{\lambda[x_0]}(x_n, \dots, x_0). \end{aligned}$$

Here $x_n^{(y)}(x_{n-1}, \dots, x_0)$ denotes $I(y, x_n, \dots, x_0)$ with an *iteration* functional I of type $\mathbb{N} \to \rho_n \to \rho_{n-1} \to \dots \to \rho_0 \to \rho_0$ defined by

$$\begin{aligned} I(0, y, z) &:= z, \\ I(x+1, y, z) &:= y(I(x, y, z)). \end{aligned}$$

THEOREM 4.3.2. *For $n \geq 1$ we have*

$$F^{n+1}_\alpha(F^n_\beta) = F^n_{\beta + \omega^\alpha}$$

provided $\beta + \omega^\alpha = \beta \# \omega^\alpha$, i.e., in the Cantor normal form of β the last summand ω^{β_0} (if it exists) has an exponent $\beta_0 \geq \alpha$.

PROOF. By induction on α. *Case* $\alpha = 0$.

$$\begin{aligned} F^{n+1}_0(F^n_\beta, x_{n-1}, \dots, x_0) &= (F^n_\beta)^{(x_0)}(x_{n-1}, \dots, x_0) \\ &= F^n_{\beta+1}(x_{n-1}, \dots, x_0). \end{aligned}$$

Case α successor.

$$\begin{aligned} F^{n+1}_\alpha(F^n_\beta, x_{n-1}, \dots, x_0) &= (F^{n+1}_{\alpha-1})^{(x_0)}(F^n_\beta, x_{n-1}, \dots, x_0) \\ &= F^n_{\beta + \omega^{\alpha-1} \cdot x_0}(x_{n-1}, \dots, x_0) \quad \text{by IH} \\ &:= F^n_{(\beta + \omega^\alpha)[x_0]}(x_{n-1}, \dots, x_0) \\ &:= F^n_{\beta + \omega^\alpha}(x_{n-1}, \dots, x_0). \end{aligned}$$

Case α limit.

$$
\begin{aligned}
F_\alpha^{n+1}(F_\beta^n, x_{n-1}, \dots, x_0) &= F_{\alpha[x_0]}^{n+1}(F_\beta^n, x_{n-1}, \dots, x_0) \\
&= F^n_{\beta+\omega^{\alpha[x_0]}}(x_{n-1}, \dots, x_0) \quad \text{by IH} \\
&= F^n_{(\beta+\omega^\alpha)[x_0]}(x_{n-1}, \dots, x_0) \\
&= F^n_{\beta+\omega^\alpha}(x_{n-1}, \dots, x_0). \qquad \square
\end{aligned}
$$

The result just proved indicates the computational complexity involved in the use of finite types. The functionals $(F_\alpha^{n+1})_{\alpha<\varepsilon_0}$ and in particular the functions $(F_\alpha^1)_{\alpha<\varepsilon_0}$ can be built from *iteration* functionals (and $F_0(x) = 2^x$) by application alone. In the resulting representation of the functions $(F_\alpha)_{\alpha<\varepsilon_0}$ we do not need the fundamental sequences $\lambda[x]$. The application pattern for F_α corresponds to the Cantor normal form of α.