CHAPTER 2

Model Theory

Model theory is an established branch of mathematical logic. It uses tools from logic to study questions in algebra. In model theory it is common to disregard the distinction between strong and weak existential quantifiers; we shall do the same in the present chapter. Also, the restriction to countable languages that we have maintained until now is given up. Moreover one makes fee use of other concepts and axioms from set theory like the *axiom of choice* (for the weak existential quantifier), most often in the form of *Zorn's lemma*.

2.1. Ultraproducts

2.1.1. Filters and ultrafilters. Let $M \neq \emptyset$ be a set. $F \subseteq \mathcal{P}(M)$ is called *filter* on M if

(a) $M \in F$ and $\emptyset \notin F$;
(b) if $X \in F$ and $X \subseteq Y \subseteq M$, then $Y \in F$;
(c) $X, Y \in F$ entails $X \cap Y \in F$.

F is called *ultrafilter* if for all $X \in \mathcal{P}(M)$

$$X \in F \text{ or } M \setminus X \in F.$$

The intuition here is that the elements X of a filter F are considered to be "big". For instance, for M infinite the set $F = \{ X \subseteq M \mid M \setminus X \text{ finite} \}$ is a filter (called *Fréchet*-filter).

LEMMA. *Suppose F is an ultrafilter and $X \cup Y \in F$. Then $X \in F$ or $Y \in F$.*

PROOF. If both X and Y are not in F, then $M \setminus X$ and $M \setminus Y$ are in F, hence also $(M \setminus X) \cap (M \setminus Y)$, which is $M \setminus (X \cup Y)$. This contradicts the assumption $X \cup Y \in F$. □

Let $M \neq \emptyset$ be a set and $S \subseteq \mathcal{P}(M)$. S has the *finite intersection property* if $X_1 \cap \dots \cap X_n \neq \emptyset$ for all $X_1, \dots, X_n \in S$ and all $n \in \mathbb{N}$.

LEMMA. *If S has the finite intersection property, then there exists a filter F on M such that $F \supseteq S$.*

Proof. $F := \{ X \mid X \supseteq X_1 \cap \dots \cap X_n \text{ for some } X_1, \dots, X_n \in S \}$. □

Lemma. *Let $M \neq \emptyset$ be a set and F a filter on M. Then there is an ultrafilter U on M such that $U \supseteq F$.*

Proof. By Zorn's lemma (which will be proved from the axiom of choice later, in the chapter on set theory), there is a maximal filter U with $F \subseteq U$. We claim that U is an ultrafilter. So let $X \subseteq M$ and assume $X \notin U$ and $M \setminus X \notin U$. Since U is maximal, $U \cup \{X\}$ cannot have the finite intersection property; hence there is a $Y \in U$ such that $Y \cap X = \emptyset$. Similary we obtain $Z \in U$ such that $Z \cap (M \setminus X) = \emptyset$. But then $Y \cap Z = \emptyset$, a contradiction. □

2.1.2. Products and ultraproducts. Let $I \neq \emptyset$ be a set and $D_i \neq \emptyset$ sets for $i \in I$. Let

$$\prod_{i \in I} D_i := \{ \alpha \mid \alpha \text{ is a function, } \mathrm{dom}(\alpha) = I \text{ and } \alpha(i) \in D_i \text{ for all } i \in I \}.$$

Observe that, by the *axiom of choice*, $\prod_{i \in I} D_i \neq \emptyset$. We write $\alpha \in \prod_{i \in I} D_i$ as $\langle \alpha(i) \mid i \in I \rangle$.

Now let $I \neq \emptyset$ be a set, F a filter on I and $\mathcal{M}_i$ models for $i \in I$. Then the *F-product* $\mathcal{M} = \prod_{i \in I}^F \mathcal{M}_i$ is defined by

(a) $|\mathcal{M}| := \prod_{i \in I} |\mathcal{M}_i|$ (notice that $|\mathcal{M}| \neq \emptyset$).

(b) for an n-ary relation symbol R and $\alpha_1, \dots, \alpha_n \in |\mathcal{M}|$ let

$$R^{\mathcal{M}}(\alpha_1, \dots, \alpha_n) := (\{ i \in I \mid R^{\mathcal{M}_i}(\alpha_1(i), \dots, \alpha_n(i)) \} \in F).$$

(c) for an n-ary function symbol f and $\alpha_1, \dots, \alpha_n \in |\mathcal{M}|$ let

$$f^{\mathcal{M}}(\alpha_1, \dots, \alpha_n) := \langle f^{\mathcal{M}_i}(\alpha_1(i), \dots, \alpha_n(i)) \mid i \in I \rangle.$$

For an ultrafilter U we call $\mathcal{M} = \prod_{i \in I}^U \mathcal{M}_i$ the *U-ultraproduct* of the $\mathcal{M}_i$.

Theorem (Fundamental theorem on ultraproducts, Łoś (1955)). *Let $\mathcal{M} = \prod_{i \in I}^U \mathcal{M}_i$ be a U-ultraproduct, A a formula and η an assignment in $|\mathcal{M}|$. Then*

$$\mathcal{M} \models A[\eta] \leftrightarrow \{ i \in I \mid \mathcal{M}_i \models A[\eta_i] \} \in U,$$

where η_i is the assignment induced by $\eta_i(x) = \eta(x)(i)$ for $i \in I$.

Proof. We first prove a similar property for terms.

$$t^{\mathcal{M}}[\eta] = \langle t^{\mathcal{M}_i}[\eta_i] \mid i \in I \rangle. \tag{2.1}$$

The proof is by induction on t. For a variable the claim follows from the definition. *Case* $f(t_1, \dots, t_n)$. For simplicity assume $n = 1$; so we consider ft. We obtain

$$\begin{aligned} (ft)^{\mathcal{M}}[\eta] &= f^{\mathcal{M}}(t^{\mathcal{M}}[\eta]) \\ &= f^{\mathcal{M}} \langle t^{\mathcal{M}_i}[\eta_i] \mid i \in I \rangle \quad \text{by IH} \end{aligned}$$

$$= \langle\, (ft)^{\mathcal{M}_i}[\eta_i] \mid i \in I \,\rangle.$$

Case $R(t_1, \dots, t_n)$. For simplicity assume $n = 1$; so consider Rt. We obtain

$$\begin{aligned} \mathcal{M} \models Rt[\eta] &\leftrightarrow R^{\mathcal{M}}(t^{\mathcal{M}}[\eta]) \\ &\leftrightarrow \{\, i \in I \mid R^{\mathcal{M}_i}(t^{\mathcal{M}}[\eta](i)) \,\} \in U \\ &\leftrightarrow \{\, i \in I \mid R^{\mathcal{M}_i}(t^{\mathcal{M}_i}[\eta_i]) \,\} \in U \qquad \text{by (2.1)} \\ &\leftrightarrow \{\, i \in I \mid \mathcal{M}_i \models Rt[\eta_i] \,\} \in U. \end{aligned}$$

Case $A \to B$.

$\mathcal{M} \models (A \to B)[\eta]$

$\leftrightarrow$ if $\mathcal{M} \models A[\eta]$, then $\mathcal{M} \models B[\eta]$

$\leftrightarrow$ if $\{\, i \in I \mid \mathcal{M}_i \models A[\eta_i] \,\} \in U$, then $\{\, i \in I \mid \mathcal{M}_i \models B[\eta_i] \,\} \in U$ by IH

$\leftrightarrow \{\, i \in I \mid \mathcal{M}_i \models A[\eta_i] \,\} \notin U$ or $\{\, i \in I \mid \mathcal{M}_i \models B[\eta_i] \,\} \in U$

$\leftrightarrow \{\, i \in I \mid \mathcal{M}_i \models \neg A[\eta_i] \,\} \in U$ or $\{\, i \in I \mid \mathcal{M}_i \models B[\eta_i] \,\} \in U$ for U is an ultrafilter

$\leftrightarrow \{\, i \in I \mid \mathcal{M}_i \models (A \to B)[\eta_i] \,\} \in U.$

The case $A \wedge B$ is easy.

Case $\forall_x A$.

$$\begin{aligned} &\mathcal{M} \models (\forall_x A)[\eta] \\ &\leftrightarrow \forall_{\alpha \in |\mathcal{M}|}(\mathcal{M} \models A[\eta_x^{\alpha}]) \\ &\leftrightarrow \forall_{\alpha \in |\mathcal{M}|}(\{\, i \in I \mid \mathcal{M}_i \models A[(\eta_i)_x^{\alpha(i)}] \,\} \in U) \quad \text{by IH} \\ &\leftrightarrow \{\, i \in I \mid \forall_{a \in |\mathcal{M}_i|}(\mathcal{M}_i \models A[(\eta_i)_x^{a}]) \,\} \in U \qquad \text{see below} \qquad (2.2) \\ &\leftrightarrow \{\, i \in I \mid \mathcal{M}_i \models (\forall_x A)[\eta_i] \,\} \in U. \end{aligned}$$

It remains to show (2.2). Let

$$X := \{\, i \in I \mid \forall_{a \in |\mathcal{M}_i|}(\mathcal{M}_i \models A[(\eta_i)_x^{a}]) \,\}$$

and $Y_\alpha := \{\, i \in I \mid \mathcal{M}_i \models A[(\eta_i)_x^{\alpha(i)}] \,\}$ for $\alpha \in |\mathcal{M}|$.

$\leftarrow$. Let $\alpha \in |\mathcal{M}|$ and $X \in U$. Clearly $X \subseteq Y_\alpha$, hence also $Y_\alpha \in U$.

$\to$. Let $Y_\alpha \in U$ for all α. Assume $X \notin U$. Since U is an ultrafilter,

$$I \setminus X = \{\, i \in I \mid \exists_{a \in |\mathcal{M}_i|}(\mathcal{M}_i \not\models A[(\eta_i)_x^{a}]) \,\} \in U.$$

We choose by the axiom of choice an $\alpha_0 \in |\mathcal{M}|$ such that

$$\alpha_0(i) = \begin{cases} \text{some } a \in |\mathcal{M}_i| \text{ such that } \mathcal{M}_i \not\models A[(\eta_i)_x^{a}] & \text{if } i \in I \setminus X, \\ \text{an arbitrary} \in |\mathcal{M}_i| & \text{otherwise.} \end{cases}$$

Then $Y_{\alpha_0} \cap (I \setminus X) = \emptyset$, contradicting $Y_{\alpha_0}, I \setminus X \in U$. $\square$

If we choose $\mathcal{M}_i = \mathcal{N}$ constant, then $\mathcal{M} = \prod_{i \in I}^{U} \mathcal{N}$ satisfies the same closed formulas as $\mathcal{N}$ (such models will be called *elementary equivalent*; the notation is $\mathcal{M} \equiv \mathcal{N}$). $\prod_{i \in I}^{U} \mathcal{N}$ is called an *ultrapower* of $\mathcal{N}$.

2.1.3. General compactness and completeness. Recall that the underlying language may be uncountable.

COROLLARY (General compactness theorem). *Let Γ be a set of formulas. If every finite subset of Γ is satisfiable, then so is Γ.*

PROOF. Let $I := \{ i \subseteq \Gamma \mid i \text{ finite} \}$. For $i \in I$ let $\mathcal{M}_i$ be a model of i under the assignment η_i. For $A \in \Gamma$ let $Z_A := \{ i \in I \mid A \in i \} = \{ i \subseteq \Gamma \mid i \text{ finite and } A \in i \}$. Then $F := \{ Z_A \mid A \in \Gamma \}$ has the finite intersection property (for $\{A_1, \dots, A_n\} \in Z_{A_1} \cap \dots \cap Z_{A_n}$). By the lemmata in 2.1.1 there is an ultrafilter U on I such that $F \subseteq U$. We consider the ultraproduct $\mathcal{M} := \prod_{i \in I}^{U} \mathcal{M}_i$ and the product assigment η defined by $\eta(x)(i) := \eta_i(x)$, and show $\mathcal{M} \models \Gamma[\eta]$. So let $A \in \Gamma$. By Łoś's theorem it suffices to show

$$X_A := \{ i \in I \mid \mathcal{M}_i \models A[\eta_i] \} \in U.$$

But this follows from $Z_A \subseteq X_A$ and $Z_A \in F \subseteq U$. $\square$

For every set Γ of formulas let $L(\Gamma)$ be the set of all function and relation symbols occurring in Γ. If $\mathcal{L}'$ is a sublanguage of $\mathcal{L}$, $\mathcal{M}'$ an $\mathcal{L}'$-model and $\mathcal{M}$ an $\mathcal{L}$-model, then $\mathcal{M}$ is called an *expansion* of $\mathcal{M}'$ (and $\mathcal{M}'$ a *reduct* of $\mathcal{M}$) if $|\mathcal{M}'| = |\mathcal{M}|$, $f^{\mathcal{M}'} = f^{\mathcal{M}}$ for all function symbols and $R^{\mathcal{M}'} = R^{\mathcal{M}}$ for all relation symbols in the language $\mathcal{L}'$. The (uniquely determined) $\mathcal{L}'$-reduct of $\mathcal{M}$ is denoted by $\mathcal{M} {\restriction} \mathcal{L}'$. If $\mathcal{M}$ is an expansion of $\mathcal{M}'$ and η an assignment in $|\mathcal{M}'|$, then clearly $t^{\mathcal{M}'}[\eta] = t^{\mathcal{M}}[\eta]$ for every $\mathcal{L}'$-term t and $\mathcal{M}' \models A[\eta]$ if and only if $\mathcal{M} \models A[\eta]$, for every $\mathcal{L}'$-formula A.

COROLLARY (General completeness theorem). *Let $\Gamma \cup \{A\}$ be a set of formulas. Assume that for all models $\mathcal{M}$ and assignments η,*

$$\mathcal{M} \models \Gamma[\eta] \to \mathcal{M} \models A[\eta].$$

Then $\Gamma \vdash_c A$.

PROOF. By assumption $\Gamma \cup \{\neg A\}$ is not satisfiable. Hence by the general compactness theorem there is a finite subset $\Gamma' \subseteq \Gamma$ such that already $\Gamma' \cup \{\neg A\}$ is not satisfiable. Let $\mathcal{L}$ be the underlying (possibly uncountable) language, and $\mathcal{L}'$ the countable sublanguage containing only function and relation symbols from Γ'. By the remark above $\Gamma' \cup \{\neg A\}$ is not satisfiable w.r.t. $\mathcal{L}'$ as well. By the completeness theorem for countable languages we obtain $\Gamma' \vdash_c A$, hence $\Gamma \vdash_c A$. $\square$

2.2. Complete Theories and Elementary Equivalence

We assume in this section that our underlying language $\mathcal{L}$ contains a binary relation symbol $=$.

2.2.1. Equality axioms. The set $\mathrm{Eq}_{\mathcal{L}}$ of $\mathcal{L}$-equality axioms consists of (the universal closures of)

$$x = x \quad \text{(reflexivity)},$$
$$x = y \to y = x \quad \text{(symmetry)},$$
$$x = y \to y = z \to x = z \quad \text{(transitivity)},$$
$$x_1 = y_1 \to \cdots \to x_n = y_n \to f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n),$$
$$x_1 = y_1 \to \cdots \to x_n = y_n \to R(x_1, \ldots, x_n) \to R(y_1, \ldots, y_n),$$

for all n-ary function symbols f and relation symbols R of the language $\mathcal{L}$.

LEMMA (Equality). (a) $\mathrm{Eq}_{\mathcal{L}} \vdash t = s \to r(t) = r(s)$.
(b) $\mathrm{Eq}_{\mathcal{L}} \vdash t = s \to (A(t) \leftrightarrow A(s))$.

PROOF. (a). Induction on r. (b). Induction on A. □

An $\mathcal{L}$-model $\mathcal{M}$ satisfies the equality axioms if and only if $=^{\mathcal{M}}$ is a *congruence relation* (i.e., an equivalence relation compatible with the functions and relations of $\mathcal{M}$). In this section we assume that all $\mathcal{L}$-models $\mathcal{M}$ considered satisfy the equality axioms. The coincidence lemma then also holds with $=^{\mathcal{M}}$ instead of $=$:

LEMMA (Coincidence). *Let η and ξ be assignments in $|\mathcal{M}|$ such that* $\mathrm{dom}(\eta) = \mathrm{dom}(\xi)$ *and* $\eta(x) =^{\mathcal{M}} \xi(x)$ *for all* $x \in \mathrm{dom}(\eta)$. *Then*
(a) $t^{\mathcal{M}}[\eta] =^{\mathcal{M}} t^{\mathcal{M}}[\xi]$ *if* $\mathrm{vars}(t) \subseteq \mathrm{dom}(\eta)$ *and*
(b) $\mathcal{M} \models A[\eta] \leftrightarrow \mathcal{M} \models A[\xi]$ *if* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\eta)$.

PROOF. Induction on t and A, respectively. □

2.2.2. Cardinality of models. Let $\mathcal{M}/{=^{\mathcal{M}}}$ be the *quotient model*, whose carrier set consists of congruence classes. We call a model $\mathcal{M}$ *infinite* (*countable*, *of cardinality* n) if $|\mathcal{M}/{=^{\mathcal{M}}}|$ is infinite (countable, of cardinality n). By an *axiom system* Γ we mean a set of closed formulas such that $\mathrm{Eq}_{L(\Gamma)} \subseteq \Gamma$. A *model* of an axiom system Γ is an $\mathcal{L}$-model $\mathcal{M}$ such that $L(\Gamma) \subseteq \mathcal{L}$ and $\mathcal{M} \models \Gamma$. For sets Γ of closed formulas we write

$$\mathrm{Mod}_{\mathcal{L}}(\Gamma) := \{\, \mathcal{M} \mid \mathcal{M} \text{ is an } \mathcal{L}\text{-model and } \mathcal{M} \models \Gamma \cup \mathrm{Eq}_{\mathcal{L}} \,\}.$$

Clearly Γ is satisfiable if and only if Γ has a model.

THEOREM. *If an axiom system has arbitrarily large finite models, then it has an infinite model.*

PROOF. Let Γ be such an axiom system. Suppose $x_0, x_1, x_2, \dots$ are distinct variables and

$$\Gamma' := \Gamma \cup \{\, x_i \neq x_j \mid i, j \in \mathbb{N} \text{ such that } i < j \,\}.$$

By assumption every finite subset of Γ' is satisfiable, hence by the general compactness theorem so is Γ'. Then we have $\mathcal{M}$ and η such that $\mathcal{M} \models \Gamma'[\eta]$ and therefore $\eta(x_i) \neq^{\mathcal{M}} \eta(x_j)$ for $i < j$. Hence $\mathcal{M}$ is infinite. □

2.2.3. Complete theories, elementary equivalence. Let $\overline{\mathcal{L}}$ be the set of all closed $\mathcal{L}$-formulas. By a *theory* T we mean an axiom system closed under $\vdash_c$, that is, $\mathrm{Eq}_{L(T)} \subseteq T$ and

$$T = \{\, A \in \overline{L(T)} \mid T \vdash_c A \,\}.$$

A theory T is called *complete* if for every formula $A \in \overline{L(T)}$, $T \vdash_c A$ or $T \vdash_c \neg A$.

For every $\mathcal{L}$-model $\mathcal{M}$ (satisfying the equality axioms) the set of all closed $\mathcal{L}$-formulas A such that $\mathcal{M} \models A$ clearly is a theory; it is called the *theory of* $\mathcal{M}$ and denoted by $\mathrm{Th}(\mathcal{M})$.

Two $\mathcal{L}$-models $\mathcal{M}$ and $\mathcal{M}'$ are called *elementarily equivalent* (written $\mathcal{M} \equiv \mathcal{M}'$) if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{M}')$. Two $\mathcal{L}$-models $\mathcal{M}$ and $\mathcal{M}'$ are called *isomorphic* (written $\mathcal{M} \cong \mathcal{M}'$) if there is a map $\pi\colon |\mathcal{M}| \to |\mathcal{M}'|$ inducing a bijection between $|\mathcal{M}/{=^{\mathcal{M}}}|$ and $|\mathcal{M}'/{=^{\mathcal{M}'}}|$, that is,

$$\forall_{a,b \in |\mathcal{M}|}(a =^{\mathcal{M}} b \leftrightarrow \pi(a) =^{\mathcal{M}'} \pi(b)),$$
$$\forall_{a' \in |\mathcal{M}'|}\exists_{a \in |\mathcal{M}|}(\pi(a) =^{\mathcal{M}'} a'),$$

such that for all $a_1, \dots, a_n \in |\mathcal{M}|$

$$\pi(f^{\mathcal{M}}(a_1, \dots, a_n)) =^{\mathcal{M}'} f^{\mathcal{M}'}(\pi(a_1), \dots, \pi(a_n)),$$
$$R^{\mathcal{M}}(a_1, \dots, a_n) \leftrightarrow R^{\mathcal{M}'}(\pi(a_1), \dots, \pi(a_n))$$

for all n-ary function symbols f and relation symbols R of the language $\mathcal{L}$.

We collect some simple properties of the notions of the theory of a model $\mathcal{M}$ and of elementary equivalence.

LEMMA. (a) $\mathrm{Th}(\mathcal{M})$ *ist complete.*

(b) *If* Γ *is an axiom system such that* $L(\Gamma) \subseteq \mathcal{L}$, *then*

$$\{\, A \in \overline{\mathcal{L}} \mid \Gamma \vdash_c A \,\} = \bigcap \{\, \mathrm{Th}(\mathcal{M}) \mid \mathcal{M} \in \mathrm{Mod}_{\mathcal{L}}(\Gamma) \,\}.$$

(c) $\mathcal{M} \equiv \mathcal{M}' \leftrightarrow \mathcal{M} \models \mathrm{Th}(\mathcal{M}')$.

(d) *If* $\mathcal{L}$ *is countable, then for every* $\mathcal{L}$*-model* $\mathcal{M}$ *there is a countable* $\mathcal{L}$*-model* $\mathcal{M}'$ *such that* $\mathcal{M} \equiv \mathcal{M}'$.

PROOF. (a). Let $\mathcal{M}$ be an $\mathcal{L}$-model and $A \in \overline{\mathcal{L}}$. Then $\mathcal{M} \models A$ or $\mathcal{M} \models \neg A$, hence $\mathrm{Th}(\mathcal{M}) \vdash_c A$ or $\mathrm{Th}(\mathcal{M}) \vdash_c \neg A$.

(b). For all $A \in \overline{\mathcal{L}}$ we have

$$\begin{aligned} \Gamma \vdash_c A &\leftrightarrow \text{for all } \mathcal{L}\text{-models } \mathcal{M},\ (\mathcal{M} \models \Gamma \to \mathcal{M} \models A) \\ &\leftrightarrow \text{for all } \mathcal{L}\text{-models } \mathcal{M},\ (\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}}(\Gamma) \to A \in \mathrm{Th}(\mathcal{M})) \\ &\leftrightarrow A \in \bigcap \{\, \mathrm{Th}(\mathcal{M}) \mid \mathcal{M} \in \mathrm{Mod}_{\mathcal{L}}(\Gamma) \,\}. \end{aligned}$$

(c). For $\to$ assume $\mathcal{M} \equiv \mathcal{M}'$ and $A \in \mathrm{Th}(\mathcal{M}')$. Then $\mathcal{M}' \models A$, hence $\mathcal{M} \models A$. For $\leftarrow$ assume $\mathcal{M} \models \mathrm{Th}(\mathcal{M}')$. Then clearly $\mathrm{Th}(\mathcal{M}') \subseteq \mathrm{Th}(\mathcal{M})$. For the converse inclusion let $A \in \mathrm{Th}(\mathcal{M})$. If $A \notin \mathrm{Th}(\mathcal{M}')$, then $\neg A \in \mathrm{Th}(\mathcal{M}')$ by (a) and hence $\mathcal{M} \models \neg A$, contradicting $A \in \mathrm{Th}(\mathcal{M})$.

(d). Let $\mathcal{L}$ be countable and $\mathcal{M}$ an $\mathcal{L}$-model. Then $\mathrm{Th}(\mathcal{M})$ is satisfiable and therefore by the theorem of Löwenheim and Skolem possesses a satisfying $\mathcal{L}$-model $\mathcal{M}'$ with the countable carrier set $\mathrm{Ter}_{\mathcal{L}}$. By (c), $\mathcal{M} \equiv \mathcal{M}'$. □

Moreover, we can characterize complete theories as follows:

THEOREM. *Let T be a theory and $\mathcal{L} = L(T)$. Then the following are equivalent.*

(a) *T is complete.*
(b) *For every model $\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}}(T)$, $\mathrm{Th}(\mathcal{M}) = T$.*
(c) *Any two models $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}_{\mathcal{L}}(T)$ are elementarily equivalent.*

PROOF. (a) $\to$ (b). Let T be complete and $\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}}(T)$. Then $\mathcal{M} \models T$, hence $T \subseteq \mathrm{Th}(\mathcal{M})$. For the converse assume $A \in \mathrm{Th}(\mathcal{M})$. Then $\neg A \notin \mathrm{Th}(\mathcal{M})$, hence $\neg A \notin T$ and therefore $A \in T$.

(b) $\to$ (c) is clear.

(c) $\to$ (a). Let $A \in \overline{\mathcal{L}}$ and $T \nvdash_c A$. Then there is a model $\mathcal{M}_0$ of $T \cup \{\neg A\}$. Now let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}}(T)$ be arbitrary. By (c) we have $\mathcal{M} \equiv \mathcal{M}_0$, hence $\mathcal{M} \models \neg A$. Therefore $T \vdash_c \neg A$. □

2.2.4. Elementary equivalence and isomorphism.

LEMMA. *Let π be an isomorphism between $\mathcal{M}$ and $\mathcal{M}'$. Then for all terms t and formulas A and for every sufficiently big assignment η in $|\mathcal{M}|$*

(a) *$\pi(t^{\mathcal{M}}[\eta]) =^{\mathcal{M}'} t^{\mathcal{M}'}[\pi \circ \eta]$ and*
(b) *$\mathcal{M} \models A[\eta] \leftrightarrow \mathcal{M}' \models A[\pi \circ \eta]$. In particular,*

$$\mathcal{M} \cong \mathcal{M}' \to \mathcal{M} \equiv \mathcal{M}'.$$

Proof. (a). Induction on t. For simplicity we only consider the case of a unary function symbol.

$$\begin{aligned}
\pi(x^{\mathcal{M}}[\eta]) &= \pi(\eta(x)) = x^{\mathcal{M}'}[\pi \circ \eta] \\
\pi((ft)^{\mathcal{M}}[\eta]) &= \pi(f^{\mathcal{M}}(t^{\mathcal{M}}[\eta])) \\
&=^{\mathcal{M}'} f^{\mathcal{M}'}(\pi(t^{\mathcal{M}}[\eta])) \\
&=^{\mathcal{M}'} f^{\mathcal{M}'}(t^{\mathcal{M}'}[\pi \circ \eta]) \\
&= (ft)^{\mathcal{M}'}[\pi \circ \eta].
\end{aligned}$$

(b). Induction on A. For simplicity we only consider the case of a unary relation symbol P and the case $\forall_x A$.

$$\begin{aligned}
\mathcal{M} \models (Pr)[\eta] &\leftrightarrow P^{\mathcal{M}}(r^{\mathcal{M}}[\eta]) \\
&\leftrightarrow P^{\mathcal{M}'}(\pi(r^{\mathcal{M}}[\eta])) \\
&\leftrightarrow P^{\mathcal{M}'}(r^{\mathcal{M}'}[\pi \circ \eta]) \\
&\leftrightarrow \mathcal{M}' \models (Pr)[\pi \circ \eta],
\end{aligned}$$

$$\begin{aligned}
\mathcal{M} \models \forall_x A[\eta] &\leftrightarrow \forall_{a \in |\mathcal{M}|}(\mathcal{M} \models A[\eta_x^a]) \\
&\leftrightarrow \forall_{a \in |\mathcal{M}|}(\mathcal{M}' \models A[\pi \circ \eta_x^a]) \\
&\leftrightarrow \forall_{a \in |\mathcal{M}|}(\mathcal{M}' \models A[(\pi \circ \eta)_x^{\pi(a)}]) \\
&\leftrightarrow \forall_{a' \in |\mathcal{M}'|}(\mathcal{M}' \models A[(\pi \circ \eta)_x^{a'}]) \\
&\leftrightarrow \mathcal{M}' \models \forall_x A[\pi \circ \eta]
\end{aligned}$$

□

The converse, i.e., that $\mathcal{M} \equiv \mathcal{M}'$ implies $\mathcal{M} \cong \mathcal{M}'$, is true for finite models, but not for infinite ones. This proves the impossibility to characterize models by first order axioms.

Theorem. *For every infinite model $\mathcal{M}$ there is an elementarily equivalent model $\mathcal{M}_0$ not isomorphic to $\mathcal{M}$.*

Proof. Let $=^{\mathcal{M}}$ be the equality on $D := |\mathcal{M}|$, and let $\mathcal{P}(D)$ denote the power set of D. For every $\alpha \in \mathcal{P}(D)$ choose a new constant c_α. In the language $\mathcal{L}' := \mathcal{L} \cup \{ c_\alpha \mid \alpha \in \mathcal{P}(D) \}$ we consider the axiom system

$$\Gamma := \mathrm{Th}(\mathcal{M}) \cup \{ c_\alpha \neq c_\beta \mid \alpha, \beta \in \mathcal{P}(D) \text{ and } \alpha \neq \beta \} \cup \mathrm{Eq}_{\mathcal{L}'}.$$

Every finite subset of Γ is satisfiable by an appropriate expansion of $\mathcal{M}$. Hence by the general compactness theorem also Γ is satisfiable, say by $\mathcal{M}_0'$. Let $\mathcal{M}_0 := \mathcal{M}_0' \restriction \mathcal{L}$. We may assume that $=^{\mathcal{M}_0}$ is the equality on $|\mathcal{M}_0|$. $\mathcal{M}_0$ is not isomorphic to $\mathcal{M}$, for otherwise we would have an injection of $\mathcal{P}(D)$ into D and therefore a contradiction. □

2.3. Applications

2.3.1. Non-standard models. By what we just proved it is impossible to characterize an infinite model by a first order axiom system up to isomorphism. However, if we extend first order logic by also allowing quantification over sets X, we can formulate the following *Peano axioms*

$$\forall_n(\mathrm{S}n \neq 0),$$
$$\forall_{n,m}(\mathrm{S}n = \mathrm{S}m \to n = m),$$
$$\forall_X(0 \in X \to \forall_n(n \in X \to \mathrm{S}n \in X) \to \forall_n(n \in X)).$$

One can show easily that $(\mathbb{N}, 0, \mathrm{S})$ is up to isomorphism the unique model of the Peano axioms. A model which is elementarily equivalent, but not isomorphic to $\mathcal{N} := (\mathbb{N}, 0, \mathrm{S})$, is called a *non-standard model* of $\mathcal{N}$. In such non-standard models the principle of complete induction does not hold for all subsets of $|\mathcal{N}|$.

THEOREM. *There are countable non-standard models of the natural numbers.*

PROOF. Let x be a variable and $\Gamma := \mathrm{Th}(\mathcal{N}) \cup \{\, x \neq \underline{n} \mid n \in \mathbb{N} \,\}$, where $\underline{0} := 0$ and $\underline{n+1} := \mathrm{S}\underline{n}$. Clearly every finite subset of Γ is satisfiable, hence by compactness also Γ. By the theorem of Löwenheim and Skolem we then have a countable or finite $\mathcal{M}$ and an assignment η such that $\mathcal{M} \models \Gamma[\eta]$. Because of $\mathcal{M} \models \mathrm{Th}(\mathcal{N})$ we have $\mathcal{M} \equiv \mathcal{N}$ by 2.2.3; hence $\mathcal{M}$ is countable. Moreover $\eta(x) \neq^{\mathcal{M}} \underline{n}^{\mathcal{M}}$ for all $n \in \mathbb{N}$, hence $\mathcal{M} \not\cong \mathcal{N}$. □

2.3.2. Archimedian ordered fields. We now consider some easy applications to well-known axiom systems. The axioms of *field theory* are (the equality axioms and)

$$\begin{aligned} &x + (y + z) = (x + y) + z, && x \cdot (y \cdot z) = (x \cdot y) \cdot z, \\ &0 + x = x, && 1 \cdot x = x, \\ &(-x) + x = 0, && x \neq 0 \to x^{-1} \cdot x = 1, \\ &x + y = y + x, && x \cdot y = y \cdot x, \end{aligned}$$

and also

$$(x + y) \cdot z = (x \cdot z) + (y \cdot z),$$
$$1 \neq 0.$$

Fields are the models of this axiom system.

In the theory of ordered fields one has in addition a binary relation symbol $<$ and as axioms

$$x \not< x,$$

$$x < y \to y < z \to x < z,$$
$$x < y \lor x = y \lor y < x,$$
$$x < y \to x + z < y + z,$$
$$0 < x \to 0 < y \to 0 < x \cdot y.$$

Ordered fields are the models of this extended axiom system. An ordered field is called *archimedian ordered* if for every element a of the field there is a natural number n such that a is less than the n-fold multiple of the 1 in the field.

THEOREM. *For every archimedian ordered field there is an elementarily equivalent ordered field that is not archimedian ordered.*

PROOF. Let $\mathcal{K}$ be an archimedian ordered field, x a variable and

$$\Gamma := \mathrm{Th}(\mathcal{K}) \cup \{\, \underline{n} < x \mid n \in \mathbb{N} \,\}.$$

Clearly every finite subset of Γ is satisfiable, hence by the general compactness theorem also Γ. Therefore we have $\mathcal{M}$ and η such that $\mathcal{M} \models \Gamma[\eta]$. Because of $\mathcal{M} \models \mathrm{Th}(\mathcal{K})$ we obtain $\mathcal{M} \equiv \mathcal{K}$ and hence $\mathcal{M}$ is an ordered field. Moreover $1^{\mathcal{M}} \cdot n <^{\mathcal{M}} \eta(x)$ for all $n \in \mathbb{N}$, hence $\mathcal{M}$ is not archimedian ordered. □

2.3.3. Axiomatizable models. A class $\mathcal{S}$ of $\mathcal{L}$-models is (*finitely*) *axiomatizable* if there is a (finite) axiom system Γ such that $\mathcal{S} = \mathrm{Mod}_{\mathcal{L}}(\Gamma)$. Clearly $\mathcal{S}$ is finitely axiomatizable if and only if $\mathcal{S} = \mathrm{Mod}_{\mathcal{L}}(\{A\})$ for some formula A. If for every $\mathcal{M} \in \mathcal{S}$ there is an elementarily equivalent $\mathcal{M}' \notin \mathcal{S}$, then $\mathcal{S}$ cannot possibly be axiomatizable. By the theorem above we can conclude that the class of archimedian ordered fields is not axiomatizable. It also follows that the class of non archimedian ordered fields is not axiomatizable.

LEMMA. *Let $\mathcal{S}$ be a class of $\mathcal{L}$-models and Γ an axiom system.*

(a) *$\mathcal{S}$ is finitely axiomatizable if and only if $\mathcal{S}$ and the complement of $\mathcal{S}$ are axiomatizable.*
(b) *If $\mathrm{Mod}_{\mathcal{L}}(\Gamma)$ is finitely axiomatizable, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\mathrm{Mod}_{\mathcal{L}}(\Gamma_0) = \mathrm{Mod}_{\mathcal{L}}(\Gamma)$.*

PROOF. (a). Let $\mathcal{S}^{\mathrm{C}}$ denote the complement of $\mathcal{S}$. For $\to$ assume $\mathcal{S} = \mathrm{Mod}_{\mathcal{L}}(\{A\})$. Then $\mathcal{M} \in \mathcal{S}^{\mathrm{C}} \leftrightarrow \mathcal{M} \models \neg A$, hence $\mathcal{S}^{\mathrm{C}} = \mathrm{Mod}_{\mathcal{L}}(\{\neg A\})$.

For the converse. assume $\mathcal{S} = \mathrm{Mod}_{\mathcal{L}}(\Gamma_1)$ and $\mathcal{S}^{\mathrm{C}} = \mathrm{Mod}_{\mathcal{L}}(\Gamma_2)$. Then $\Gamma_1 \cup \Gamma_2$ is not satisfiable, hence there is a finite $\Gamma \subseteq \Gamma_1$ such that $\Gamma \cup \Gamma_2$ is not satisfiable. One obtains

$$\mathcal{M} \in \mathcal{S} \to \mathcal{M} \models \Gamma \to \mathcal{M} \not\models \Gamma_2 \to \mathcal{M} \notin \mathcal{S}^{\mathrm{C}} \to \mathcal{M} \in \mathcal{S}.$$

Hence $\mathcal{S} = \mathrm{Mod}_{\mathcal{L}}(\Gamma)$.

(b). Let $\mathrm{Mod}_{\mathcal{L}}(\Gamma) = \mathrm{Mod}_{\mathcal{L}}(\{A\})$. Then $\Gamma \vdash_c A$, hence also $\Gamma_0 \vdash_c A$ for a finite $\Gamma_0 \subseteq \Gamma$. One obtains

$$\mathcal{M} \models \Gamma \to \mathcal{M} \models \Gamma_0 \to \mathcal{M} \models A \to \mathcal{M} \models \Gamma.$$

Hence $\mathrm{Mod}_{\mathcal{L}}(\Gamma_0) = \mathrm{Mod}_{\mathcal{L}}(\Gamma)$. □

2.3.4. Dense linear orders without end points. Finally we consider as an example of a complete theory the theory DO of dense linear orders without end points. The axioms are (the equality axioms and)

$$\begin{array}{ll} x \not< x, & x < y \to \exists_z (x < z \wedge z < y), \\ x < y \to y < z \to x < z, & \exists_y (x < y), \\ x < y \vee x = y \vee y < x, & \exists_y (y < x). \end{array}$$

Lemma. *Every countable model of* DO *is isomorphic to the model* $(\mathbb{Q}, <)$ *of rational numbers.*

Proof. Let $\mathcal{M} = (D, \prec)$ be a countable model of DO; we can assume that $=^{\mathcal{M}}$ is the equality on D. Let $D = \{\, b_n \mid n \in \mathbb{N} \,\}$ and $\mathbb{Q} = \{\, a_n \mid n \in \mathbb{N} \,\}$, where we may assume $a_n \neq a_m$ and $b_n \neq b_m$ for $n < m$. We define recursively functions $f_n \subseteq \mathbb{Q} \times D$ as follows. Let $f_0 := \{(a_0, b_0)\}$. Assume we have already constructed f_n.

Case $n+1 = 2m$. Let j be minimal such that $b_j \notin \mathrm{ran}(f_n)$. Choose $a_i \notin \mathrm{dom}(f_n)$ such that for all $a \in \mathrm{dom}(f_n)$ we have $a_i < a \leftrightarrow b_j < f_n(a)$; such an a_i exists, since $\mathcal{M}$ and $(\mathbb{Q}, <)$ are models of DO. Let $f_{n+1} := f_n \cup \{(a_i, b_j)\}$.

Case $n+1 = 2m+1$. This is treated similarly. Let i be minimal such that $a_i \notin \mathrm{dom}(f_n)$. Choose $b_j \notin \mathrm{ran}(f_n)$ such that for all $a \in \mathrm{dom}(f_n)$ we have $a_i < a \leftrightarrow b_j < f_n(a)$; such a b_j exists, since $\mathcal{M}$ and $(\mathbb{Q}, <)$ are models of DO. Let $f_{n+1} := f_n \cup \{(a_i, b_j)\}$.

Then $\{b_0, \dots, b_m\} \subseteq \mathrm{ran}(f_{2m})$ and $\{a_0, \dots, a_{m+1}\} \subseteq \mathrm{dom}(f_{2m+1})$ by construction, and $f := \bigcup_n f_n$ is an isomorphism of $(\mathbb{Q}, <)$ onto $\mathcal{M}$. □

Theorem. *The theory* DO *is complete, and* $\mathrm{DO} = \mathrm{Th}(\mathbb{Q}, <)$.

Proof. Clearly $(\mathbb{Q}, <)$ is a model of DO. Hence by 2.2.3 it suffices to show that for every model $\mathcal{M}$ of DO we have $\mathcal{M} \equiv (\mathbb{Q}, <)$. So let $\mathcal{M}$ model of DO. By 2.2.3 there is a countable $\mathcal{M}'$ such that $\mathcal{M} \equiv \mathcal{M}'$. By the preceding lemma $\mathcal{M}' \cong (\mathbb{Q}, <)$, hence $\mathcal{M} \equiv \mathcal{M}' \equiv (\mathbb{Q}, <)$. □

A further example of a complete theory is the theory of algebraically closed fields. For a proof of this fact and for many more subjects of model theory we refer to the literature (e.g., Chang and Keisler (1990)).