CHAPTER 2

Models

It is an obvious question to ask whether the logical rules we have been considering suffice, i.e. whether we have forgotten some necessary rules. To answer this question we first have to fix the *meaning* of a formula, i.e. we have to provide a semantics.

This is rather straightforward for classical logic: we can take the usual notion of a structure (or model, or (universal) algebra). However, for minimal and intuitionistic logic we need a more refined notion: we shall use so-called Beth-structures here. Using this concept of a model we will prove soundness and completeness for both, minimal and intuitionistic logic. As a corollary we will obtain completeness of classical logic, w.r.t. the standard notion of a structure.

1. Structures for Classical Logic

1.1. Structures. We define the notion of a structure (more accurately $\mathcal{L}$-structure) and define what the value of a term and the meaning of a formula in such a structure should be.

DEFINITION. $\mathcal{M} = (D, I)$ is a *pre-structure* (or $\mathcal{L}$-pre-structure), if D a non-empty set (the *carrier set* or the *domain* of $\mathcal{M}$) and I is a map (*interpretation*) assigning to every n-ary function symbol f of $\mathcal{L}$ a function

$$I(f)\colon D^n \to D.$$

In case $n = 0$, $I(f)$ is an element of D. $\mathcal{M} = (D, I_0, I_1)$ is a *structure* (or $\mathcal{L}$-structure), if (D, I_0) is a pre-structure and I_1 a map assigning to every n-ary relation symbol R of $\mathcal{L}$ an n-ary relation

$$I_1(R) \subseteq D^n.$$

In case $n = 0$, $I_1(R)$ is one of the truth values 1 and 0; in particular we require $I_1(\bot) = 0$.

If $\mathcal{M} = (D, I)$ or (D, I_0, I_1), then we often write $|\mathcal{M}|$ for the carrier set D of $\mathcal{M}$ and $f^{\mathcal{M}}$, $R^{\mathcal{M}}$ for the interpretations $I_0(f)$, $I_1(R)$ of the function and relation symbols.

An *assignment* (or variable assignment) in D is a map η assigning to every variable $x \in \mathsf{dom}(\eta)$ a value $\eta(x) \in D$. Finite assignments will be written as $[x_1 := a_1, \dots, x_n := a_n]$ (or else as $[a_1/x_1, \dots, a_n/x_n]$), with distinct $x_1, \dots, x_n$. If η is an assignment in D and $a \in D$, let η_x^a be the assignment in D mapping x to a and coinciding with η elsewhere, so

$$\eta_x^a(y) := \begin{cases} \eta(y), & \text{if } y \neq x \\ a, & \text{if } y = x. \end{cases}$$

Let a pre-structure $\mathcal{M}$ and an assignment η in $|\mathcal{M}|$ be given. We define a homomorphic extension of η (denoted by η as well) to the set Set $\mathsf{Ter}_{\mathcal{L}}$ of $\mathcal{L}$-terms t such that $\mathsf{vars}(t) \subseteq \mathsf{dom}(\eta)$ by

$$\begin{aligned} \eta(c) &:= c^{\mathcal{M}}, \\ \eta(f(t_1,\dots,t_n)) &:= f^{\mathcal{M}}(\eta(t_1),\dots,\eta(t_n)). \end{aligned}$$

Observe that the extension of η depends on $\mathcal{M}$; therefore we may also write $t^{\mathcal{M}}[\eta]$ for $\eta(t)$.

For every structure $\mathcal{M}$, assignment η in $|\mathcal{M}|$ and formula A with $\mathsf{FV}(A) \subseteq \mathsf{dom}(\eta)$ we define $\mathcal{M} \models A[\eta]$ (read: A is *valid* in $\mathcal{M}$ under the assignment η) by recursion on A.

$$\begin{aligned}
\mathcal{M} \models R(t_1,\dots,t_n)[\eta] \quad &:\Longleftrightarrow \quad (t_1^{\mathcal{M}}[\eta],\dots,t_n^{\mathcal{M}}[\eta]) \in I_1(R) \quad \text{for } R \text{ not 0-ary.} \\
\mathcal{M} \models R[\eta] \quad &:\Longleftrightarrow \quad I_1(R) = 1 \quad \text{for } R \text{ 0-ary.} \\
\mathcal{M} \models (A \wedge B)[\eta] \quad &:\Longleftrightarrow \quad \mathcal{M} \models A[\eta] \text{ and } \mathcal{M} \models B[\eta]. \\
\mathcal{M} \models (A \vee B)[\eta] \quad &:\Longleftrightarrow \quad \mathcal{M} \models A[\eta] \text{ or } \mathcal{M} \models B[\eta]. \\
\mathcal{M} \models (A \to B)[\eta] \quad &:\Longleftrightarrow \quad \text{if } \mathcal{M} \models A[\eta], \text{ then } \mathcal{M} \models B[\eta]. \\
\mathcal{M} \models (\forall x A)[\eta] \quad &:\Longleftrightarrow \quad \text{for all } a \in |\mathcal{M}| \text{ we have } \mathcal{M} \models A[\eta_x^a]. \\
\mathcal{M} \models (\exists x A)[\eta] \quad &:\Longleftrightarrow \quad \text{there is an } a \in |\mathcal{M}| \text{ such that } \mathcal{M} \models A[\eta_x^a].
\end{aligned}$$

Because of $I_1(\bot) = 0$ we have in particular $\mathcal{M} \not\models \bot[\eta]$.

If Γ is a set of formulae, we write $\mathcal{M} \models \Gamma[\eta]$, if for all $A \in \Gamma$ we have $\mathcal{M} \models A[\eta]$. If $\mathcal{M} \models A[\eta]$ for all assignments η in $|\mathcal{M}|$, we write $\mathcal{M} \models A$.

1.2. Coincidence and Substitution Lemma.

LEMMA (Coincidence). *Let $\mathcal{M}$ be a structure, t a term, A a formula and η, ξ assignments in $|\mathcal{M}|$.*

(a) *If $\eta(x) = \xi(x)$ for all $x \in \mathsf{vars}(t)$, then $\eta(t) = \xi(t)$.*
(b) *If $\eta(x) = \xi(x)$ for all $x \in \mathsf{FV}(A)$, then $\mathcal{M} \models A[\eta]$ iff $\mathcal{M} \models A[\xi]$.*

PROOF. Induction on terms and formulae. □

LEMMA (Substitution). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, t, r $\mathcal{L}$-terms, A an $\mathcal{L}$-formula and η an assignment in $|\mathcal{M}|$. Then*

(a) $\eta(r[x := t]) = \eta_x^{\eta(t)}(r)$.
(b) $\mathcal{M} \models A[x := t][\eta] \iff \mathcal{M} \models A[\eta_x^{\eta(t)}]$.

PROOF. (a). Induction on r. (b). Induction on A. We restrict ourselves to the cases of an atomic formula and a universal formula; the other cases are easier.

Case $R(s_1,\dots,s_n)$. For simplicity assume $n = 1$. Then

$$\begin{aligned}
\mathcal{M} \models R(s)[x := t][\eta] &\iff \mathcal{M} \models R(s[x := t])[\eta] \\
&\iff \eta(s[x := t]) \in R^{\mathcal{M}} \\
&\iff \eta_x^{\eta(t)}(s) \in R^{\mathcal{M}} \quad \text{by (a)} \\
&\iff \mathcal{M} \models R(s)[\eta_x^{\eta(t)}].
\end{aligned}$$

Case $\forall yA$. We may assume $y \neq x$ and $y \notin \mathsf{vars}(t)$.

$$\begin{aligned}
&\mathcal{M} \models (\forall yA)[x := t][\eta] \\
&\iff \mathcal{M} \models (\forall yA[x := t])[\eta] \\
&\iff \text{for all } a \in |\mathcal{M}|,\ \mathcal{M} \models A[x := t][\eta_y^a] \\
&\iff \text{for all } a \in |\mathcal{M}|,\ \mathcal{M} \models A[(\eta_y^a)_x^b] \text{ with } b := \eta_y^a(t) = \eta(t) \\
&\qquad \text{(by IH and the coincidence lemma)} \\
&\iff \text{for all } a \in |\mathcal{M}|,\ \mathcal{M} \models A[(\eta_x^b)_y^a], \quad \text{(because } x \neq y) \\
&\iff \mathcal{M} \models (\forall yA)[\eta_x^b]
\end{aligned}$$

This completes the proof. □

1.3. Soundness. We prove the soundness theorem: it says that every formula derivable in classical logic is valid in an arbitrary structure.

THEOREM (Soundness). *Let* $\Gamma \vdash_c B$. *If* $\mathcal{M}$ *is a structure and* η *an assignment in* $|\mathcal{M}|$, *then* $\mathcal{M} \models \Gamma[\eta]$ *entails* $\mathcal{M} \models B[\eta]$.

PROOF. Induction on derivations. The given derivation of B from Γ can only have finitely many free assumptions; hence we may assume $\Gamma = \{A_1, \dots, A_n\}$.

Case $u\colon B$. Then $B \in \Gamma$ and the claim is obvious.

Case $\mathsf{Stab}_R\colon \forall\vec{x}.\neg\neg R\vec{x} \to R\vec{x}$. Again the claim is clear, since $\mathcal{M} \models \neg\neg A[\eta]$ means the same as $\mathcal{M} \models A[\eta]$.

Case $\to^-$. Assume $\mathcal{M} \models \Gamma[\eta]$. We must show $\mathcal{M} \models B[\eta]$. By IH, $\mathcal{M} \models (A \to B)[\eta]$ and $\mathcal{M} \models A[\eta]$. The claim follows from the definition of $\models$.

Case $\to^+$. Assume $\mathcal{M} \models \Gamma[\eta]$. We must show $\mathcal{M} \models (A \to B)[\eta]$. So assume in addition $\mathcal{M} \models A[\eta]$. We must show $\mathcal{M} \models B[\eta]$. By IH (with $\Gamma \cup \{A\}$ instead of Γ) this clearly holds.

Case $\forall^+$. Assume $\mathcal{M} \models \Gamma[\eta]$. We must show $\mathcal{M} \models A[\eta_x^a]$. We may assume that all assumptions $A_1, \dots, A_n$ actually in the given derivation. Since because of the variable condition for $\forall^+$ the variable x does not appear free in any of the formulae $A_1, \dots, A_n$, we have by the coincidence lemma $\mathcal{M} \models \Gamma[\eta_x^a]$. The IH (with η_x^a instead of η) yields $\mathcal{M} \models A[\eta_x^a]$.

Case $\forall^-$. Assume $\mathcal{M} \models \Gamma[\eta]$. We must show $\mathcal{M} \models A[x := t][\eta]$, i.e. by the substitution lemma $\mathcal{M} \models A[\eta_x^b]$ with $b := \eta(t)$. By IH, $\mathcal{M} \models (\forall xA)[\eta]$, i.e. $\mathcal{M} \models A[\eta_x^a]$ for all $a \in |\mathcal{M}|$. With $\eta(t)$ for a the claim follows.

The other cases are proved similarly. □

2. Beth-Structures for Minimal Logic

2.1. Beth-Structures. Consider a partially ordered set of "possible worlds". The worlds are represented as nodes in a finitely branching tree. They may be thought of as possible states such that all nodes "above" a node k are the ways in which k may develop in the future. The worlds are increasing, that is, if an atomic formula $R\vec{t}$ true is in a world k, then $R\vec{t}$ is true in all worlds "above" k.

More formally, each Beth-structure is based on a finitely branching tree T. A node k over a set S is a finite sequence $k = \langle a_0, a_1, \dots, a_{n-1}\rangle$ of elements of S; $\mathsf{lh}(k)$ is the length of k. We write $k \preceq k'$ if k is the initial segment of k'. A tree on S is a set of nodes closed under initial segments. A tree T is finitely branching if every node in T has finitely many immediate successors.

A tree T is *unbounded* if for every $n \in \mathbb{N}$ there is a node $k \in T$ such that $\mathsf{lh}(k) = n$. A *branch* of T is a linearly ordered subtree of T. A *leaf* is a node without successors in T.

For the proof of the completeness theorem, a Beth-structure based on a complete binary tree (i.e. the complete tree over $\{0, 1\}$) will suffice. The nodes will be all the finite sequences of 0's and 1's, and the ordering is as above. The root is the empty sequence and $k0$ is the sequence k with the postfix 0. Similarly for $k1$.

DEFINITION. Let $(T, \preceq)$ be a finitely branching tree. $\mathcal{B} = (D, I_0, I_1)$ is a *$\mathcal{L}$-Beth-structure* on T, where D is a nonempty set, and for each n-ary function symbol in $\mathcal{L}$, I_0 assigns f a map $I_0(f)\colon D^n \to D$. For each n-ary relation symbol R in $\mathcal{L}$ and each node $k \in T$, $I_1(R, k) \subseteq D^n$ is assigned in such a way that monotonicity is preserved, that is,

$$k \preceq k' \Rightarrow I_1(R, k) \subseteq I_1(R, k').$$

If $n = 0$, then $I_1(R, k)$ is either true or false, and it follows by the monotonicity that if $k \preceq k'$ and $I_1(R, k)$ then $I_1(R, k')$.

There is no special requirement set on $I_1(\bot, k)$. In minimal logic, *falsum* $\bot$ plays a role of an ordinary propositional variable.

For an assignment η, $t^{\mathcal{B}}[\eta]$ is understood classically. The classical satisfaction relation $\mathcal{M} \models A[\eta]$ is replaced with the forcing relation in Beth-structures. It is obvious from the definition that any T can be extended to a complete tree $\bar{T}$ without leaves, in which for each leaf $k \in T$ all sequences $k0$, $k00$, $k000, \dots$ are added to T. For each node $k0\dots0$, we add $I_1(R, k0\dots0) := I_1(R, k)$.

DEFINITION. $\mathcal{B}, k \Vdash A[\eta]$ ($\mathcal{B}$ forces A at a node k for an assignment η) is defined inductively as follows. We write $k \Vdash A[\eta]$ when it is clear from the context what the underlying structure $\mathcal{B}$ is, and we write $\forall k' {\succeq_n} k\, A$ for $\forall k' {\succeq} k. \mathsf{lh}(k') = \mathsf{lh}(k) + n \to A$.

$$\begin{array}{lll}
k \Vdash R(t_1, \dots, t_p)[\eta] & :\Longleftrightarrow & \exists n \forall k' {\succeq_n} k\, (t_1^{\mathcal{B}}[\eta], \dots, t_p^{\mathcal{B}}[\eta]) \in I_1(R, k'), \\
 & & \text{if } R \text{ is not 0-ary.} \\
k \Vdash R[\eta] & :\Longleftrightarrow & \exists n \forall k' {\succeq_n} k\, I_1(R, k') = 1 \quad \text{if } R \text{ is 0-ary.} \\
k \Vdash (A \vee B)[\eta] & :\Longleftrightarrow & \exists n \forall k' {\succeq_n} k. k' \Vdash A[\eta] \text{ or } k' \Vdash B[\eta]. \\
k \Vdash (\exists x A)[\eta] & :\Longleftrightarrow & \exists n \forall k' {\succeq_n} k \exists a {\in} |\mathcal{B}|\, k' \Vdash A[\eta_x^a]. \\
k \Vdash (A \to B)[\eta] & :\Longleftrightarrow & \forall k' {\succeq} k. k' \Vdash A[\eta] \Rightarrow k' \Vdash B[\eta]. \\
k \Vdash (A \wedge B)[\eta] & :\Longleftrightarrow & k \Vdash A[\eta] \text{ and } k \Vdash B[\eta]. \\
k \Vdash (\forall x A)[\eta] & :\Longleftrightarrow & \forall a {\in} |\mathcal{B}|\, k \Vdash A[\eta_x^a].
\end{array}$$

The clauses for atoms, disjunction and existential quantifier include a concept of a "bar", in $\bar{T}$.

2.2. Covering Lemma. It is easily seen (from the definition and using monotonicity) that from $k \Vdash A[\eta]$ and $k \preceq k'$ we can conclude $k' \Vdash A[\eta]$. The converse is also true:

LEMMA (Covering Lemma).

$$\forall k' \succeq_n k \, k' \Vdash A[\eta] \Rightarrow k \Vdash A[\eta].$$

PROOF. Induction on A. We write $k \Vdash A$ for $k \Vdash A[\eta]$.
Case $R\vec{t}$. Assume

$$\exists n \forall k' \succeq_n k \, k' \Vdash R\vec{t},$$

hence by definition

$$\exists n \forall k' \succeq_n k \exists m \forall k'' \succeq_m k' \vec{t}^{\mathcal{B}}[\eta] \in I_1(R, k'').$$

Since T is a finitely branching tree,

$$\exists m \forall k' \succeq_m k \, \vec{t}^{\mathcal{B}}[\eta] \in I_1(R, k').$$

Hence $k \Vdash R\vec{t}$.

The cases $A \vee B$ and $\exists x A$ are handled similarly.

Case $A \to B$. Let $k' \Vdash A \to B$ for all $k' \succeq k$ with $\mathsf{lh}(k') = \mathsf{lh}(k) + n$. We show

$$\forall l \succeq k . l \Vdash A \Rightarrow l \Vdash B.$$

Let $l \succeq k$ and $l \Vdash A$. We show that $l \Vdash B$. We apply the IH to B and $m := \max(\mathsf{lh}(k) + n, \mathsf{lh}(l))$. So assume $l' \succeq l$ and $\mathsf{lh}(l') = m$. It is sufficient to show $l' \Vdash B$. If $\mathsf{lh}(l') = \mathsf{lh}(l)$, then $l' = l$ and we are done. If $\mathsf{lh}(l') = \mathsf{lh}(k) + n > \mathsf{lh}(l)$, then l' is an extension of l as well as of k and length $\mathsf{lh}(k) + n$, and hence $l' \Vdash A \to B$ by assumption. Moreover, $l' \Vdash A$, since $l' \succeq l$ and $l \Vdash A$. It follows that $l' \Vdash B$.

The cases $A \wedge B$ and $\forall x A$ are obvious. □

2.3. Coincidence and Substitution. The coincidence and substitution lemmas hold for Beth-structures.

LEMMA (Coincidence). *Let $\mathcal{B}$ be a Beth-structure, t a term, A a formula and η, ξ assignments in $|\mathcal{B}|$.*

(a) *If $\eta(x) = \xi(x)$ for all $x \in \mathsf{vars}(t)$, then $\eta(t) = \xi(t)$.*
(b) *If $\eta(x) = \xi(x)$ for all $x \in \mathsf{FV}(A)$, then $\mathcal{B}, k \Vdash A[\eta] \iff \mathcal{B}, k \Vdash A[\xi]$.*

PROOF. Induction on terms and formulae. □

LEMMA (Substitution). *Let $\mathcal{B}$ be a Beth-structure, t, r terms, A a formula and η an assignment in $|\mathcal{B}|$. Then*

(a) $\eta(r[x := t]) = \eta_x^{\eta(t)}(r)$.
(b) $\mathcal{B}, k \Vdash A[x := t][\eta] \iff \mathcal{B}, k \Vdash A[\eta_x^{\eta(t)}]$.

PROOF. Induction on terms and formulae. □

2.4. Soundness. As usual, we proceed to prove soundness theorem.

THEOREM (Soundness). *Let* $\Gamma \cup \{A\}$ *be a set of formulae such that* $\Gamma \vdash A$. *Then, if* $\mathcal{B}$ *is a Beth-structure,* k *a node and* η *an assignment in* $|\mathcal{B}|$, *it follows that* $\mathcal{B}, k \Vdash \Gamma[\eta]$ *entails* $\mathcal{B}, k \Vdash A[\eta]$.

PROOF. Induction on derivations.

We begin with the axiom schemes $\vee_0^+$, $\vee_1^+$, $\vee^-$, $\exists^+$ and $\exists^-$. $k \Vdash C[\eta]$ is abbreviated $k \Vdash C$, when η is known from the context.

Case $\vee_0^+ : A \to A \vee B$. We show $k \Vdash A \to A \vee B$. Assume for $k' \succeq k$ that $k' \Vdash A$. Show: $k' \Vdash A \vee B$. This follows from the definition, since $k' \Vdash A$. The case $\vee_1^+ : B \to A \vee B$ is symmetric.

Case $\vee^- : A \vee B \to (A \to C) \to (B \to C) \to C$. We show that $k \Vdash A \vee B \to (A \to C) \to (B \to C) \to C$. Assume for $k' \succeq k$ that $k' \Vdash A \vee B$, $k' \Vdash A \to C$ and $k' \Vdash B \to C$ (we can safely assume that k' is the same for all three premises). Show that $k' \Vdash C$. By definition, there is an n s.t. for all $k'' \succeq_n k'$, $k'' \Vdash A$ or $k'' \Vdash B$. In both cases it follows that $k'' \Vdash C$, since $k' \Vdash A \to C$ and $k' \Vdash B \to C$. By the covering lemma, $k' \Vdash C$.

Case $\exists^+ : A \to \exists x A$. Show that $k \Vdash (A \to \exists x A)[\eta]$. Assume that $k' \succeq k$ and $k' \Vdash A[\eta]$. Show that $k' \Vdash (\exists x A)[\eta]$. Since $\eta = \eta_x^{\eta(x)}$ there is an $a \in |\mathcal{B}|$ (namely $a := \eta(x)$) such that $k' \Vdash A[\eta_x^a]$. Hence, $k' \Vdash (\exists x A)[\eta]$.

Case $\exists^- : \exists x A \to (\forall x. A \to B) \to B$ and $x \notin \mathsf{FV}(B)$. We show that $k \Vdash (\exists x A \to (\forall x. A \to B) \to B)[\eta]$. Assume that $k' \succeq k$ and $k' \Vdash (\exists x A)[\eta]$ and $k' \Vdash (\forall x. A \to B)[\eta]$. We show $k' \Vdash B[\eta]$. By definition, there is an n such that for all $k'' \succeq_n k'$ we have $a \in |\mathcal{B}|$ and $k'' \Vdash A[\eta_x^a]$. From $k' \Vdash (\forall x. A \to B)[\eta]$ follows that $k'' \Vdash B[\eta_x^a]$, and since $x \notin \mathsf{FV}(B)$, from the coincidence lemma, $k'' \Vdash B[\eta]$. Then, finally, by the covering lemma $k' \Vdash B[\eta]$.

Case $\to^+$. Let $k \Vdash \Gamma$ hold. We show that $k \Vdash A \to B$. Assume $k' \succeq k$ and $k' \Vdash A$. Our goal is $k' \Vdash B$. We have $k' \Vdash \Gamma \cup \{A\}$. Thus, $k' \Vdash B$ by IH.

Case $\to^-$. Let $k \Vdash \Gamma$ hold. The IH gives us $k \Vdash A \to B$ and $k \Vdash A$. Hence $k \Vdash B$.

Case $\forall^+$. Let $k \Vdash \Gamma[\eta]$ and $x \notin \mathsf{FV}(\Gamma)$ hold. Show that $k \Vdash (\forall x A)[\eta]$, i.e. $k \Vdash A[\eta_x^a]$ for an arbitrary $a \in |\mathcal{B}|$. We have

$$\begin{array}{ll} k \Vdash \Gamma[\eta_x^a] & \text{by the coincidence lemma, since } x \notin \mathsf{FV}(\Gamma) \\ k \Vdash A[\eta_x^a] & \text{by IH.} \end{array}$$

Case $\forall^-$. Let $k \Vdash \Gamma[\eta]$. We show that $k \Vdash A[x := t][\eta]$. We have

$$\begin{array}{ll} k \Vdash (\forall x A)[\eta] & \text{by IH} \\ k \Vdash A[\eta_x^{\eta(t)}] & \text{by definition} \\ k \Vdash A[x := t][\eta] & \text{by the substitution lemma.} \end{array}$$

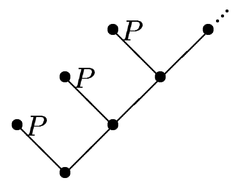
This concludes the proof. □

2.5. Counter Models. With soundness at hand, it is easy to build counter models for derivations not valid in minimal or intuitionistic logic.

A *Beth-structure* $\mathcal{B} = (D, I_0, I_1)$ *for intuitionistic logic* is a Beth-structure in which $\bot$ is never forced, i.e. $I_1(\bot, k) = 0$ for all k. Thus, in Beth-structures for intuitionistic logic we have

$$\begin{aligned} k \Vdash \neg A &\iff \forall k' \succeq k\, k' \not\Vdash A, \\ k \Vdash \neg\neg A &\iff \forall k' \succeq k\, k' \not\Vdash \neg A \\ &\iff \forall k' \succeq k \exists k'' \succeq k'\, k'' \Vdash A. \end{aligned}$$

As an example, we show that $\not\vdash_i \neg\neg P \to P$. We describe the desired Beth-structure by means of a diagram below. Next to each node, we write the propositions forced on that node.



The it is easily seen that

$$\langle\rangle \not\Vdash P, \qquad \langle\rangle \Vdash \neg\neg P.$$

Thus $\langle\rangle \not\Vdash \neg\neg P \to P$ and hence $\not\vDash \neg\neg P \to P$. Since for each R and all k, $k \Vdash \mathsf{Efq}_R$, it also follows that $\not\vDash_i \neg\neg P \to P$. The model also shows that the Pierce formula $((P \to Q) \to P) \to P$ is invalid in intuitionistic logic.

3. Completeness of Minimal and Intuitionistic Logic

Next, we show the converse of soundness theorem, for minimal as well as intuitionistic logic.

3.1. Completeness of Minimal Logic.

THEOREM (Completeness). *Let $\Gamma \cup \{A\}$ be a set of formulae. Then the following propositions are equivalent.*

(a) $\Gamma \vdash A$.

(b) $\Gamma \Vdash A$, *i.e. for all Beth-structures $\mathcal{B}$, nodes k and assignments η*

$$\mathcal{B}, k \Vdash \Gamma[\eta] \Rightarrow \mathcal{B}, k \Vdash A[\eta].$$

PROOF. Soundness is one direction. For the other direction we employ a technique developed by Harvey Friedman and construct a Beth-structure $\mathcal{B}$ (over the set T_{01} of all finite 0-1-sequences k ordered by the initial segment relation $k \preceq k'$) with the property that $\Gamma \vdash B$ is equivalent to $\mathcal{B}, \langle\rangle \Vdash B[\mathsf{id}]$.

In order to define $\mathcal{B}$, we will need an enumeration $A_0, A_1, A_2, \ldots$ of $\mathcal{L}$-formulae, in which each formula occurs countably many times. We also fix an enumeration $x_0, x_1, \ldots$ of variables. Let $\Gamma = \bigcup_n \Gamma_n$ be the union of finite sets Γ_n such that $\Gamma_n \subseteq \Gamma_{n+1}$. With each node $k \in T_{01}$, we associate a finite set Δ_k of formulae by induction on the length of k.

Let $\Delta_{\langle\rangle} := \emptyset$. Take a node k such that $\mathsf{lh}(k) = n$ and suppose that Δ_k is already defined. Write $\Delta \vdash_n B$ to mean that there is a derivation of length $\leq n$ of B from Δ. We define Δ_{k0} and Δ_{k1} as follows:

Case 1. $\Gamma_n, \Delta_k \not\vdash_n A_n$. Then let

$$\Delta_{k0} := \Delta_k \quad \text{and} \quad \Delta_{k1} := \Delta_k \cup \{A_n\}.$$

Case 2. $\Gamma_n, \Delta_k \vdash_n A_n = A'_n \vee A''_n$. Then let

$$\Delta_{k0} := \Delta_k \cup \{A_n, A'_n\} \quad \text{and} \quad \Delta_{k1} := \Delta_k \cup \{A_n, A''_n\}.$$

Case 3. $\Gamma_n, \Delta_k \vdash_n A_n = \exists x A'_n$. Then let

$$\Delta_{k0} := \Delta_{k1} := \Delta_k \cup \{A_n, A'_n[x := x_i]\}.$$

x_i is the first variable $\notin \mathsf{FV}(\Gamma_n, A_n, \Delta_k)$.

Case 4. $\Gamma_n, \Delta_k \vdash_n A_n$, and A_n is neither a disjunction nor an existentially quantified formula. Then let

$$\Delta_{k0} := \Delta_{k1} := \Delta_k \cup \{A_n\}.$$

Obviously $k \preceq k'$ implies that $\Delta_k \subseteq \Delta_{k'}$. We note that

$$\forall k' {\succeq_n} k \, \Gamma, \Delta_{k'} \vdash B \Rightarrow \Gamma, \Delta_k \vdash B. \tag{6}$$

It is sufficient to show

$$\Gamma, \Delta_{k0} \vdash B \quad \text{and} \quad \Gamma, \Delta_{k1} \vdash B \Rightarrow \Gamma, \Delta_k \vdash B.$$

In cases 1 and 4, this is obvious. For cases 2 and 3, it follows immediately from the axiom schemes $\vee^-$ and $\exists^-$.

Next, we show

$$\Gamma, \Delta_k \vdash B \Rightarrow \exists n \forall k' {\succeq_n} k \, B \in \Delta_{k'}. \tag{7}$$

We choose $n \geq \mathsf{lh}(k)$ such that $B = A_n$ and $\Gamma_n, \Delta_k \vdash_n A_n$. For all $k' \succeq k$, if $\mathsf{lh}(k') = n + 1$ then $A_n \in \Delta_{k'}$ (cf. the cases 2-4).

Using the sets Δ_k we can define an $\mathcal{L}$-Beth-structure $\mathcal{B}$ as $(\mathsf{Ter}_{\mathcal{L}}, I_0, I_1)$ (where $\mathsf{Ter}_{\mathcal{L}}$ denotes the set of terms of $\mathcal{L}$) and the canonical $I_0(f)\vec{t} := f\vec{t}$ and

$$\vec{t} \in I_1(R, k) \quad :\Longleftrightarrow \quad R\vec{t} \in \Delta_k.$$

Obviously, $t^{\mathcal{B}}[\mathsf{id}] = t$ for all $\mathcal{L}$-terms t.

We show that

$$\Gamma, \Delta_k \vdash B \iff \mathcal{B}, k \Vdash B[\mathsf{id}], \tag{8}$$

by induction on the complexity of B. For $\mathcal{B}, k \Vdash B[\mathsf{id}]$ we write $k \Vdash B$.

Case $R\vec{t}$. The following propositions are equivalent.

$\Gamma, \Delta_k \vdash R\vec{t}$	
$\exists n \forall k' {\succeq_n} k \, R\vec{t} \in \Delta_{k'}$	by (7) and (6)
$\exists n \forall k' {\succeq_n} k \, \vec{t} \in I_1(R, k')$	by definition of $\mathcal{B}$
$k \Vdash R\vec{t}$	by definition of $\Vdash$, since $t^{\mathcal{B}}[\mathsf{id}] = t$.

Case $B \vee C$. $\Rightarrow$. Let $\Gamma, \Delta_k \vdash B \vee C$. Choose an $n \geq \mathsf{lh}(k)$ such that $\Gamma_n, \Delta_k \vdash_n A_n = B \vee C$. Then, for all $k' \succeq k$ s.t. $\mathsf{lh}(k') = n$ it follows that

$$\Delta_{k'0} = \Delta_{k'} \cup \{B \vee C, B\} \quad \text{and} \quad \Delta_{k'1} = \Delta_{k'} \cup \{B \vee C, C\},$$

and by IH

$$k'0 \Vdash B \quad \text{and} \quad k'1 \Vdash C.$$

By definition, we have $k \Vdash B \vee C$. $\Leftarrow$.

$$k \Vdash B \vee C$$

$$\exists n \forall k' {\succeq_n} k \, .k' \Vdash B \ \text{ or } \ k' \Vdash C$$

$$\exists n \forall k' {\succeq_n} k\,.\Gamma, \Delta_{k'} \vdash B \ \text{ or } \ \Gamma, \Delta_{k'} \vdash C \quad \text{by IH}$$
$$\exists n \forall k' {\succeq_n} k\, \Gamma, \Delta_{k'} \vdash B \vee C$$
$$\Gamma, \Delta_k \vdash B \vee C \quad \text{by (6).}$$

The case $B \wedge C$ is evident.

Case $B \to C$. $\Rightarrow$. Let $\Gamma, \Delta_k \vdash B \to C$. We must show $k \Vdash B \to C$, i.e.,

$$\forall k' {\succeq} k . k' \Vdash B \Rightarrow k' \Vdash C.$$

Let $k' \succeq k$ be such that $k' \Vdash B$. By IH, it follows that $\Gamma, \Delta_{k'} \vdash B$, and $\Gamma, \Delta_{k'} \vdash C$ follows by assumption. Then again by IH $k' \Vdash C$.

$\Leftarrow$. Let $k \Vdash B \to C$, i.e. $\forall k' {\succeq} k . k' \Vdash B \Rightarrow k' \Vdash C$. We show that $\Gamma, \Delta_k \vdash B \to C$. At this point, we apply (6). Choose an $n \geq \mathsf{lh}(k)$ such that $B = A_n$. Let $k' \succeq_m k$ be such that $m := n - \mathsf{lh}(k)$. We show that $\Gamma, \Delta_{k'} \vdash B \to C$. If $\Gamma, \Delta_{k'} \vdash_n A_n$, then $k' \Vdash B$ by IH, and $k' \Vdash C$ by assumption, hence $\Gamma, \Delta_{k'} \vdash C$ again by IH and thus $\Gamma, \Delta_{k'} \vdash B \to C$.

If $\Gamma, \Delta_{k'} \nvdash_n A_n$ then by definition $\Delta_{k'1} = \Delta_{k'} \cup \{B\}$, hence $\Gamma, \Delta_{k'1} \vdash B$, and $k'1 \Vdash B$ by IH. Now $k'1 \Vdash C$ by assumption, and finally $\Gamma, \Delta_{k'1} \vdash C$ by IH. From $\Delta_{k'1} = \Delta_{k'} \cup \{B\}$, it follows that $\Gamma, \Delta_{k'} \vdash B \to C$.

Case $\forall x B$. The following propositions are equivalent.

$$\Gamma, \Delta_k \vdash \forall x B$$
$$\forall t {\in} \mathsf{Ter}_{\mathcal{L}}\, \Gamma, \Delta_k \vdash B[x := t]$$
$$\forall t {\in} \mathsf{Ter}_{\mathcal{L}}\, k \Vdash B[x := t] \quad \text{by IH}$$
$$\forall t {\in} \mathsf{Ter}_{\mathcal{L}}\, k \Vdash B[\mathsf{id}_x^t] \quad \text{by the substitution lemma, since } t^{\mathcal{B}}[\mathsf{id}] = t$$
$$k \Vdash \forall x B \quad \text{by definition of } \Vdash.$$

Case $\exists x B$. This case is similar to the case $\vee$. The proof proceeds as follows. $\Rightarrow$. Let $\Gamma, \Delta_k \vdash \exists x B$. Choose an $n \geq \mathsf{lh}(k)$ such that $\Gamma_n, \Delta_k \vdash_n A_n = \exists x B$. Then, for all $k' \succeq k$ such that $\mathsf{lh}(k') = n$ it follows that

$$\Delta_{k'0} = \Delta_{k'1} = \Delta_k \cup \{\exists x B, B[x := x_i]\}$$

where x_i is not free in $\Delta_k \cup \{\exists x B\}$. Hence by IH

$$k'0 \Vdash B[x := x_i] \quad \text{and} \quad k'1 \Vdash B[x := x_i].$$

It follows by definition that $k \Vdash \exists x B$. $\Leftarrow$.

$$k \Vdash \exists x B$$
$$\exists n \forall k' {\succeq_n} k \exists t {\in} \mathsf{Ter}_{\mathcal{L}}\, k' \Vdash B[\mathsf{id}_x^t]$$
$$\exists n \forall k' {\succeq_n} k \exists t {\in} \mathsf{Ter}_{\mathcal{L}}\, k' \Vdash B[x := t]$$
$$\exists n \forall k' {\succeq_n} k \exists t {\in} \mathsf{Ter}_{\mathcal{L}}\, \Gamma, \Delta_{k'} \vdash B[x := t] \quad \text{by IH}$$
$$\exists n \forall k' {\succeq_n} k\, \Gamma, \Delta_{k'} \vdash \exists x B$$
$$\Gamma, \Delta_k \vdash \exists x B \quad \text{by (6).}$$

Now, we are in a position to finalize the proof of the completeness theorem. We apply (b) to the Beth-structure $\mathcal{B}$ constructed above from Γ, the empty node $\langle\rangle$ and the assignment $\eta = \mathsf{id}$. Then $\mathcal{B}, \langle\rangle \Vdash \Gamma[\mathsf{id}]$ by (8), hence $\mathcal{B}, \langle\rangle \Vdash A[\mathsf{id}]$ by assumption and therefore $\Gamma \vdash A$ by (8) again. $\square$

3.2. Completeness of Intuitionistic Logic. Completeness of intuitionistic logic follows as a corollary.

COROLLARY. *Let $\Gamma \cup \{A\}$ be a set of formulae. The following propositions are equivalent.*

(a) $\Gamma \vdash_i A$.
(b) $\Gamma, \mathsf{Efq} \Vdash A$, *i.e., for all Beth-structures $\mathcal{B}$ for the intuitionistic logic, nodes k and assignments η*

$$\mathcal{B}, k \Vdash \Gamma[\eta] \Rightarrow \mathcal{B}, k \Vdash A[\eta]. \qquad \square$$

4. Completeness of Classical Logic

We give a proof of completeness of classical logic relying on the completeness proof for minimal logic above. Write $\Gamma \models A$ to mean that, for all structures $\mathcal{M}$ and assignments η,

$$\mathcal{M} \models \Gamma[\eta] \Rightarrow \mathcal{M} \models A[\eta].$$

4.1. The Completeness Theorem.

THEOREM (Completeness). *Let $\Gamma \cup \{A\}$ be a set of formulae (in $\mathcal{L}$). The following propositions are equivalent.*

(a) $\Gamma \vdash_c A$.
(b) $\Gamma \models A$.

PROOF. Soundness is one direction. For the other direction, we adapt the completeness of minimal logic.

Evidently, it is sufficient to treat formulae without $\vee$, $\exists$ and $\wedge$ (by Lemma 2.4).

Let $\Gamma \nvdash_c A$, i.e., $\Gamma, \mathsf{Stab} \nvdash A$. By the completeness theorem of minimal logic, there is a Beth-structure $\mathcal{B} = (\mathsf{Ter}_{\mathcal{L}}, I_0, I_1)$ on the complete binary tree T_{01} and a node l_0 such that $l_0 \Vdash \Gamma, \mathsf{Stab}$ and $l_0 \nVdash A$ (we write $k \Vdash B$ for $\mathcal{B}, k \Vdash B[\mathsf{id}]$).

A node k is *consistent* if $k \nVdash \bot$, and *stable* if $k \Vdash \mathsf{Stab}$. Let k be a stable node, and B a formula (without $\vee$, $\exists$ and $\wedge$). Then, $\mathsf{Stab} \vdash \neg\neg B \to B$ by the stability lemma. Hence, $k \Vdash \neg\neg B \to B$, and

$$\begin{aligned} k \nVdash B &\iff k \nVdash \neg\neg B \\ &\iff \exists k' \succeq k. k' \text{ consistent and } k' \Vdash \neg B. \end{aligned} \tag{9}$$

Let α be a branch in the underlying tree T_{01}. We define

$$\begin{aligned} \alpha \Vdash A \quad &:\iff \quad \exists k \in \alpha\, k \Vdash A, \\ \alpha \text{ is consistent} \quad &:\iff \quad \alpha \nVdash \bot, \\ \alpha \text{ is stable} \quad &:\iff \quad \exists k \in \alpha\, k \Vdash \mathsf{Stab}. \end{aligned}$$

Note that

$$\text{from } \alpha \Vdash \vec{A} \text{ and } \vdash \vec{A} \to B \text{ it follows that } \alpha \Vdash B. \tag{10}$$

To see this, consider $\alpha \Vdash \vec{A}$. Then $k \Vdash \vec{A}$ for a $k \in \alpha$, since α is linearly ordered. From $\vdash \vec{A} \to B$ it follows that $k \Vdash B$, i.e., $\alpha \Vdash B$.

A branch α is *generic* (in the sense that it generates a classical model) if it is consistent and stable, if in addition for all formulae B

$$\alpha \Vdash B \text{ or } \alpha \Vdash \neg B, \tag{11}$$

and for all formulae $\forall\vec{y}B$ (with $\vec{y}$ not empty) where B is not universally quantified

$$\forall \vec{s}{\in}\mathsf{Ter}_{\mathcal{L}}\, \alpha \Vdash B[\vec{y} := \vec{s}] \Rightarrow \alpha \Vdash \forall\vec{y}B \tag{12}$$

For a branch α, we define a classical structure $\mathcal{M}^\alpha = (\mathsf{Ter}_{\mathcal{L}}, I_0, I_1^\alpha)$ as

$$I_1^\alpha(R) := \bigcup_{k\in\alpha} I_1(R,k) \quad \text{for } R \neq \bot.$$

We show that for every generic branch α and each formula B with all connectives in $\{\to, \forall\}$

$$\alpha \Vdash B \iff \mathcal{M}^\alpha \models B. \tag{13}$$

The proof is by induction on the logical complexity of B.

Case $R\vec{t}$, $R \neq \bot$. Then the proposition holds for all α.

Case $\bot$. We have $\alpha \nVdash \bot$ for all consistent α.

Case $B \to C$. $\Rightarrow$. Let $\alpha \Vdash B \to C$ and $\mathcal{M}^\alpha \models B$. We must show that $\mathcal{M}^\alpha \models C$. Note that $\alpha \Vdash B$ by IH, hence $\alpha \Vdash C$, hence $\mathcal{M}^\alpha \models C$ again by IH. $\Leftarrow$. Let $\mathcal{M}^\alpha \models B \to C$. If $\mathcal{M}^\alpha \models B$, then $\mathcal{M}^\alpha \models C$, hence $\alpha \Vdash C$ by IH and therefore $\alpha \Vdash B \to C$. If $\mathcal{M}^\alpha \not\models B$, then $\alpha \nVdash B$ by IH, hence $\alpha \Vdash \neg B$ by (11) and therefore $\alpha \Vdash B \to C$, since α is stable (and $\vdash (\neg\neg C \to C) \to \bot \to C$).

Case $\forall\vec{y}B$ ($\vec{y}$ not empty) where B is not universally quantified. The following propositions are equivalent.

$$\begin{aligned} &\alpha \Vdash \forall\vec{y}B \\ &\forall \vec{s}{\in}\mathsf{Ter}_{\mathcal{L}}\, \alpha \Vdash B[\vec{y} := \vec{s}] && \text{by (12)} \\ &\forall \vec{s}{\in}\mathsf{Ter}_{\mathcal{L}}\, \mathcal{M}^\alpha \models B[\vec{y} := \vec{s}] && \text{by IH} \\ &\mathcal{M}^\alpha \models \forall\vec{y}B. \end{aligned}$$

We show that for each consistent stable node k, there is a generic branch containing k. For the purposes of the proof, we let $A_0, A_1, \dots$ be an enumeration of formulae. We define a sequence $k = k_0 \preceq k_1 \preceq k_2 \dots$ of consistent stable nodes inductively. Let $k_0 := k$. Assume that k_n is defined. We write A_n in the form $\forall\vec{y}B$ ($\vec{y}$ possibly empty) and B is not a universal formula. In case $k_n \Vdash \forall\vec{y}B$ let $k_{n+1} := k_n$. Otherwise we have $k_n \nVdash B[\vec{y} := \vec{s}]$ for some $\vec{s}$, and by (9) there is a consistent node $k' \succeq k_n$ such that $k' \Vdash \neg B[\vec{y} := \vec{s}]$. Let $k_{n+1} := k'$. Since $k_n \preceq k_{n+1}$, the node k_{n+1} is stable.

Let $\alpha := \{\, l \mid \exists n\; l \preceq k_n \,\}$, hence $k \in \alpha$. We show that α is generic. Clearly α is consistent and stable. The propositions (11) and (12) can be proved simultaneously. Let $C = \forall\vec{y}B$, where B is not a universal formula, and choose n, $C = A_n$. In case $k_n \Vdash \forall\vec{y}B$ we are done. Otherwise we have $k_n \nVdash B[\vec{y} := \vec{s}]$ for some $\vec{s}$, and by construction $k_{n+1} \Vdash \neg B[\vec{y} := \vec{s}]$. For (11) we get $k_{n+1} \Vdash \neg\forall\vec{y}B$ (since $\vdash \forall\vec{y}B \to B[\vec{y} := \vec{s}]$), and (12) follows from the consistency of α.

We are now in a position to give a proof of completeness. Since $l_0 \nVdash A$ and l_0 is stable, (9) yields a consistent node $k \succeq l_0$ such that $k \Vdash \neg A$.

Evidently, k is stable as well. By the proof above, there is a generic branch α such that $k \in \alpha$. Since $k \Vdash \neg A$ it follows that $\alpha \Vdash \neg A$, hence $\mathcal{M}^\alpha \models \neg A$ by (13). Moreover, $\alpha \Vdash \Gamma$, and $\mathcal{M}^\alpha \models \Gamma$ follow by (13). Then, $\Gamma \not\models A$. □

4.2. Compactness, Löwenheim-Skolem Theorem. The completeness theorem has many important corollaries. We mention only two. A set Γ of $\mathcal{L}$-formulae is *consistent* if $\Gamma \nvdash_c \bot$, and *satisfiable* if there is an $\mathcal{L}$-structure $\mathcal{M}$ and an assignment η in $|\mathcal{M}|$ such that $\mathcal{M} \models B[\eta]$ for all $B \in \Gamma$.

COROLLARY. *Let Γ be a set of $\mathcal{L}$-formulae.*

(a) *If Γ is consistent, then Γ is satisfiable.*
(b) *(Compactness theorem). If each finite subset of Γ is satisfiable, Γ is satisfiable.*

PROOF. (a). From $\Gamma \nvdash_c \bot$ we obtain $\Gamma \not\models \bot$ by the completeness theorem, and this implies satisfiability of Γ.

(b). Otherwise we have $\Gamma \models \bot$, hence $\Gamma \vdash_c \bot$ by the completeness theorem, hence also $\Gamma_0 \vdash_c \bot$ for a finite subset $\Gamma_0 \subseteq \Gamma$, and therefore $\Gamma_0 \models \bot$ contrary to our assumption that Γ_0 has a model. □

COROLLARY (Löwenheim and Skolem). *Let Γ be a set of $\mathcal{L}$-formulae (we assume that $\mathcal{L}$ is countable). If Γ is satisfiable, then Γ is satisfiable on an $\mathcal{L}$-structure with a countable carrier set.*

PROOF. We make use of the proof of the completeness theorem with $A = \bot$. It either yields $\Gamma \vdash_c \bot$ (which is excluded by assumption), or else a model of $\Gamma \cup \{\neg\bot\}$, whose carrier set is the countable set $\mathsf{Ter}_{\mathcal{L}}$. □

5. Uncountable Languages

We give a second proof of the completeness theorem for classical logic, which works for uncountable languages as well. This proof makes use of the *axiom of choice* (in the form of *Zorns lemma*).

5.1. Filters and Ultrafilters. Let $M \neq \emptyset$ be a set. $F \subseteq \mathcal{P}(M)$ is called *filter* on M, if

(a) $M \in F$ and $\emptyset \notin F$;
(b) if $X \in F$ and $X \subseteq Y \subseteq M$, then $Y \in F$;
(c) $X, Y \in F$ entails $X \cap Y \in F$.

F is called *ultrafilter*, if for all $X \in \mathcal{P}(M)$

$$X \in F \text{ or } M \setminus X \in F.$$

The intuition here is that the elements X of a filter F are considered to be "big". For instance, for M infinite the set $F = \{ X \subseteq M \mid M \setminus X \text{ finite} \}$ is a filter.

LEMMA. *Suppose F is an ultrafilter and $X \cup Y \in F$. Then $X \in F$ or $Y \in F$.*

PROOF. If both X and Y are not in F, then $M \setminus X$ and $M \setminus Y$ are in F, hence also $(M \setminus X) \cap (M \setminus Y)$, which is $M \setminus (X \cup Y)$. This contradicts the assumption $X \cup Y \in F$. □

Let $M \neq \emptyset$ be a set and $S \subseteq \mathcal{P}(M)$. S has the *finite intersection property*, if $X_1 \cap \cdots \cap X_n \neq \emptyset$ for all $X_1, \ldots, X_n \in S$ and all $n \in \mathbb{N}$.

LEMMA. *If S has the finite intersection property, then there exists a filter F on M such that $F \supseteq S$.*

PROOF. $F := \{ X \mid X \supseteq X_1 \cap \cdots \cap X_n$ for some $X_1, \ldots, X_n \in S \}$. □

LEMMA. *Let $M \neq \emptyset$ be a set and F a filter on M. Then there is an ultrafilter U on M such that $U \supseteq F$.*

PROOF. By Zorns lemma (which will be proved - from the axiom of choice - in Chapter 5), there is a maximal filter U with $F \subseteq U$. We claim that U is an ultrafilter. So let $X \subseteq M$ and assume $X \notin U$ and $M \setminus X \notin U$. Since U is maximal, $U \cup \{X\}$ cannot have the finite intersection property; hence there is a $Y \in U$ such that $Y \cap X = \emptyset$. Similary we obtain $Z \in U$ such that $Z \cap (M \setminus X) = \emptyset$. But then $Y \cap Z = \emptyset$, a contradiction. □

5.2. Products and Ultraproducts. Let $M \neq \emptyset$ be a set and $A_i \neq \emptyset$ sets for $i \in M$. Let

$$\prod_{i \in M} A_i := \{ \alpha \mid \alpha \text{ is a function, } \mathsf{dom}(\alpha) = M \text{ and } \alpha(i) \in A_i \text{ for all } i \in M \}.$$

Observe that, by the *axiom of choice*, $\prod_{i \in M} A_i \neq \emptyset$. We write $\alpha \in \prod_{i \in M} A_i$ as $\langle \alpha(i) \mid i \in M \rangle$.

Now let $M \neq \emptyset$ be a set, F a filter on M and $\mathcal{A}_i$ structures for $i \in M$. Then the *F-product structure* $\mathcal{A} = \prod_{i \in M}^F \mathcal{A}_i$ is defined by

(a) $|\mathcal{A}| := \prod_{i \in M} |\mathcal{A}_i|$ (notice that $|\mathcal{A}| \neq \emptyset$).
(b) for an n-ary relation symbol R and $\alpha_1, \ldots, \alpha_n \in |\mathcal{A}|$ let

$$R^{\mathcal{A}}(\alpha_1, \ldots, \alpha_n) \quad :\Longleftrightarrow \quad \{ i \in M \mid R^{\mathcal{A}_i}(\alpha_1(i), \ldots, \alpha_n(i)) \} \in F.$$

(c) for an n-ary function symbol f and $\alpha_1, \ldots, \alpha_n \in |\mathcal{A}|$ let

$$f^{\mathcal{A}}(\alpha_1, \ldots, \alpha_n) := \langle f^{\mathcal{A}_i}(\alpha_1(i), \ldots, \alpha_n(i)) \mid i \in M \rangle.$$

For an ultrafilter U we call $\mathcal{A} = \prod_{i \in M}^U \mathcal{A}_i$ the *U-ultraproduct* of the $\mathcal{A}_i$ for $i \in M$.

5.3. The Fundamental Theorem on Ultraproducts. The properties of ultrafilters correspond in a certain sense to the definition of the consequence relation $\models$. For example, for an ultrafilter U we have

$$\begin{aligned} \mathcal{M} \models (A \vee B)[\eta] &\iff \mathcal{M} \models A[\eta] \text{ or } \mathcal{M} \models B[\eta] \\ X \cup Y \in U &\iff X \in U \text{ or } Y \in U \end{aligned}$$

and

$$\begin{aligned} \mathcal{M} \models \neg A[\eta] &\iff \mathcal{M} \not\models A[\eta] \\ X \notin U &\iff M \setminus X \in U. \end{aligned}$$

This is the background of the following theorem.

THEOREM (Fundamental Theorem on Ultraproducts, Łoś 1955). *Let* $\mathcal{A} = \prod_{i\in M}^{U} \mathcal{A}_i$ *be an* U*-ultraproduct,* A *a formula and* η *an assignment in* $|\mathcal{A}|$ *. Then we have*

$$\mathcal{A} \models A[\eta] \iff \{\, i \in M \mid \mathcal{A}_i \models A[\eta_i] \,\} \in U,$$

where η_i *is the assignment induced by* $\eta_i(x) = \eta(x)(i)$ *for* $i \in M$*.*

PROOF. We first prove a similar property for terms.

$$t^{\mathcal{A}}[\eta] = \langle\, t^{\mathcal{A}_i}[\eta_i] \mid i \in M \,\rangle. \tag{14}$$

The proof is by induction on t. For a variable the claim follows from the definition. **Case** $ft_1 \dots t_n$. For simplicity assume $n = 1$; so we consider ft. We obtain

$$\begin{aligned}(ft)^{\mathcal{A}}[\eta] &= f^{\mathcal{A}}(t^{\mathcal{A}}[\eta])\\ &= f^{\mathcal{A}}(\langle\, t^{\mathcal{A}_i}[\eta_i] \mid i \in M \,\rangle) \quad \text{by IH}\\ &= \langle\, (ft)^{\mathcal{A}_i}[\eta_i] \mid i \in M \,\rangle.\end{aligned}$$

Case $Rt_1 \dots t_n$. For simplicity assume $n = 1$; so consider Rt. We obtain

$$\begin{aligned}\mathcal{A} \models Rt[\eta] &\iff R^{\mathcal{A}}(t^{\mathcal{A}}[\eta])\\ &\iff \{\, i \in M \mid R^{\mathcal{A}_i}(t^{\mathcal{A}}[\eta](i)) \,\} \in U\\ &\iff \{\, i \in M \mid R^{\mathcal{A}_i}(t^{\mathcal{A}_i}[\eta_i]) \,\} \in U \qquad \text{by (14)}\\ &\iff \{\, i \in M \mid \mathcal{A}_i \models Rt[\eta_i] \,\} \in U.\end{aligned}$$

Case $A \to B$.

$$\begin{aligned}&\mathcal{A} \models (A \to B)[\eta]\\ &\iff \text{if } \mathcal{A} \models A[\eta], \text{ then } \mathcal{A} \models B[\eta]\\ &\iff \text{if } \{\, i \in M \mid \mathcal{A}_i \models A[\eta_i] \,\} \in U, \text{ then } \{\, i \in M \mid \mathcal{A}_i \models B[\eta_i] \,\} \in U\\ &\quad \text{by IH}\\ &\iff \{\, i \in M \mid \mathcal{A}_i \models A[\eta_i] \,\} \notin U \text{ or } \{\, i \in M \mid \mathcal{A}_i \models B[\eta_i] \,\} \in U\\ &\iff \{\, i \in M \mid \mathcal{A}_i \models \neg A[\eta_i] \,\} \in U \text{ or } \{\, i \in M \mid \mathcal{A}_i \models B[\eta_i] \,\} \in U\\ &\quad \text{for } U \text{ is an ultrafilter}\\ &\iff \{\, i \in M \mid \mathcal{A}_i \models (A \to B)[\eta_i] \,\} \in U.\end{aligned}$$

Case $\forall x A$.

$$\begin{aligned}&\mathcal{A} \models (\forall x A)[\eta]\\ &\iff \text{for all } \alpha \in |\mathcal{A}|,\ \mathcal{A} \models A[\eta_x^{\alpha}]\\ &\iff \text{for all } \alpha \in |\mathcal{A}|,\ \{\, i \in M \mid \mathcal{A}_i \models A[(\eta_i)_x^{\alpha(i)}] \,\} \in U \quad \text{by IH}\\ (15)\quad &\iff \{\, i \in M \mid \text{for all } a \in |\mathcal{A}_i|,\ \mathcal{A}_i \models A[(\eta_i)_x^{a}] \,\} \in U \qquad \text{see below}\\ &\iff \{\, i \in M \mid \mathcal{A}_i \models (\forall x A)[\eta_i] \,\} \in U.\end{aligned}$$

It remains to show (15). Let $X := \{\, i \in M \mid \text{for all } a \in |\mathcal{A}_i|,\ \mathcal{A}_i \models A[(\eta_i)_x^{a}] \,\}$ and $Y_\alpha := \{\, i \in M \mid \mathcal{A}_i \models A[(\eta_i)_x^{\alpha(i)}] \,\}$ for $\alpha \in |\mathcal{A}|$.

$\Leftarrow$. Let $\alpha \in |\mathcal{A}|$ and $X \in U$. Clearly $X \subseteq Y_\alpha$, hence also $Y_\alpha \in U$.

$\Rightarrow$. Let $Y_\alpha \in U$ for all α. Assume $X \notin U$. Since U is an ultrafilter,

$$M \setminus X = \{\, i \in M \mid \text{there is an } a \in |\mathcal{A}_i| \text{ such that } \mathcal{A}_i \not\models A[(\eta_i)_x^a] \,\} \in U.$$

We choose by the axiom of choice an $\alpha_0 \in |\mathcal{A}|$ such that

$$\alpha_0(i) = \begin{cases} \text{some } a \in |\mathcal{A}_i| \text{ such that } \mathcal{A}_i \not\models A[(\eta_i)_x^a] & \text{if } i \in M \setminus X, \\ \text{an arbitrary} \in |\mathcal{A}_i| & \text{otherwise.} \end{cases}$$

Then $Y_{\alpha_0} \cap (M \setminus X) = \emptyset$, contradicting $Y_{\alpha_0}, M \setminus X \in U$. □

If we choose $\mathcal{A}_i = \mathcal{B}$ constant, then $\mathcal{A} = \prod_{i \in M}^U \mathcal{B}$ satisfies the same formulae as $\mathcal{B}$ (such structures will be called *elementary equivalent* in section 6; the notation is $\mathcal{A} \equiv \mathcal{B}$). $\prod_{i \in M}^U \mathcal{B}$ is called an *ultrapower* of $\mathcal{B}$.

5.4. General Compactness and Completeness.

Corollary (General Compactness Theorem). *Every finitely satisfiable set Γ of formulae is satisfiable.*

Proof. Let $M := \{\, i \subseteq \Gamma \mid i \text{ finite} \,\}$. For $i \in M$ let $\mathcal{A}_i$ be a model of i under the assignment η_i. For $A \in \Gamma$ let $Z_A := \{\, i \in M \mid A \in i \,\} = \{\, i \subseteq \Gamma \mid i \text{ finite and } A \in i \,\}$. Then $F := \{\, Z_A \mid A \in \Gamma \,\}$ has the finite intersection property (for $\{A_1, \dots, A_n\} \in Z_{A_1} \cap \dots Z_{A_n}$). By the lemmata in 5.1 there is an ultrafilter U on M such that $F \subseteq U$. We consider $\mathcal{A} := \prod_{i \in M}^U \mathcal{A}_i$ and the product assigment η such that $\eta(x)(i) := \eta_i(x)$, and show $\mathcal{A} \models \Gamma[\eta]$. So let $A \in \Gamma$. By the theorem it suffices to show $X_A := \{\, i \in M \mid \mathcal{A}_i \models A[\eta_i] \,\} \in U$. But this follows form $Z_A \subseteq X_A$ and $Z_A \in F \subseteq U$. □

An immediate consequence is that if $\Gamma \models A$, then there exists a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models A$.

For every set Γ of formulae let $L(\Gamma)$ be the set of all function and relation symbols occurring in Γ. If $\mathcal{L}$ is a sublanguage of $\mathcal{L}'$, $\mathcal{M}$ an $\mathcal{L}$-structure and $\mathcal{M}'$ an $\mathcal{L}'$-structure, then $\mathcal{M}'$ is called an *expansion* of $\mathcal{M}$ (and $\mathcal{M}$ a *reduct* of $\mathcal{M}'$), if $|\mathcal{M}| = |\mathcal{M}'|$, $f^{\mathcal{M}} = f^{\mathcal{M}'}$ for all function symbols and $R^{\mathcal{M}} = R^{\mathcal{M}'}$ for all relation symbols in the language $\mathcal{L}$. The (uniquely determined) $\mathcal{L}$-reduct of $\mathcal{M}'$ is denoted by $\mathcal{M}' \restriction \mathcal{L}$. If $\mathcal{M}'$ is an expansion of $\mathcal{M}$ and η an assignment in $|\mathcal{M}|$, then clearly $t^{\mathcal{M}}[\eta] = t^{\mathcal{M}'}[\eta]$ for every $\mathcal{L}$-term t and $\mathcal{M} \models A[\eta]$ iff $\mathcal{M}' \models A[\eta]$ for every $\mathcal{L}$-formula A. Hence the validity of $\Gamma \models A$ does not depend on the underlying language $\mathcal{L}$, as long as $L(\Gamma \cup \{A\}) \subseteq \mathcal{L}$ (or more precisely $\subseteq \mathsf{Fun}_{\mathcal{L}} \cup \mathsf{Rel}_{\mathcal{L}}$).

Corollary (General Completeness Theorem). *Let $\Gamma \cup \{A\}$ be a set of formulae, where the underlying language may be uncountable. Then*

$$\Gamma \vdash_c A \iff \Gamma \models A.$$

Proof. One direction again is the soundness theorem. For the converse we can assume (by the first remark above) that for some finite $\Gamma' \subseteq \Gamma$ we have $\Gamma' \models A$. But then we have $\Gamma' \models A$ in a countable language (by the second remark above). By the completeness theorem for countable languages we obtain $\Gamma' \vdash_c A$, hence also $\Gamma \vdash_c A$. □

6. Basics of Model Theory

In this section we will (as is common in model theory) also allow uncountable languages $\mathcal{L}$. As we have just seen, completeness as well as compactness hold for such languages as well.

6.1. Equality Axioms. We first consider *equality axioms*. So we assume in this section that our underlying language $\mathcal{L}$ contains a binary relation symbol $=$. The set $\mathsf{Eq}_{\mathcal{L}}$ of $\mathcal{L}$-equality axioms consists of (the universal closures of)

$$x = x \quad \text{(reflexivity)},$$
$$x = y \to y = x \quad \text{(symmetry)},$$
$$x = y \to y = z \to x = z \quad \text{(transitivity)},$$
$$x_1 = y_1 \to \cdots \to x_n = y_n \to fx_1 \dots x_n = fy_1 \dots y_n,$$
$$x_1 = y_1 \to \cdots \to x_n = y_n \to Rx_1 \dots x_n \to Ry_1 \dots y_n,$$

for all n-ary function symbols f and relation symbols R of the language $\mathcal{L}$.

LEMMA (Equality). (a) $\mathsf{Eq}_{\mathcal{L}} \vdash t = s \to r[x := t] = r[x := s]$.
(b) $\mathsf{Eq}_{\mathcal{L}} \vdash t = s \to (A[x := t] \leftrightarrow A[x := s])$.

PROOF. (a). Induction on r. (b). Induction on A; we only consider the case $\forall yA$. Then $(\forall yA)[x := r] = \forall yA[x := r]$, and by IH we have $\mathsf{Eq}_{\mathcal{L}} \vdash t = s \to A[x := t] \to A[x := s]$. This entails the claim. □

An $\mathcal{L}$-structure $\mathcal{M}$ satisfies the equality axioms iff $=^{\mathcal{M}}$ is a *congruence relation* (i.e., an equivalence relation compatible with the functions and relations of $\mathcal{M}$). In this section we assume that all $\mathcal{L}$-structures considered $\mathcal{M}$ satisfy the equality axioms. The coincidence lemma then also holds with $=^{\mathcal{M}}$ instead of $=$:

LEMMA (Coincidence). *Let η and ξ be assignments in $|\mathcal{M}|$ such that* $\mathsf{dom}(\eta) = \mathsf{dom}(\xi)$ *and* $\eta(x) =^{\mathcal{M}} \xi(x)$ *for all* $x \in \mathsf{dom}(\eta)$. *Then*
(a) $t^{\mathcal{M}}[\eta] =^{\mathcal{M}} t^{\mathcal{M}}[\xi]$ *if* $\mathsf{vars}(t) \subseteq \mathsf{dom}(\eta)$ *and*
(b) $\mathcal{M} \models A[\eta] \iff \mathcal{M} \models A[\xi]$ *if* $\mathsf{FV}(A) \subseteq \mathsf{dom}(\eta)$.

PROOF. Induction on t and A, respectively. □

6.2. Cardinality of Models. Let $\mathcal{M}/{=^{\mathcal{M}}}$ be the *quotient structure*, whose carrier set consists of congruence classes. We call a structure $\mathcal{M}$ *infinite* (*countable*, *of cardinality* n), if $\mathcal{M}/{=^{\mathcal{M}}}$ is infinite (countable, of cardinality n).

By an *axiom system* Γ we understand a set of closed formulae such that $\mathsf{Eq}_{L(\Gamma)} \subseteq \Gamma$. A *model* of an axiom system Γ is an $\mathcal{L}$-structure $\mathcal{M}$ such that $L(\Gamma) \subseteq \mathcal{L}$ and $\mathcal{M} \models \Gamma$. For sets Γ of closed formulae we write

$$\mathsf{Mod}_{\mathcal{L}}(\Gamma) := \{\, \mathcal{M} \mid \mathcal{M} \text{ is an } \mathcal{L}\text{-structure and } \mathcal{M} \models \Gamma \cup \mathsf{Eq}_{\mathcal{L}} \,\}.$$

Clearly Γ is satisfiable iff Γ has a model.

THEOREM. *If an axiom system has arbitrarily large finite models, then it has an infinite model.*

PROOF. Let Γ be such an axiom system. Suppose $x_0, x_1, x_2, \dots$ are distinct variables and

$$\Gamma' := \Gamma \cup \{ x_i \neq x_j \mid i, j \in \mathbb{N} \text{ such that } i < j \}.$$

By assumption every finite subset of Γ' is satisfiable, hence by the general compactness theorem so is Γ'. Then we have $\mathcal{M}$ and η such that $\mathcal{M} \models \Gamma'[\eta]$ and therefore $\eta(x_i) \neq^{\mathcal{M}} \eta(x_j)$ for $i < j$. Hence $\mathcal{M}$ is infinite. □

6.3. Complete Theories, Elementary Equivalence. Let $\overline{\mathcal{L}}$ be the set of all closed $\mathcal{L}$-formulae. By a *theory* T we mean an axiom system closed under $\vdash_c$, so $\mathsf{Eq}_{L(T)} \subseteq T$ and

$$T = \{ A \in \overline{L(T)} \mid T \vdash_c A \}.$$

A theory T is called *complete*, if for every formula $A \in \overline{L(T)}$, $T \vdash_c A$ or $T \vdash_c \neg A$.

For every $\mathcal{L}$-structure $\mathcal{M}$ (satisfying the equality axioms) the set of all closed $\mathcal{L}$-formulae A such that $\mathcal{M} \models A$ clearly is a theory; it is called the *theory of* $\mathcal{M}$ and denoted by $\mathsf{Th}(\mathcal{M})$.

Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{M}'$ are called *elementarily equivalent* (written $\mathcal{M} \equiv \mathcal{M}'$), if $\mathsf{Th}(\mathcal{M}) = \mathsf{Th}(\mathcal{M}')$. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{M}'$ are called *isomorphic* (written $\mathcal{M} \cong \mathcal{M}'$), if there is a map $\pi \colon |\mathcal{M}| \to |\mathcal{M}'|$ inducing a bijection between $|\mathcal{M}/{=^{\mathcal{M}}}|$ and $|\mathcal{M}'/{=^{\mathcal{M}'}}|$, so

$$\forall a, b {\in} |\mathcal{M}|. a =^{\mathcal{M}} b \iff \pi(a) =^{\mathcal{M}'} \pi(b),$$
$$(\forall a' {\in} |\mathcal{M}'|)(\exists a {\in} |\mathcal{M}|)\, \pi(a) =^{\mathcal{M}'} a',$$

such that for all $a_1, \dots, a_n \in |\mathcal{M}|$

$$\pi(f^{\mathcal{M}}(a_1, \dots, a_n)) =^{\mathcal{M}'} f^{\mathcal{M}'}(\pi(a_1), \dots, \pi(a_n)),$$
$$R^{\mathcal{M}}(a_1, \dots, a_n) \iff R^{\mathcal{M}'}(\pi(a_1), \dots, \pi(a_n))$$

for all n-ary function symbols f and relation symbols R of the language $\mathcal{L}$.

We first collect some simple properties of the notions of the theory of a structure $\mathcal{M}$ and of elementary equivalence.

LEMMA. (a) $\mathsf{Th}(\mathcal{M})$ *ist complete.*

(b) *If* Γ *is an axiom system such that* $L(\Gamma) \subseteq \mathcal{L}$, *then*

$$\{ A \in \overline{\mathcal{L}} \mid \Gamma \vdash_c A \} = \bigcap \{ \mathsf{Th}(\mathcal{M}) \mid \mathcal{M} \in \mathsf{Mod}_{\mathcal{L}}(\Gamma) \}.$$

(c) $\mathcal{M} \equiv \mathcal{M}' \iff \mathcal{M} \models \mathsf{Th}(\mathcal{M}')$.

(d) *If* $\mathcal{L}$ *is countable, then for every* $\mathcal{L}$*-structure* $\mathcal{M}$ *there is a countable* $\mathcal{L}$*-structure* $\mathcal{M}'$ *such that* $\mathcal{M} \equiv \mathcal{M}'$.

PROOF. (a). Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \in \overline{\mathcal{L}}$. Then $\mathcal{M} \models A$ or $\mathcal{M} \models \neg A$, hence $\mathsf{Th}(\mathcal{M}) \vdash_c A$ or $\mathsf{Th}(\mathcal{M}) \vdash_c \neg A$.

(b). For all $A \in \overline{\mathcal{L}}$ we have

$$\begin{aligned} \Gamma \vdash_c A &\iff \Gamma \models A \\ &\iff \text{for all } \mathcal{L}\text{-structures } \mathcal{M}, (\mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models A) \\ &\iff \text{for all } \mathcal{L}\text{-structures } \mathcal{M}, (\mathcal{M} \in \mathsf{Mod}_{\mathcal{L}}(\Gamma) \Rightarrow A \in \mathsf{Th}(\mathcal{M})) \\ &\iff A \in \bigcap \{ \mathsf{Th}(\mathcal{M}) \mid \mathcal{M} \in \mathsf{Mod}_{\mathcal{L}}(\Gamma) \}. \end{aligned}$$

(c). $\Rightarrow$. Assume $\mathcal{M} \equiv \mathcal{M}'$ and $A \in \mathsf{Th}(\mathcal{M}')$. Then $\mathcal{M}' \models A$, hence $\mathcal{M} \models A$.

$\Leftarrow$. Assume $\mathcal{M} \models \mathsf{Th}(\mathcal{M}')$. Then clearly $\mathsf{Th}(\mathcal{M}') \subseteq \mathsf{Th}(\mathcal{M})$. For the converse inclusion let $A \in \mathsf{Th}(\mathcal{M})$. If $A \notin \mathsf{Th}(\mathcal{M}')$, by (a) we would also have $\neg A \in \mathsf{Th}(\mathcal{M}')$, hence $\mathcal{M} \models \neg A$ contradicting $A \in \mathsf{Th}(\mathcal{M})$.

(d). Let $\mathcal{L}$ be countable and $\mathcal{M}$ an $\mathcal{L}$-structure. Then $\mathsf{Th}(\mathcal{M})$ is satisfiable and therefore by the theorem of Löwenheim and Skolem possesses a satisfying $\mathcal{L}$-structure $\mathcal{M}'$ with a countable carrier set $\mathsf{Ter}_{\mathcal{L}}$. By (c), $\mathcal{M} \equiv \mathcal{M}'$. □

Moreover, we can characterize complete theories as follows:

THEOREM. *Let T be a theory and $\mathcal{L} = L(T)$. Then the following are equivalent.*

(a) *T is complete.*
(b) *For every model $\mathcal{M} \in \mathsf{Mod}_{\mathcal{L}}(T)$, $\mathsf{Th}(\mathcal{M}) = T$.*
(c) *Any two models $\mathcal{M}, \mathcal{M}' \in \mathsf{Mod}_{\mathcal{L}}(T)$ are elementarily equivalent.*

PROOF. (a) $\Rightarrow$ (b). Let T be complete and $\mathcal{M} \in \mathsf{Mod}_{\mathcal{L}}(T)$. Then $\mathcal{M} \models T$, hence $T \subseteq \mathsf{Th}(\mathcal{M})$. For the converse assume $A \in \mathsf{Th}(\mathcal{M})$. Then $\neg A \notin \mathsf{Th}(\mathcal{M})$, hence $\neg A \notin T$ and therefore $A \in T$.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). Let $A \in \overline{\mathcal{L}}$ and $T \nvdash_c A$. Then there is a model $\mathcal{M}_0$ of $T \cup \{\neg A\}$. Now let $\mathcal{M} \in \mathsf{Mod}_{\mathcal{L}}(T)$ be arbitrary. By (c) we have $\mathcal{M} \equiv \mathcal{M}_0$, hence $\mathcal{M} \models \neg A$. Therefore $T \vdash_c \neg A$. □

6.4. Elementary Equivalence and Isomorphism.

LEMMA. *Let π be an isomorphism between $\mathcal{M}$ and $\mathcal{M}'$. Then for all terms t and formulae A and for every sufficiently big assignment η in $|\mathcal{M}|$*

(a) $\pi(t^{\mathcal{M}}[\eta]) =^{\mathcal{M}'} t^{\mathcal{M}'}[\pi \circ \eta]$ *and*
(b) $\mathcal{M} \models A[\eta] \iff \mathcal{M}' \models A[\pi \circ \eta]$. *In particular,*

$$\mathcal{M} \cong \mathcal{M}' \Rightarrow \mathcal{M} \equiv \mathcal{M}'.$$

PROOF. (a). Induction on t. For simplicity we only consider the case of a unary function symbol.

$$\begin{aligned}
\pi(x^{\mathcal{M}}[\eta]) &= \pi(\eta(x)) = x^{\mathcal{M}'}[\pi \circ \eta] \\
\pi(c^{\mathcal{M}}[\eta]) &= \pi(c^{\mathcal{M}}) =^{\mathcal{M}'} c^{\mathcal{M}'} \\
\pi((ft)^{\mathcal{M}}[\eta]) &= \pi(f^{\mathcal{M}}(t^{\mathcal{M}}[\eta])) \\
&=^{\mathcal{M}'} f^{\mathcal{M}'}(\pi(t^{\mathcal{M}}[\eta])) \\
&=^{\mathcal{M}'} f^{\mathcal{M}'}(t^{\mathcal{M}'}[\pi \circ \eta]) \\
&= (ft)^{\mathcal{M}'}[\pi \circ \eta].
\end{aligned}$$

(b). Induction on A. For simplicity we only consider the case of a unary relation symbol and the case $\forall x A$.

$$\begin{aligned}
\mathcal{M} \models Rt[\eta] &\iff R^{\mathcal{M}}(t^{\mathcal{M}}[\eta]) \\
&\iff R^{\mathcal{M}'}(\pi(t^{\mathcal{M}}[\eta])) \\
&\iff R^{\mathcal{M}'}(t^{\mathcal{M}'}[\pi \circ \eta])
\end{aligned}$$

$$
\begin{aligned}
&\iff \mathcal{M}' \models Rt[\pi \circ \eta],\\
\mathcal{M} \models \forall x A[\eta] &\iff \text{for all } a \in |\mathcal{M}|,\ \mathcal{M} \models A[\eta_x^a]\\
&\iff \text{for all } a \in |\mathcal{M}|,\ \mathcal{M}' \models A[\pi \circ \eta_x^a]\\
&\iff \text{for all } a \in |\mathcal{M}|,\ \mathcal{M}' \models A[(\pi \circ \eta)_x^{\pi(a)}]\\
&\iff \text{for all } a' \in |\mathcal{M}'|,\ \mathcal{M}' \models A[(\pi \circ \eta)_x^{a'}]\\
&\iff \mathcal{M}' \models \forall x A[\pi \circ \eta]
\end{aligned}
$$

This concludes the proof. □

The converse, i.e. that $\mathcal{M} \equiv \mathcal{M}'$ implies $\mathcal{M} \cong \mathcal{M}'$, is true for finite structures (see exercise sheet 9), but not for infinite ones:

THEOREM. *For every infinite structure $\mathcal{M}$ there is an elementarily equivalent structure $\mathcal{M}_0$ not isomorphic to $\mathcal{M}$.*

PROOF. Let $=^{\mathcal{M}}$ be the equality on $M := |\mathcal{M}|$, and let $\mathcal{P}(M)$ denote the power set of M. For every $\alpha \in \mathcal{P}(M)$ choose a new constant c_α. In the language $\mathcal{L}' := \mathcal{L} \cup \{ c_\alpha \mid \alpha \in \mathcal{P}(M) \}$ we consider the axiom system

$$\Gamma := \mathsf{Th}(\mathcal{M}) \cup \{ c_\alpha \neq c_\beta \mid \alpha, \beta \in \mathcal{P}(M) \text{ and } \alpha \neq \beta \} \cup \mathsf{Eq}_{\mathcal{L}'}.$$

Every finite subset of Γ is satisfiable by an appropriate expansion of $\mathcal{M}$. Hence by the general compactness theorem also Γ is satisfiable, say by $\mathcal{M}_0'$. Let $\mathcal{M}_0 := \mathcal{M}_0' \restriction \mathcal{L}$. We may assume that $=^{\mathcal{M}_0}$ is the equality on $|\mathcal{M}_0|$. $\mathcal{M}_0$ is not isomorphic to $\mathcal{M}$, for otherwise we would have an injection of $\mathcal{P}(M)$ into M and therefore a contradiction. □

6.5. Non Standard Models. By what we just proved it is impossible to characterize an infinite structure by a first order axiom system up to isomorphism. However, if we extend first order logic by also allowing quantification over sets X, we can formulate the following *Peano axioms*

$$
\begin{aligned}
&\forall n\, \mathsf{S}(n) \neq 0,\\
&\forall n \forall m. \mathsf{S}(n) = \mathsf{S}(m) \to n = m,\\
&\forall X. 0 \in X \to (\forall n. n \in X \to \mathsf{S}(n) \in X) \to \forall n\, n \in X.
\end{aligned}
$$

One can show easily that $(\mathbb{N}, 0, \mathsf{S})$ is up to isomorphism the unique model of the Peano axioms. A structure which is elementarily equivalent, but not isomorphic to $\mathcal{N} := (\mathbb{N}, 0, \mathsf{S})$, is called a *non standard model* of the natural numbers. In non standard models of the natural numbers the principle of complete induction does not hold for all sets $X \subseteq \mathbb{N}$.

Similarly, a structure which is elementarily equivalent, but not isomorphic to $(\mathbb{R}, 0, 1, +, \cdot, <)$ is called a non standard model of the reals. In every non standard model of the reals the completeness axiom

$$\forall X. \emptyset \neq X \text{ bounded} \to \exists y. y = \sup(X)$$

does not hold for all sets $X \subseteq \mathbb{R}$.

THEOREM. *There are countable non standard models of the natural numbers.*

PROOF. Let x be a variable and

$$\Gamma := \mathsf{Th}(\mathcal{N}) \cup \{\, x \neq \underline{n} \mid n \in \mathbb{N} \},$$

where $\underline{0} := 0$ and $\underline{n+1} := \mathsf{S}\underline{n}$. Clearly every finite subset of Γ is satisfiable, hence by compactness also Γ. By the theorem of Löwenheim and Skolem we then have a countable or finite $\mathcal{M}$ and an assignment η such that $\mathcal{M} \models \Gamma[\eta]$. Because of $\mathcal{M} \models \mathsf{Th}(\mathcal{N})$ we have $\mathcal{M} \equiv \mathcal{N}$ by 6.3; hence $\mathcal{M}$ is countable. Moreover $\eta(x) \neq^{\mathcal{M}} \underline{n}^{\mathcal{M}}$ for all $n \in \mathbb{N}$, hence $\mathcal{M} \not\cong \mathcal{N}$. $\square$

6.6. Archimedian Ordered Fields. We now consider some easy applications to well-known axiom systems.

The axioms of *field theory* are (the equality axioms and)

$$\begin{array}{ll} x + (y + z) = (x + y) + z, & x \cdot (y \cdot z) = (x \cdot y) \cdot z, \\ 0 + x = x, & 1 \cdot x = x, \\ (-x) + x = 0, & x \neq 0 \to x^{-1} \cdot x = 1, \\ x + y = y + x, & x \cdot y = y \cdot x, \end{array}$$

and also

$$\begin{array}{l} (x + y) \cdot z = (x \cdot z) + (y \cdot z), \\ 1 \neq 0. \end{array}$$

Fields are the models of this axiom system.

In the theory of *ordered fields* one has in addition a binary relation symbol $<$ and as axioms

$$\begin{array}{l} x \not< x, \\ x < y \to y < z \to x < z, \\ x < y \vee^{\mathsf{cl}} x = y \vee^{\mathsf{cl}} y < x, \\ x < y \to x + z < y + z, \\ 0 < x \to 0 < y \to 0 < x \cdot y. \end{array}$$

Ordered fields are the models of this extended axiom system. An ordered field is called *archimedian ordered*, if for every element a of the field there is a natural number n such that a is less than the n-fold multiple of the 1 in the field.

THEOREM. *For every archimedian ordered field there is an elementarily equivalent ordered field that is not archimedian ordered.*

PROOF. Let $\mathcal{K}$ be an archimedian ordered field, x a variable and

$$\Gamma := \mathsf{Th}(\mathcal{K}) \cup \{\, \underline{n} < x \mid n \in \mathbb{N} \}.$$

Clearly every finite subset of Γ is satisfiable, hence by the general compactness theorem also Γ. Therefore we have $\mathcal{M}$ and η such that $\mathcal{M} \models \Gamma[\eta]$. Because of $\mathcal{M} \models \mathsf{Th}(\mathcal{K})$ we obtain $\mathcal{M} \equiv \mathcal{K}$ and hence $\mathcal{M}$ is an ordered field. Moreover $1^{\mathcal{M}} \cdot n <^{\mathcal{M}} \eta(x)$ for all $n \in \mathbb{N}$, hence $\mathcal{M}$ is not archimedian ordered. $\square$

6.7. Axiomatizable Structures. A class $\mathcal{S}$ of $\mathcal{L}$-structures is called (*finitely*) *axiomatizable*, if there is a (finite) axiom system Γ such that $\mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\Gamma)$. Clearly $\mathcal{S}$ is finitely axiomatizable iff $\mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\{A\})$ for some formula A. If for every $\mathcal{M} \in \mathcal{S}$ there is an elementarily equivalent $\mathcal{M}' \notin \mathcal{S}$, then $\mathcal{S}$ cannot possibly be axiomatizable. By the theorem above we can conclude that the class of archimedian ordered fields is not axiomatizable. It also follows that the class of non archimedian ordered fields is not axiomatizable.

LEMMA. *Let $\mathcal{S}$ be a class of $\mathcal{L}$-structures and Γ an axiom system.*

(a) *$\mathcal{S}$ is finitely axiomatizable iff $\mathcal{S}$ and the complement of $\mathcal{S}$ are axiomatizable.*
(b) *If $\mathsf{Mod}_{\mathcal{L}}(\Gamma)$ is finitely axiomatizable, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\mathsf{Mod}_{\mathcal{L}}(\Gamma_0) = \mathsf{Mod}_{\mathcal{L}}(\Gamma)$.*

PROOF. (a). Let $1 - \mathcal{S}$ denote the complement of $\mathcal{S}$.

$\Rightarrow$. Let $\mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\{A\})$. Then $\mathcal{M} \in 1 - \mathcal{S} \iff \mathcal{M} \models \neg A$, hence $1 - \mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\{\neg A\})$.

$\Leftarrow$. Let $\mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\Gamma_1)$ and $1 - \mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\Gamma_2)$. Then $\Gamma_1 \cup \Gamma_2$ is not satisfiable, hence there is a finite $\Gamma \subseteq \Gamma_1$ such that $\Gamma \cup \Gamma_2$ is not satisfiable. One obtains

$$\mathcal{M} \in \mathcal{S} \Rightarrow \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \not\models \Gamma_2 \Rightarrow \mathcal{M} \notin 1 - \mathcal{S} \Rightarrow \mathcal{M} \in \mathcal{S},$$

hence $\mathcal{S} = \mathsf{Mod}_{\mathcal{L}}(\Gamma)$.

(b). Let $\mathsf{Mod}_{\mathcal{L}}(\Gamma) = \mathsf{Mod}_{\mathcal{L}}(\{A\})$. Then $\Gamma \models A$, hence also $\Gamma_0 \models A$ for a finite $\Gamma_0 \subseteq \Gamma$. One obtains

$$\mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \Gamma_0 \Rightarrow \mathcal{M} \models A \Rightarrow \mathcal{M} \models \Gamma,$$

hence $\mathsf{Mod}_{\mathcal{L}}(\Gamma_0) = \mathsf{Mod}_{\mathcal{L}}(\Gamma)$. □

6.8. Complete Linear Orders Without End Points. Finally we consider as an example of a complete theory the theory DO of complete linear orders without end points. The axioms are (the equality axioms and)

$$\begin{array}{ll} x \not< x, & x < y \to \exists^{\mathsf{cl}} z. x < z \land z < y, \\ x < y \to y < z \to x < z, & \exists^{\mathsf{cl}} y\, x < y, \\ x < y \lor^{\mathsf{cl}} x = y \lor^{\mathsf{cl}} y < x, & \exists^{\mathsf{cl}} y\, y < x. \end{array}$$

LEMMA. *Every countable model of DO is isomorphic to the structure $(\mathbb{Q}, <)$ of rational numbers.*

PROOF. Let $\mathcal{M} = (M, \prec)$ be a countable model of DO; we can assume that $=^{\mathcal{M}}$ is the equality on M. Let $M = \{\, b_n \mid n \in \mathbb{N} \,\}$ and $\mathbb{Q} = \{\, a_n \mid n \in \mathbb{N} \,\}$, where we may assume $a_n \neq a_m$ and $b_n \neq b_m$ for $n < m$. We define recursively functions $f_n \subseteq \mathbb{Q} \times M$ as follows. Let $f_0 := \{(a_0, b_0)\}$. Assume we have already constructed f_n.

Case $n+1 = 2m$. Let j be minimal such that $b_j \notin \mathsf{ran}(f_n)$. Choose $a_i \notin \mathsf{dom}(f_n)$ such that for all $a \in \mathsf{dom}(f_n)$ we have $a_i < a \leftrightarrow b_j < f_n(a)$; such an a_i exists, since $\mathcal{M}$ and $(\mathbb{Q}, <)$ are models of DO. Let $f_{n+1} := f_n \cup \{(a_i, b_j)\}$.

Case $n + 1 = 2m + 1$. This is treated similarly. Let i be minimal such that $a_i \notin \mathsf{dom}(f_n)$. Choose $b_j \notin \mathsf{ran}(f_n)$ such that for all $a \in \mathsf{dom}(f_n)$ we have $a_i < a \leftrightarrow b_j < f_n(a)$; such a b_j exists, since $\mathcal{M}$ and $(\mathbb{Q}, <)$ are models of DO. Let $f_{n+1} := f_n \cup \{(a_i, b_j)\}$.

Then $\{b_0, \dots, b_m\} \subseteq \mathsf{ran}(f_{2m})$ and $\{a_0, \dots, a_{m+1}\} \subseteq \mathsf{dom}(f_{2m+1})$ by construction, and $f := \bigcup_n f_n$ is an isomorphism of $(\mathbb{Q}, <)$ onto $\mathcal{M}$. $\square$

THEOREM. *The theory* DO *is complete, and* $\mathsf{DO} = \mathsf{Th}(\mathbb{Q}, <)$.

PROOF. Clearly $(\mathbb{Q}, <)$ is a model of DO. Hence by 6.3 it suffices to show that for every model $\mathcal{M}$ of DO we have $\mathcal{M} \equiv (\mathbb{Q}, <)$. So let $\mathcal{M}$ model of DO. By 6.3 there is a countable $\mathcal{M}'$ such that $\mathcal{M} \equiv \mathcal{M}'$. By the preceding lemma $\mathcal{M}' \cong (\mathbb{Q}, <)$, hence $\mathcal{M} \equiv \mathcal{M}' \equiv (\mathbb{Q}, <)$. $\square$

A further example of a complete theory is the theory of algebraically closed fields. For a proof of this fact and for many more subjects of model theory we refer to the literature (e.g., the book of Chang and Keisler [**6**]).

7. Notes

The completeness theorem for classical logic has been proved by Gödel [**10**] in 1930. He did it for countable languages; the general case has been treated 1936 by Malzew [**17**]. Löwenheim and Skolem proved their theorem even before the completeness theorem was discovered: Löwenheim in 1915 [**16**] und Skolem in 1920 [**24**].

Beth-structures for intuitionistic logic have been introduced by Beth in 1956 [**1**]; however, the completeness proofs given there were in need of correction. 1959 Beth revised his paper in [**2**].