CHAPTER 5

Real analysis

We are interested in *exact real numbers*, as opposed to floating point numbers. The final goal is to develop the basics of real analysis in such a way that from a proof of an existence formula one can extract a program. For instance, from a proof of the intermediate value theorem we want to extract a program that, given an arbitrary error bound $\frac{1}{2^p}$, computes a rational x where the given function is zero up to the error bound.

5.1. Exact real arithmetic

5.1.1. Cauchy sequences, equality. We shall view a real as a Cauchy sequence of rationals with a separately given modulus.

DEFINITION 5.1.1. A real number x is a pair $((a_n)_{n\in\mathbb{N}}, M)$ with $a_n \in \mathbb{Q}$ and $M\colon \mathbb{P} \to \mathbb{N}$ such that $(a_n)_n$ is a *Cauchy sequence* with modulus M, that is

$$|a_n - a_m| \leq \frac{1}{2^p} \quad \text{for } n, m \geq M(p)$$

and M is weakly increasing (that is $M(p) \leq M(q)$ for $p \leq q$). M is called *Cauchy modulus* of x.

We shall loosely speak of a real $(a_n)_n$ if the Cauchy modulus M is clear from the context or inessential. Every rational a is tacitly understood as the real represented by the constant sequence $a_n = a$ with the constant modulus $M(p) = 0$.

DEFINITION 5.1.2. Two reals $x := ((a_n)_n, M)$, $y := ((b_n)_n, N)$ are called *equivalent* (or *equal* and written $x = y$, if the context makes clear what is meant), if

$$|a_{M(p+1)} - b_{N(p+1)}| \leq \frac{1}{2^p} \quad \text{for all } p \in \mathbb{P}.$$

We want to show that this is an equivalence relation. Reflexivity and symmetry are clear. For transitivity we use the following lemma:

LEMMA 5.1.3 (RealEqChar). *For reals $x := ((a_n)_n, M)$, $y := ((b_n)_n, N)$ the following are equivalent:*

(a) *$x = y$;*

(b) $\forall_p \exists_{n_0} \forall_{n \geq n_0} (|a_n - b_n| \leq \frac{1}{2^p})$.

Proof. (a) implies (b). For $n \geq M(p+2), N(p+2)$ we have

$$|a_n - b_n| \leq |a_n - a_{M(p+2)}| + |a_{M(p+2)} - b_{N(p+2)}| + |b_{N(p+2)} - b_n| \\ \leq \frac{1}{2^{p+2}} + \frac{1}{2^{p+1}} + \frac{1}{2^{p+2}}.$$

(b) implies (a). Let $q \in \mathbb{P}$, and $n \geq n_0, M(p+1), N(p+1)$ with n_0 provided for q by (b). Then

$$|a_{M(p+1)} - b_{N(p+1)}| \leq |a_{M(p+1)} - a_n| + |a_n - b_n| + |b_n - b_{N(p+1)}| \\ \leq \frac{1}{2^{p+1}} + \frac{1}{2^q} + \frac{1}{2^{p+1}}.$$

The claim follows, because this holds for every $q \in \mathbb{P}$. □

Remark 5.1.4 (RealSeqEqToEq). An immediate consequence is that any two reals with the same Cauchy sequence (but possibly different moduli) are equal.

Lemma 5.1.5 (RealEqTrans). *Equality between reals is transitive.*

Proof. Let $(a_n)_n$, $(b_n)_n$, $(c_n)_n$ be the Cauchy sequences for x, y, z. Assume $x = y$, $y = z$ and pick n_1, n_2 for $p+1$ according to the lemma above. Then $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| \leq \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}}$ for $n \geq n_1, n_2$. □

5.1.2. The Archimedian property. For every function on the reals we certainly want compatibility with equality. This however is not always the case; here is an important example.

Lemma 5.1.6 (RealBound). *For every real $x := ((a_n)_n, M)$ we can find p_x such that $|a_n| \leq 2^{p_x}$ for all n.*

Proof. Let $n_0 := M(1)$ and p_x be such that $\max\{\, |a_n| \mid n \leq n_0 \,\} + \frac{1}{2} \leq 2^{p_x}$. Then $|a_n| \leq 2^{p_x}$ for all n. □

Clearly this assignment of p_x to x is not compatible with equality.

5.1.3. Nonnegative and positive reals. A real $x := ((a_n)_n, M)$ is called *nonnegative* (written $x \in \mathbb{R}^{0+}$) if

$$-\frac{1}{2^p} \leq a_{M(p)} \quad \text{for all } p \in \mathbb{P}.$$

It is *p-positive* (written $x \in_p \mathbb{R}^+$, or $x \in \mathbb{R}^+$ if p is not needed) if

$$\frac{1}{2^p} \leq a_{M(p+1)}.$$

We want to show that both properties are compatible with equality. First we prove a useful characterization of nonnegative reals.

LEMMA 5.1.7 (RealNNegChar). *For a real $x := ((a_n)_n, M)$ the following are equivalent:*

(a) $x \in \mathbb{R}^{0+}$;
(b) $\forall_p \exists_{n_0} \forall_{n \geq n_0} (-\frac{1}{2^p} \leq a_n)$.

PROOF. (a) implies (b). For $n \geq M(p+1)$ we have

$$\begin{aligned} -\frac{1}{2^p} &\leq -\frac{1}{2^{p+1}} + a_{M(p+1)} \\ &= -\frac{1}{2^{p+1}} + (a_{M(p+1)} - a_n) + a_n \\ &\leq -\frac{1}{2^{p+1}} + \frac{1}{2^{p+1}} + a_n. \end{aligned}$$

(b) implies (a). Let $q \in \mathbb{P}$ and $n \geq n_0, M(p)$ with n_0 provided by (b) (for q). Then

$$\begin{aligned} -\frac{1}{2^p} - \frac{1}{2^q} &\leq -\frac{1}{2^p} + a_n \\ &= -\frac{1}{2^p} + (a_n - a_{M(p)}) + a_{M(p)} \\ &\leq -\frac{1}{2^p} + \frac{1}{2^p} + a_{M(p)}. \end{aligned}$$

The claim follows, because this holds for every q. □

LEMMA 5.1.8 (RealNNegCompat). *If $x \in \mathbb{R}^{0+}$ and $x = y$, then $y \in \mathbb{R}^{0+}$.*

PROOF. Let $x := ((a_n)_n, M)$ and $y := ((b_n)_n, N)$. Assume $x \in \mathbb{R}^{0+}$ and $x = y$, and let p be given. Pick n_0 according to the lemma above and n_1 according to the characterization of equality of reals in Lemma 5.1.3 (RealEqChar) (both for $p+1$). Then for $n \geq n_0, n_1$

$$-\frac{1}{2^p} \leq -\frac{1}{2^{p+1}} + a_n \leq (b_n - a_n) + a_n.$$

Hence $y \in \mathbb{R}^{0+}$ by definition. □

LEMMA 5.1.9 (RealPosChar). *For a real $x := ((a_n)_n, M)$ with $x \in_p \mathbb{R}^+$ we have*

$$\frac{1}{2^{p+1}} \leq a_n \quad \textit{for } M(p+1) \leq n.$$

Conversely, from $\forall_{n \geq n_0} (\frac{1}{2^q} \leq a_n)$ we can infer $x \in_{q+1} \mathbb{R}^+$.

PROOF. Assume $x \in_p \mathbb{R}^+$, that is $\frac{1}{2^p} \leq a_{M(p+1)}$. Then

$$\frac{1}{2^{p+1}} \leq -\frac{1}{2^{p+1}} + a_{M(p+1)} = -\frac{1}{2^{p+1}} + (a_{M(p+1)} - a_n) + a_n \leq a_n$$