CHAPTER 4

Computational content of proofs

We have already mentioned that (co)inductive predicates can be declared as either computationally relevant (c.r.) or non-computational (n.c.). But what is the computational content in the c.r. case? We first address this question for (co)inductive predicates, and then extend it to arbitrary formulas. Next we study in what sense a proof of a c.r. formula A provides us with concrete computational content. This can be seen as a "witness" for the validity of A, or – in the sense of Kolmogorov (1932) – a "solution" to problem A.

Finally we take a step back and reflect on what we have done. We formally define what it means for a term to "realize" the c.r. formula A. We extract from a proof M of A a term $\mathrm{et}(M)$ and (again formally) prove that it is a realizer of A. In this proof we need "invariance axioms" stating that every c.r. formula not involving realizability is invariant under realizability, formally $A \leftrightarrow \exists_z(z \mathbf{\ r\ } A)$, where $z \mathbf{\ r\ } A$ means "z realizes A".

4.1. Realizers

Assume that we have a global assignment giving for every c.r. predicate variable X of arity $\vec{\rho}$ an n.c. predicate variable $X^{\mathbf{r}}$ of arity $(\vec{\rho}, \xi)$ where ξ is the type variable associated with X. We will also introduce $I^{\mathbf{r}}/{}^{\mathrm{co}}I^{\mathbf{r}}$ for (co)inductive predicates $I/{}^{\mathrm{co}}I$. A formula or predicate C is called **r**-free if it does not contain any of these $X^{\mathbf{r}}$, $I^{\mathbf{r}}$ or ${}^{\mathrm{co}}I^{\mathbf{r}}$. A derivation M is called **r**-free if it contains **r**-free formulas only.

Definition ($C^{\mathbf{r}}$ for **r**-free predicates and formulas C). For every **r**-free predicate or formula C we define a predicate or formula $C^{\mathbf{r}}$. For n.c. C let $C^{\mathbf{r}} := C$. In case C is c.r. $C^{\mathbf{r}}$ is an n.c. predicate of arity $(\vec{\sigma}, \tau(C))$ with $\vec{\sigma}$ the arity of C. We often write $z \mathbf{\ r\ } C$ for $C^{\mathbf{r}} z$ in case C is a c.r. formula. For c.r. *predicates* X let $X^{\mathbf{r}}$ be the n.c. predicate variable provided, and

$$\{\, \vec{x} \mid A \,\}^{\mathbf{r}} := \{\, \vec{x}, z \mid z \mathbf{\ r\ } A \,\}.$$

Now consider a c.r. (co)inductive predicate

$$I/{}^{\mathrm{co}}I := (\mu/\nu)_X((K_i(X))_{i<k}$$

with associated base type ι_I given by the constructor types $(\kappa_i(\xi))_{i<k}$ where $\kappa_i(\xi) := \tau(K_i(X))$. The i-th constructor of ι_I is $\mathrm{C}_i\colon \kappa_i(\iota_I)$. Let s be a variable of type $\tau(I)$ and ϑ the substitution $\xi \mapsto \tau(I)$, $X^{\mathbf{r}} \mapsto \{\, \vec{x}, s \mid Y\vec{x}s \,\}$. We define n.c. predicates $I^{\mathbf{r}}$ and ${}^{\mathrm{co}}I^{\mathbf{r}}$ by

$$I^{\mathbf{r}}/{}^{\mathrm{co}}I^{\mathbf{r}} := (\mu/\nu)_Y((C_i \ \mathbf{r}\ K_i(X))\vartheta)_{i<k}.$$

The substitution ϑ is necessary since the arity of Y (and hence of $I^{\mathbf{r}}/{}^{\mathrm{co}}I^{\mathbf{r}}$) must be $(\vec{\rho}, \tau(I))$ and not $(\vec{\rho}, \xi)$. For c.r. *formulas* let

$$z \ \mathbf{r}\ P\vec{t} := P^{\mathbf{r}}\vec{t}z,$$

$$z \ \mathbf{r}\ (A \to B) := \begin{cases} \forall_w(w \ \mathbf{r}\ A \to zw \ \mathbf{r}\ B) & \text{if } A \text{ is c.r.} \\ A \to z \ \mathbf{r}\ B & \text{if } A \text{ is n.c.} \end{cases}$$

$$z \ \mathbf{r}\ \forall_x A := \forall_x(z \ \mathbf{r}\ A).$$

Example. As an easy example for the construction of $I^{\mathbf{r}}$ consider the predicate Even, defined by $\mu_X(K_0(X), K_1(X))$ with $K_0(X) := (0 \in X)$ and $K_1(X) := \forall_n(n \in X \to S(Sn) \in X)$. The associated base type ι_{Even} is given by the constructor types $\kappa_0(\xi) := \xi$ and $\kappa_1(\xi) := \xi \to \xi$, i.e., $\iota_{\mathrm{Even}} = \mathbb{N}$ with constructors $\mathrm{C}_0 := 0$ and $\mathrm{C}_1 := S$. Let ϑ be the substitution $\xi \mapsto \mathbb{N}$, $X^{\mathbf{r}} \mapsto \{\, n, m \mid Ynm \,\}$. Since $S \ \mathbf{r}\ K_1(X)$ is $\forall_{n,m}(X^{\mathbf{r}}nm \to X^{\mathbf{r}}(S(Sn), Sm))$ we obtain

$$I^{\mathbf{r}} := \mu_Y(Y00, \forall_{n,m}(Ynm \to Y(S(Sn), Sm)).$$

We express Kolmogorov's view of formulas as problems by means of *invariance axioms*:

Axiom (Invariance under realizability). *For* $\mathbf{r}$*-free c.r. formulas A we require as axioms*

$$\mathrm{InvAll}_A\colon \forall_z(z \ \mathbf{r}\ A \to A). \tag{16}$$

$$\mathrm{InvEx}_A\colon A \to \exists_z(z \ \mathbf{r}\ A). \tag{17}$$

Realizers of totality and cototality predicates will be of special interest for us. Notice that the types $\tau(T_\iota)$ and $\tau({}^{\mathrm{co}}T_\iota)$ are both ι. Moreover we have

Lemma 4.1.1 (Realizers of totality). *For closed base types ι the following are equivalent.*

(a) $T_\iota^{\mathbf{r}}xy$,
(b) $x \sim_\iota^{\mathrm{nc}} y$,
(c) $x \in T_\iota^{\mathrm{nc}} \wedge x \equiv y$.

Proof. (a) $\leftrightarrow$ (b). Both $T_\iota^{\mathbf{r}}xy$ and $x \sim_\iota^{\mathrm{nc}} y$ satify the same clauses. Use the respective elimination axiom in each of the two directions.

(b) $\leftrightarrow$ (c). Use Corollary 3.4.2 (page 45). ☐

LEMMA 4.1.2 (Realizers of cototality). *For closed base types ι the following are equivalent.*

(a) ${}^{\mathrm{co}}T^{\mathbf{r}}_{\iota}xy$,
(b) $x \approx^{\mathrm{nc}}_{\iota} y$,
(c) $x \in {}^{\mathrm{co}}T^{\mathrm{nc}}_{\iota} \wedge x \equiv y$.

PROOF. As an example we give the proof for $\mathbb{N}$. Since we have n.c. goals only, decorations are omitted.

(a) $\to$ (b). We use the greatest-fixed-point axiom for $\approx_{\mathbb{N}}$:

$$\forall_{n,m}(Xnm \to (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}((n' \approx_{\mathbb{N}} m' \vee Xn'm') \wedge n \equiv Sn' \wedge m \equiv Sm')) \to X \subseteq {\approx_{\mathbb{N}}}$$

and apply it with ${}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}$ for X. It suffices to prove the premise. Assume ${}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}nm$; the goal is

$$C := (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}((n' \approx_{\mathbb{N}} m' \vee {}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}n'm') \wedge n \equiv Sn' \wedge m \equiv Sm').$$

By the closure axiom $({}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}})^{-}$ we have

$$(n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}({}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}n'm' \wedge n \equiv Sn' \wedge m \equiv Sm').$$

We argue by cases (i.e., use $\vee^{-}$).

Case 1. $n \equiv 0 \wedge m \equiv 0$. Go for the l.h.s. of the disjunction C.

Case 2. $\exists_{n',m'}({}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}n'm' \wedge n \equiv Sn' \wedge m \equiv Sm')$. Go for the r.h.s. of C.

(b) $\to$ (a). Recall ${}^{\mathrm{co}}T_{\mathbb{N}} := \nu_X(0 \in X,\ \forall_{n \in X}(Sn \in X))$, hence by definition

$${}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}} := \nu_{X^{\mathbf{r}}}(X^{\mathbf{r}}00,\ \forall_{n,m}(X^{\mathbf{r}}nm \to X^{\mathbf{r}}(Sn)(Sm))).$$

We need to show $n \approx_{\mathbb{N}} m \to {}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}nm$. To this end we use the greatest-fixed-point axiom for ${}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}$:

$$\forall_{n,m}(Xnm \to (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}(n', m' \in ({}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}} \cup X) \wedge n \equiv Sn' \wedge m \equiv Sm')) \to X \subseteq {}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}}$$

and apply it with $\approx_{\mathbb{N}}$ for X. It suffices to prove the premise. Assume $n \approx_{\mathbb{N}} m$; the goal is

$$C := (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}((n', m' \in ({}^{\mathrm{co}}T^{\mathbf{r}}_{\mathbb{N}} \cup {\approx_{\mathbb{N}}}) \wedge n \equiv Sn' \wedge m \equiv Sm')).$$

By the closure axiom $(\approx_{\mathbb{N}})^{-}$ we have

$$n \approx_{\mathbb{N}} m \to (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}(n' \approx_{\mathbb{N}} m' \wedge n \equiv Sn' \wedge m \equiv Sm').$$

We argue by cases (i.e., use $\vee^{-}$).

Case 1. $n \equiv 0 \wedge m \equiv 0$. Go for the l.h.s. of the disjunction C.

Case 2. $\exists_{n',m'}(n' \approx_{\mathbb{N}} m' \wedge n \equiv Sn' \wedge m \equiv Sm')$. Go for the r.h.s. of C.

(b) $\leftrightarrow$ (c). Use the Bisimilarity axiom and Proposition 3.4.1 (page 44). $\square$

Next we study what our general definition says about realizers for the c.r. inductively defined decorated connectives.

Recall that for the sum type $\rho+\sigma$ we had the constructors $(\mathrm{InL}_{\rho\sigma})^{\rho\to\rho+\sigma}$ and $(\mathrm{InR}_{\rho\sigma})^{\sigma\to\rho+\sigma}$. In the special situation that one of the two parameter types is the unit type $\mathbb{U}$ it is common to view the sum type $\mathbb{U}+\sigma$ as a unary algebra form, with constructors DummyL of type $\mathbb{U}+\sigma$ and Inr of type $\sigma\to\mathbb{U}+\sigma$. Similarly $\rho+\mathbb{U}$ is viewed as a unary algebra form, with constructors Inl of type $\rho\to\rho+\mathbb{U}$ and DummyR of type $\rho+\mathbb{U}$.

LEMMA 4.1.3 (Realizers for $\vee$). *$z\ \mathbf{r}\ (A\vee B)$ is equivalent to*

$\exists_x(x\ \mathbf{r}\ A\wedge z\equiv\mathrm{InL}(x))\vee^{\mathrm{nc}}\exists_y(y\ \mathbf{r}\ B\wedge z\equiv\mathrm{InR}(y))$	*for A,B c.r.*
$\exists_x(x\ \mathbf{r}\ A\wedge z\equiv\mathrm{Inl}(x))\vee^{\mathrm{nc}}(B\wedge z\equiv\mathrm{DummyR})$	*for A c.r. and B n.c.*
$(A\wedge z\equiv\mathrm{DummyL})\vee^{\mathrm{nc}}\exists_y(y\ \mathbf{r}\ B\wedge z\equiv\mathrm{Inr}(y))$	*for A n.c. and B c.r.*
$(A\wedge z\equiv\mathtt{tt})\vee^{\mathrm{nc}}(B\wedge z\equiv\mathtt{ff})$	*for A,B n.c.*

PROOF. As an example we consider the case A n.c. and B c.r. Recall $\mathrm{OrR}_{X^{\mathrm{nc}},Y^{\mathrm{c}}}:=\mu_Z(X^{\mathrm{nc}}\to Z, Y^{\mathrm{c}}\to Z)$. Then

$$\mathrm{OrR}^{\mathbf{r}}_{X^{\mathrm{nc}},Y^{\mathbf{r}}}:=\mu_{Z^{\mathbf{r}}}(X^{\mathrm{nc}}\to\mathrm{DummyL}\in Z^{\mathbf{r}},\forall_y(y\ \mathbf{r}\ Y\to\mathrm{Inr}(y)\in Z^{\mathbf{r}})).$$

Now substituting X^{nc} by A and $Y^{\mathbf{r}}$ by $\{\,y\mid y\ \mathbf{r}\ B\,\}$ in the introduction axioms gives

$$(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_0\colon A\to\mathrm{DummyL}\ \mathbf{r}\ (A\vee B),$$
$$(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_1\colon \forall_y(y\ \mathbf{r}\ B\to\mathrm{Inr}(y)\ \mathbf{r}\ (A\vee B)).$$

This suffices for "$\leftarrow$": if $A\wedge z\equiv\mathrm{DummyL}$, then from $(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_0$ we obtain $z\ \mathbf{r}\ (A\vee B)$, and if we have y with $y\ \mathbf{r}\ B$ and $z\equiv\mathrm{Inr}(y)$, then from $(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_1$ we again obtain $z\ \mathbf{r}\ (A\vee B)$.

Conversely, the elimination axiom $(\mathrm{OrR}^{\mathbf{r}}_{X^{\mathrm{nc}},Y^{\mathbf{r}}})^-$ is

$$(X^{\mathrm{nc}}\to\mathrm{DummyL}\in Z)\to\forall_y(y\ \mathbf{r}\ Y\to\mathrm{Inr}(y)\in Z)\to\mathrm{OrR}^{\mathbf{r}}_{X^{\mathrm{nc}},Y^{\mathbf{r}}}\subseteq Z.$$

Substitute Z by $\{\,z\mid(A\wedge z\equiv\mathrm{DummyL})\vee^{\mathrm{nc}}\exists_y(y\ \mathbf{r}\ B\wedge z\equiv\mathrm{Inr}(y))\,\}$. Then with A for X^{nc} and $\{\,y\mid y\ \mathbf{r}\ B\,\}$ for $Y^{\mathbf{r}}$ the two premises become provable and we obtain

$$\forall_z(z\ \mathbf{r}\ (A\vee B)\to(A\wedge z\equiv\mathrm{DummyL})\vee^{\mathrm{nc}}\exists_y(y\ \mathbf{r}\ B\wedge z\equiv\mathrm{Inr}(y))).\qquad\square$$

Similarly we have

LEMMA 4.1.4 (Realizers for $\wedge$). *$z\ \mathbf{r}\ (A\wedge B)$ is equivalent to*

$z\equiv\langle\mathrm{lft}(z),\mathrm{rht}(z)\rangle\wedge(\mathrm{lft}(z)\ \mathbf{r}\ A)\wedge(\mathrm{rht}(z)\ \mathbf{r}\ B)$	*for A c.r. and B c.r.*
$(z\ \mathbf{r}\ A)\wedge B$	*for A c.r. and B n.c.*
$A\wedge(z\ \mathbf{r}\ B)$	*for A n.c. and B c.r.*

PROOF. Exercise. $\square$

LEMMA 4.1.5 (Realizers for $\exists$). $z \mathbf{r} \exists_x A \leftrightarrow \exists_x (z \mathbf{r} A)$ *for* A *c.r.*

PROOF. Recall $\mathrm{Ex}_Y := \mu_X(\forall_x (x \in Y \to X))$. Then

$$\mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}} := \mu_{X^{\mathbf{r}}}(\forall_{x,z}(Y^{\mathbf{r}} xz \to X^{\mathbf{r}} z)).$$

Now substituting $Y^{\mathbf{r}}$ by $\{ x, z \mid z \mathbf{r} A \}$ in the introduction axiom gives

$$(\mathrm{Ex}^{\mathbf{r}}_{\{x,z \mid z\mathbf{r}A\}})^+_0 \colon \forall_{x,z}(z \mathbf{r} A \to z \mathbf{r} \exists_x A)$$

Conversely, the elimination axiom $(\mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}})^-$ is

$$\forall_z (z \in \mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}} \to \forall_{x,z}(Y^{\mathbf{r}} xz \to z \in X) \to z \in X).$$

which is equivalent to

$$\forall_z (z \in \mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}} \to \forall_z(\exists_x Y^{\mathbf{r}} xz \to z \in X) \to z \in X).$$

Substituting X by $\{ z \mid \exists_x (Y^{\mathbf{r}} xz) \}$ makes the middle part provable. Thus with $\{ x, z \mid z \mathbf{r} A \}$ for $Y^{\mathbf{r}}$ we obtain $\forall_z (z \mathbf{r} \exists_x A \to \exists_x (z \mathbf{r} A))$ from $(\mathrm{Ex}^{\mathbf{r}}_{\{x,z \mid z\mathbf{r}A\}})^-$. $\square$

4.2. Extracted terms, soundness

Let M be a proof in TCF of a c.r. formula A. Assume M is an $\mathbf{r}$-free proof, i.e., M contains no realizability predicates $I^{\mathbf{r}}$ or ${}^{\mathrm{co}}I^{\mathbf{r}}$. We define its *extracted term* $\mathrm{et}(M)$, of type $\tau(A)$, with the aim to express M's computational content. It will be a term built up from variables, constructors, recursion operators, destructors and corecursion operators by λ-abstraction and application.

DEFINITION (Extracted term). For an $\mathbf{r}$-free proof M of a c.r. formula A we define its extracted term $\mathrm{et}(M)$ by

$$\begin{aligned}
\mathrm{et}(u^A) &:= z_u^{\tau(A)} \quad (z_u^{\tau(A)} \text{ uniquely associated to } u^A),\\
\mathrm{et}((\lambda_{u^A} M^B)^{A\to B}) &:= \begin{cases} \lambda_{z_u} \mathrm{et}(M) & \text{if } A \text{ is c.r.}\\ \mathrm{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\mathrm{et}((M^{A\to B} N^A)^B) &:= \begin{cases} \mathrm{et}(M)\mathrm{et}(N) & \text{if } A \text{ is c.r.}\\ \mathrm{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\mathrm{et}((\lambda_x M^A)^{\forall_x A}) &:= \mathrm{et}(M),\\
\mathrm{et}((M^{\forall_x A(x)} t)^{A(t)}) &:= \mathrm{et}(M).
\end{aligned}$$

It remains to define extracted terms for the axioms. Consider a (c.r.) inductively defined predicate I. For its introduction and elimination axioms define

$$\mathrm{et}(I_i^+) := \mathrm{C}_i,$$
$$\mathrm{et}(I^-) := \mathcal{R},$$

where both the constructor C_i and the recursion operator $\mathcal{R}$ refer to the base type ι_I associated with I. For the closure and greatest-fixed-point axioms of ${}^{\mathrm{co}}I$ define

$$\mathrm{et}({}^{\mathrm{co}}I^-) := \mathcal{D},$$
$$\mathrm{et}({}^{\mathrm{co}}I_i^+) := {}^{\mathrm{co}}\mathcal{R},$$

where again both the destructor $\mathcal{D}$ and the corecursion operator ${}^{\mathrm{co}}\mathcal{R}$ refer to the base type ι_I associated with I. For the elimination axiom $(I^{\mathrm{nc}})^-$ of a one-clause-nc inductive predicate with a c.r. competitor predicate the extracted term is the identity.

From the Soundness Theorem 4.2.1 below it will follow that the term extracted from a closed **r**-free proof of a c.r. formula A realizes A. As a preparation we first attend the axioms. Let I be an inductive predicate and ι_I its associated base type. One can show that the extracted term of $I^\pm$, ${}^{\mathrm{co}}I^\pm$ realizes the respective axiom[1]. Proofs of these facts are automatically generated in Minlog.

THEOREM 4.2.1 (Soundness). *Let M be an **r**-free derivation of a formula A from assumptions $u_i \colon C_i$ $(i < n)$. Then we can derive*

$$\begin{cases} \mathrm{et}(M)\ \mathbf{r}\ A & \text{if } A \text{ is c.r.} \\ A & \text{if } A \text{ is n.c.} \end{cases}$$

from assumptions

$$\begin{cases} z_{u_i}\ \mathbf{r}\ C_i & \text{if } C_i \text{ is c.r.} \\ C_i & \text{if } C_i \text{ is n.c.} \end{cases}$$

PROOF. *Case* $u \colon A$. *Subcase* A c.r. Then $\mathrm{et}(u) = z_u$. *Subcase* A n.c. Immediate.

Case $c \colon A$. *Subcase* A c.r. The axioms have been treated above. *Subcase* A n.c. Immediate.

Case $(\lambda_{u^A} M^B)^{A \to B}$ with B c.r. We must derive $\mathrm{et}(\lambda_u M)\ \mathbf{r}\ (A \to B)$. To this end we distinguish subcases. *Subcase* A c.r. Then the goal

$$\forall_z (z\ \mathbf{r}\ A \to \mathrm{et}(M)(z)\ \mathbf{r}\ B)$$

[1] In Appendix C such proofs for some (co)inductive predicates are written out.

follows from the induction hypothesis by $\to^+$ and $\forall^+$.

Subcase A n.c. Then the goal is

$$A \to \mathrm{et}(\lambda_u M)\ \mathbf{r}\ B.$$

Recall that $\mathrm{et}(\lambda_u M) = \mathrm{et}(M)$. By induction hypothesis we have a derivation of $\mathrm{et}(M)\ \mathbf{r}\ B$ from A, which is what we want.

Case $(\lambda_{u^A} M^B)^{A \to B}$ with B n.c. We need a derivation of $A \to B$.

Subcase A c.r. By induction hypothesis we have a derivation of B from $z\ \mathbf{r}\ A$. Using the invariance axiom $A \to \exists_z(z\ \mathbf{r}\ A)$ we obtain the required derivation of B from A as follows.

$$\cfrac{\cfrac{A \to \exists_z(z\ \mathbf{r}\ A) \qquad A}{\exists_z(z\ \mathbf{r}\ A)} \qquad \begin{matrix}[z\ \mathbf{r}\ A] \\ \mid \mathrm{IH} \\ B\end{matrix}}{B}\ \exists^-$$

Subcase A n.c. By induction hypothesis we have a derivation of B from A, which is what we want.

Case $(M^{A \to B} N^A)^B$ with B c.r. We need a derivation of $\mathrm{et}(MN)\ \mathbf{r}\ B$. To this end we distinguish subcases. *Subcase* A c.r. Then $\mathrm{et}(MN) = \mathrm{et}(M)\mathrm{et}(N)$. By induction hypothesis we have derivations of $\mathrm{et}(M)\ \mathbf{r}\ (A \to B)$ and hence of

$$\forall_z(z\ \mathbf{r}\ A \ \to\ \mathrm{et}(M)z\ \mathbf{r}\ B)$$

and of $\mathrm{et}(N)\ \mathbf{r}\ A$. This gives the claim. *Subcase* A n.c. Then $\mathrm{et}(MN) = \mathrm{et}(M)$. By induction hypothesis we have derivations of $\mathrm{et}(M)\ \mathbf{r}\ (A \to B)$ and hence of

$$A \ \to\ \mathrm{et}(M)\ \mathbf{r}\ B$$

and of A. Applying the former to the latter gives $\mathrm{et}(M)\ \mathbf{r}\ B$.

Case $(M^{A \to B} N^A)^B$ with B n.c. The goal is to find a derivation of B. *Subcase* A c.r. By induction hypothesis we have derivations of $A \to B$ and of $\mathrm{et}(N)\ \mathbf{r}\ A$. Now using the invariance axiom $\forall_z(z\ \mathbf{r}\ A \to A)$ we obtain the required derivation of B by $\to^-$ from the derivation of $A \to B$ and

$$\cfrac{\cfrac{\forall_z(z\ \mathbf{r}\ A \to A) \qquad \mathrm{et}(N)}{\mathrm{et}(N)\ \mathbf{r}\ A \to A} \qquad \begin{matrix}\mid \mathrm{IH} \\ \mathrm{et}(N)\ \mathbf{r}\ A\end{matrix}}{A}$$

Subcase A n.c. By induction hypothesis we have derivations of $A \to B$ and of A, hence also a derivation of B.

Case $(\lambda_x M^A)^{\forall_x A}$ with $\forall_x A$ c.r. We need a derivation of $\mathrm{et}(\lambda_x M)\ \mathbf{r}\ \forall_x A$. By definition $\mathrm{et}(\lambda_x M) = \mathrm{et}(M)$. Hence we must derive

$$\mathrm{et}(M)\ \mathbf{r}\ \forall_x A, \quad \text{which is} \quad \forall_x(\mathrm{et}(M)\ \mathbf{r}\ A).$$

This follows from the induction hypothesis.

Case $(\lambda_x M^A)^{\forall_x A}$ with $\forall_x A$ n.c. By induction hypothesis we have a derivation of A. Apply $\forall^+$.

Case $(M^{\forall_x A(x)} t)^{A(t)}$ with $A(t)$ c.r. We must derive $\mathrm{et}(Mt)\ \mathbf{r}\ A(t)$. By definition $\mathrm{et}(Mt) = \mathrm{et}(M)$, and by induction hypothesis we can derive

$$\mathrm{et}(M)\ \mathbf{r}\ \forall_x A(x), \quad \text{which is} \quad \forall_x(\mathrm{et}(M)\ \mathbf{r}\ A(x)).$$

Case $(M^{\forall_x A(x)} t)^{A(t)}$ with $A(t)$ n.c. By induction hypothesis we have a derivation of $\forall_x A(x)$. Apply $\forall^-$. $\square$