CHAPTER 2

Partial continuous functionals

The objects studied in mathematics have types, which in many cases are function types, possibly of a higher type. Such objects in most cases are infinite, and we intend to describe them in terms of their finite approximations. An appropriate framework for such an approach are the partial continuous functionals of Scott (1982) and Ershov (1977). Continuity of a function f here means that for every approximation V of the value $f(x)$ there is an approximation U of the argument x such that $f[U]$ has more information than V. We define the partial continuous functionals via Scott's information systems.

2.1. Information systems

The basic idea of information systems is to provide an axiomatic setting to describe approximations of abstract objects by concrete, finite ones. We take an arbitrary countable set A of "bits of data" or "tokens" as a basic notion to be explained axiomatically. In order to use such data to build approximations of abstract objects, we need a notion of "consistency", which determines when the elements of a finite set of tokens are consistent with each other. We also need an "entailment relation" between consistent sets U of data and single tokens a, which intuitively expresses the fact that the information contained in U is sufficient to compute the bit of information a. The axioms below are a minor modification of Scott's (1982), due to Larsen and Winskel (1991).

2.1.1. Ideals.

Definition. An *information system* is a structure $(A, \mathrm{Con}, \vdash)$ where A is an at most countable non-empty set (the *tokens*), Con is a set of finite subsets of A (the *consistent* sets) and $\vdash$ is a subset of $\mathrm{Con} \times A$ (the *entailment* relation), which satisfy

$$
\begin{aligned}
&U \subseteq V \in \mathrm{Con} \to U \in \mathrm{Con},\\
&\{a\} \in \mathrm{Con},\\
&U \vdash a \to U \cup \{a\} \in \mathrm{Con},\\
&a \in U \in \mathrm{Con} \to U \vdash a,\\
&U \in \mathrm{Con} \to \forall_{a \in V}(U \vdash a) \to V \vdash b \to U \vdash b.
\end{aligned}
$$

The elements of Con are called *formal neighborhoods*. We use U, V, W to denote *finite* sets, and write

$$
\begin{aligned}
&U \vdash V &&\text{for} \quad U \in \mathrm{Con} \land \forall_{a \in V}(U \vdash a),\\
&a \uparrow b &&\text{for} \quad \{a, b\} \in \mathrm{Con} \qquad (a, b \text{ are } \textit{consistent}),\\
&U \uparrow V &&\text{for} \quad \forall_{a \in U, b \in V}(a \uparrow b).
\end{aligned}
$$

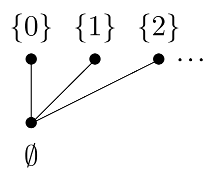
Definition. The *ideals* (also called *objects*) of an information system $\boldsymbol{A} = (A, \mathrm{Con}, \vdash)$ are defined to be those subsets x of A which satisfy

$$
\begin{aligned}
&U \subseteq x \to U \in \mathrm{Con} \quad (x \text{ is } \textit{consistent}),\\
&U \vdash a \to U \subseteq x \to a \in x \quad (x \text{ is } \textit{deductively closed}).
\end{aligned}
$$

We write $x \in |\boldsymbol{A}|$ to mean that x is an ideal of $\boldsymbol{A}$.

Examples. The *deductive closure* $\overline{U} := \{\, a \in A \mid U \vdash a \,\}$ of $U \in \mathrm{Con}$ is an ideal.

Every countable set A can be turned into a "flat" information system by letting the set of tokens be A, $\mathrm{Con} := \{\emptyset\} \cup \{\, \{a\} \mid a \in A \,\}$ and $U \vdash a$ mean $a \in U$. In this case the ideals are just the elements of Con. For $A = \mathbb{N}$ we have the following picture of the Con-sets.



A rather important example is the following, which concerns approximations of functions from a countable set A into a countable set B. The tokens are the pairs (a, b) with $a \in A$ and $b \in B$, and

$$
\begin{aligned}
&\mathrm{Con} := \{\, \{\, (a_i, b_i) \mid i < k \,\} \mid \forall_{i,j<k}(a_i = a_j \to b_i = b_j) \,\},\\
&U \vdash (a, b) := (a, b) \in U.
\end{aligned}
$$

It is easy to verify that this defines an information system whose ideals are (the graphs of) all partial functions from A to B.

2.1.2. Function spaces. We define the "function space" $\boldsymbol{A} \to \boldsymbol{B}$ between two information systems $\boldsymbol{A}$ and $\boldsymbol{B}$.

Definition. Let $\boldsymbol{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\boldsymbol{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. Define $\boldsymbol{A} \to \boldsymbol{B} = (C, \mathrm{Con}, \vdash)$ by

$$C := \mathrm{Con}_A \times B,$$

$$\{\, (U_i, b_i) \mid i \in I \,\} \in \mathrm{Con} := \forall_{J \subseteq I} \Big(\bigcup_{j \in J} U_j \in \mathrm{Con}_A \to \{\, b_j \mid j \in J \,\} \in \mathrm{Con}_B \Big).$$

For the definition of the entailment relation $\vdash$ it is helpful to first define the notion of an *application* of $W := \{\, (U_i, b_i) \mid i \in I \,\} \in \mathrm{Con}$ to $U \in \mathrm{Con}_A$:

$$\{\, (U_i, b_i) \mid i \in I \,\} U := \{\, b_i \mid U \vdash_A U_i \,\}.$$

From the definition of Con we know that this set is in Con_B. Now define $W \vdash (U, b)$ by $WU \vdash_B b$.

Remark. Clearly application is *monotone in the second argument*, in the sense that $U \vdash_A U'$ implies ($WU' \subseteq WU$, hence also) $WU \vdash_B WU'$. In fact, application is also *monotone in the first argument*, i.e.,

$$W \vdash W' \quad \text{implies} \quad WU \vdash_B W'U.$$

To see this let $W = \{\, (U_i, b_i) \mid i \in I \,\}$ and $W' = \{\, (U'_j, b'_j) \mid j \in J \,\}$. By definition $W'U = \{\, b'_j \mid U \vdash_A U'_j \,\}$. Now fix j such that $U \vdash_A U'_j$; we must show $WU \vdash_B b'_j$. By assumption $W \vdash (U'_j, b'_j)$, hence $WU'_j \vdash_B b'_j$. Because of $WU \supseteq WU'_j$ the claim follows.

Lemma 2.1.1. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are information systems, then so is $\boldsymbol{A} \to \boldsymbol{B}$ defined as above.*

Proof. Let $\boldsymbol{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\boldsymbol{B} = (B, \mathrm{Con}_B, \vdash_B)$. The first, second and fourth property of the definition are clearly satisfied. For the third, suppose

$$\{(U_1, b_1), \dots, (U_n, b_n)\} \vdash (U, b), \quad \text{i.e.,} \quad \{\, b_j \mid U \vdash_A U_j \,\} \vdash_B b.$$

We have to show that $\{(U_1, b_1), \dots, (U_n, b_n), (U, b)\} \in \mathrm{Con}$. So let $I \subseteq \{1, \dots, n\}$ and suppose

$$U \cup \bigcup_{i \in I} U_i \in \mathrm{Con}_A.$$

We must show that $\{b\} \cup \{\, b_i \mid i \in I \,\} \in \mathrm{Con}_B$. Let $J \subseteq \{1, \dots, n\}$ consist of those j with $U \vdash_A U_j$. Then also

$$U \cup \bigcup_{i \in I} U_i \cup \bigcup_{j \in J} U_j \in \mathrm{Con}_A.$$

Since

$$\bigcup_{i \in I} U_i \cup \bigcup_{j \in J} U_j \in \mathrm{Con}_A,$$

from the consistency of $\{(U_1, b_1), \dots, (U_n, b_n)\}$ we can conclude that

$$\{\, b_i \mid i \in I \,\} \cup \{\, b_j \mid j \in J \,\} \in \mathrm{Con}_B.$$

But $\{\, b_j \mid j \in J \,\} \vdash_B b$ by assumption. Hence

$$\{\, b_i \mid i \in I \,\} \cup \{\, b_j \mid j \in J \,\} \cup \{b\} \in \mathrm{Con}_B.$$

For the final property, suppose

$$W \vdash W' \quad \text{and} \quad W' \vdash (U, b).$$

We have to show $W \vdash (U, b)$, i.e., $WU \vdash_B b$. We obtain $WU \vdash_B W'U$ by monotonicity in the first argument, and $W'U \vdash_B b$ by definition. □

We shall now give an alternative characterization of the ideals in $\boldsymbol{A} \to \boldsymbol{B}$, as "approximable maps". The basic idea for approximable maps is the desire to study "information respecting" maps from $\boldsymbol{A}$ into $\boldsymbol{B}$. Such a map is given by a relation r between Con_A and B, where $(U, b) \in r$ intuitively means that whenever we are given the information $U \in \mathrm{Con}_A$, then we know that at least the token b appears in the value.

Definition. Let $\boldsymbol{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\boldsymbol{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. A relation $r \subseteq \mathrm{Con}_A \times B$ is an *approximable map* if it satisfies the following:

(a) if $(U, b_1), \dots, (U, b_n) \in r$, then $\{b_1, \dots, b_n\} \in \mathrm{Con}_B$;
(b) if $(U, b_1), \dots, (U, b_n) \in r$ and $\{b_1, \dots, b_n\} \vdash_B b$, then $(U, b) \in r$;
(c) if $(U', b) \in r$ and $U \vdash_A U'$, then $(U, b) \in r$.

Theorem 2.1.2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be information systems. Then the ideals of $\boldsymbol{A} \to \boldsymbol{B}$ are exactly the approximable maps from $\boldsymbol{A}$ to $\boldsymbol{B}$.*

Proof. Exercise.

□

2.1.3. Continuous functions. We can also characterize approximable maps in a different way, which is closer to usual characterizations of continuity[1]:

Lemma 2.1.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be information systems and $f \colon |\boldsymbol{A}| \to |\boldsymbol{B}|$ monotone (i.e., $x \subseteq y$ implies $f(x) \subseteq f(y)$). Then the following are equivalent.*

[1]In fact, approximable maps are exactly the continuous functions w.r.t. the so-called Scott topology. However, we will not enter this subject here.