CHAPTER 1

Logic

The main subject of Mathematical Logic is mathematical proof. In this introductory chapter we deal with the basics of formalizing such proofs and, via normalization, analysing their structure. The system we pick for the representation of proofs is Gentzen's natural deduction from (1935). Our reasons for this choice are twofold. First, as the name says this is a *natural* notion of formal proof, which means that the way proofs are represented corresponds very much to the way a careful mathematician writing out all details of an argument would proceed anyway. Second, formal proofs in natural deduction are closely related (via the so-called Curry-Howard correspondence) to terms in typed lambda calculus. This provides us not only with a compact notation for logical derivations (which otherwise tend to become somewhat unmanagable tree-like structures), but also opens up a route to applying the computational techniques which underpin lambda calculus.

An underlying theme of this chapter is to bring out the constructive content of logic, particularly in regard to the relationship between minimal and classical logic. For us the latter is most appropriately viewed as a subsystem of the former. This approach will reveal some interesting aspects of proofs, e.g., that it is possible and useful to distinguish between existential proofs that actually construct witnessing objects, and others that don't.

As an example for a non-constructive existence proof, consider the following proposition.

There are irrational numbers a, b such that a^b is rational.

This can be proved as follows, by cases.

Case $\sqrt{2}^{\sqrt{2}}$ is rational. Choose $a = \sqrt{2}$ and $b = \sqrt{2}$. Then a, b are irrational and by assumption a^b is rational.

Case $\sqrt{2}^{\sqrt{2}}$ is irrational. Choose $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then by assumption a, b are irrational and

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$$

is rational.

As long as we have not decided whether $\sqrt{2}^{\sqrt{2}}$ is rational, we do not know which numbers a, b we must take. Hence we have an example of an existence proof which does not provide an instance.

1.1. Natural deduction

The rules of natural deduction come in pairs: we have an introduction and an elimination rule for the logical connectives $\to$ and $\forall$. The resulting system is called *minimal logic*; it was introduced by Kolmogorov (1932), Gentzen (1935) and Johansson (1937). Notice that no negation is yet present. If we go on and require *ex-falso-quodlibet* for the nullary propositional symbol $\bot$ ("falsum") we can embed *intuitionistic logic* with negation as $A \to \bot$. To embed classical logic, we need to go further and add as an axiom schema the principle of *indirect proof*, also called *stability* ($\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x})$ for relation symbols R).

1.1.1. Introduction and elimination rules for $\to$ and $\forall$. First we have an assumption rule, allowing to write down an arbitrary formula A together with a marker u:

$$u\colon A \qquad \text{assumption.}$$

The other rules of natural deduction split into introduction rules (I-rules for short) and elimination rules (E-rules) for the logical connectives $\to$ and $\forall$. For implication $\to$ there is an introduction rule $\to^+$ and an elimination rule $\to^-$ also called *modus ponens*. The left premise $A \to B$ in $\to^-$ is called the *major* (or *main*) premise, and the right premise A the *minor* (or *side*) premise. Note that with an application of the $\to^+$-rule *all* assumptions above it marked with $u\colon A$ are cancelled (which is denoted by putting square brackets around these assumptions), and the u then gets written alongside. There may of course be other uncancelled assumptions $v\colon A$ of the same formula A, which may get cancelled at a later stage.

$$\dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{A \to B}\to^+ u \qquad\qquad \dfrac{\begin{matrix}\mid M\\ A \to B\end{matrix} \qquad \begin{matrix}\mid N\\ A\end{matrix}}{B}\to^-$$

For the universal quantifier $\forall$ there is an introduction rule $\forall^+$ (again marked, but now with the bound variable x) and an elimination rule $\forall^-$ whose right premise is the term t to be substituted. The rule $\forall^+ x$ with conclusion $\forall_x A$ is subject to the following *(eigen-)variable condition*: the derivation M of the premise A must not contain any open assumption having x as a free variable.

$$\begin{array}{c} \mid M \\ \dfrac{A}{\forall_x A}\,\forall^+ x \end{array} \qquad \begin{array}{c} \mid M \\ \dfrac{\forall_x A(x) \qquad t}{A(t)}\,\forall^- \end{array}$$

As an example we derive $(A \to B \to C) \to (A \to B) \to A \to C$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u\colon A \to B \to C \qquad w\colon A}{B \to C} \qquad \dfrac{v\colon A \to B \qquad w\colon A}{B}}{C}}{A \to C}\,\to^+ w}{(A \to B) \to A \to C}\,\to^+ v}{(A \to B \to C) \to (A \to B) \to A \to C}\,\to^+ u$$

Since in many cases the rule used is determined by the conclusion, we may suppress in such cases the name of the rule. As an example involving the universal quantifier we derive $\forall_x(A \to B) \to A \to \forall_x B$, under the assumption that x is not free in A (written $x \notin \mathrm{FV}(A)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u\colon \forall_x(A \to B) \qquad x}{A \to B} \qquad v\colon A}{B}}{\forall_x B}\,\forall^+ x}{A \to \forall_x B}\,\to^+ v}{\forall_x(A \to B) \to A \to \forall_x B}\,\to^+ u$$

Note that the variable condition is satisfied: x is not free in A (and also not free in $\forall_x(A \to B)$).

Definition. A formula A is called *derivable* (in *minimal logic*), written $\vdash A$, if there is a derivation of A (without free assumptions) using the natural deduction rules. A formula B is called derivable from assumptions $A_1, \dots, A_n$, if there is a derivation of B with free assumptions among $A_1, \dots, A_n$. Let Γ be a (finite or infinite) set of formulas. We write $\Gamma \vdash B$ if the formula B is derivable from finitely many assumptions $A_1, \dots, A_n \in \Gamma$.

1.1.2. Properties of negation. Recall that negation was defined by $\neg A := (A \to \bot)$. The following can easily be derived.

$$A \to \neg\neg A,$$
$$\neg\neg\neg A \to \neg A.$$

However, $\neg\neg A \to A$ is in general *not* derivable.

LEMMA. *The following are derivable.*

$$(A \to B) \to \neg B \to \neg A,$$
$$\neg(A \to B) \to \neg B,$$
$$\neg\neg(A \to B) \to \neg\neg A \to \neg\neg B,$$
$$(\bot \to B) \to (\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B),$$
$$\neg\neg\forall_x A \to \forall_x \neg\neg A.$$

Derivations are left as an exercise.

We have not yet considered the logical connectives "or" (disjunction) and "exists". They are special from a constructive point of view, since a proposition with such a connective should have "computational content". For this reason we do not consider them as logical concepts, but rather as (special cases of) inductive definitions which will be dealt with later in this course.

However, there is a substitute, the so-called weak (or "classical") form of disjunction end existence. We write them with a tilde to mark the difference from the strong form of these connectives. They are defined by

$$A \tilde{\vee} B := \neg A \to \neg B \to \bot, \qquad \tilde{\exists}_x A := \neg\forall_x \neg A.$$

Since $\tilde{\exists}_x \tilde{\exists}_y A$ unfolds into an awkward formula we extend the $\tilde{\exists}$-terminology to lists of variables:

$$\tilde{\exists}_{x_1,\dots,x_n} A := \forall_{x_1,\dots,x_n}(A \to \bot) \to \bot.$$

Moreover let

$$\tilde{\exists}_{x_1,\dots,x_n}(A_1 \tilde{\wedge} \dots \tilde{\wedge} A_m) := \forall_{x_1,\dots,x_n}(A_1 \to \dots \to A_m \to \bot) \to \bot.$$

This allows to stay in the $\to, \forall$ part of the language. Notice that $\tilde{\wedge}$ only makes sense in this context, i.e., in connection with $\tilde{\exists}$.

1.1.3. Intuitionistic and classical derivability. In the definition of derivability falsity $\bot$ plays no role. We may change this and require *ex-falso-quodlibet* axioms, of the form

$$\forall_{\vec{x}}(\bot \to R\vec{x}\,)$$

with R a relation symbol distinct from $\bot$. Let Efq denote the set of all such axioms. A formula A is called *intuitionistically derivable*, written $\vdash_i A$, if $\mathrm{Efq} \vdash A$. We write $\Gamma \vdash_i B$ for $\Gamma \cup \mathrm{Efq} \vdash B$.

We may even go further and require *stability* axioms, of the form

$$\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x}\,)$$

with R again a relation symbol distinct from $\bot$. Let Stab denote the set of all these axioms. A formula A is called *classically derivable*, written $\vdash_c A$, if $\mathrm{Stab} \vdash A$. We write $\Gamma \vdash_c B$ for $\Gamma \cup \mathrm{Stab} \vdash B$.

It is easy to see that intuitionistically (i.e., from Efq) we can derive $\bot \to A$ for an *arbitrary* formula A, using the introduction rules for the connectives.

Theorem (Stability, or principle of indirect proof).

(a) $\vdash (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$.
(b) $\vdash (\neg\neg A \to A) \to \neg\neg\forall_x A \to A$.
(c) $\vdash_c \neg\neg A \to A$.

Proof. (a) For simplicity, in the derivation to be constructed we leave out applications of $\to^+$ at the end.

$$\dfrac{u\colon \neg\neg B \to B \qquad \dfrac{\dfrac{v\colon \neg\neg(A \to B) \qquad \dfrac{\dfrac{u_1\colon \neg B \qquad \dfrac{u_2\colon A \to B \qquad w\colon A}{B}}{\bot}}{\neg(A \to B)}\,{\to^+ u_2}}{\bot}}{\neg\neg B}\,{\to^+ u_1}}{B}$$

(b)

$$\dfrac{u\colon \neg\neg A \to A \qquad \dfrac{\dfrac{v\colon \neg\neg\forall_x A \qquad \dfrac{\dfrac{u_1\colon \neg A \qquad \dfrac{u_2\colon \forall_x A \qquad x}{A}}{\bot}}{\neg\forall_x A}\,{\to^+ u_2}}{\bot}}{\neg\neg A}\,{\to^+ u_1}}{A}$$

(c) Induction on A. The case $R\vec{t}$ with R distinct from $\bot$ is given by Stab. In the case $\bot$ the desired derivation is

$$\dfrac{v\colon (\bot \to \bot) \to \bot \qquad \dfrac{u\colon \bot}{\bot \to \bot}\,{\to^+ u}}{\bot}$$

In the cases $A \to B$ and $\forall_x A$ use (a) and (b), respectively. $\square$

Using stability we can prove some well-known facts about the interaction of weak disjunction and the weak existential quantifier with implication. We first prove a more refined claim, stating to what extent we need to go beyond minimal logic.

Lemma. *The following are derivable.*

(1) $(\tilde{\exists}_x A \to B) \to \forall_x (A \to B)$ *if* $x \notin \mathrm{FV}(B)$,

(2) $(\neg\neg B \to B) \to \quad \forall_x (A \to B) \to \tilde{\exists}_x A \to B$ *if* $x \notin \mathrm{FV}(B)$,

(3) $(\bot \to B[x{:=}c]) \to (A \to \tilde{\exists}_x B) \to \tilde{\exists}_x (A \to B)$ *if* $x \notin \mathrm{FV}(A)$,

(4) $\tilde{\exists}_x (A \to B) \to A \to \tilde{\exists}_x B$ *if* $x \notin \mathrm{FV}(A)$.

The last two items can also be seen as simplifying a weakly existentially quantified implication whose premise does not contain the quantified variable. In case the conclusion does not contain the quantified variable we have

(5) $(\neg\neg B \to B) \to \quad \tilde{\exists}_x (A \to B) \to \forall_x A \to B$ *if* $x \notin \mathrm{FV}(B)$,

(6) $\forall_x (\neg\neg A \to A) \to (\forall_x A \to B) \to \tilde{\exists}_x (A \to B)$ *if* $x \notin \mathrm{FV}(B)$.

Proof. (1)

$$
\begin{prooftree}
\AxiomC{$\tilde{\exists}_x A \to B$}
\AxiomC{$u_1 \colon \forall_x \neg A$}
\AxiomC{x}
\BinaryInfC{$\neg A$}
\AxiomC{A}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_1$}
\UnaryInfC{$\neg \forall_x \neg A$}
\BinaryInfC{B}
\end{prooftree}
$$

(2)

$$
\begin{prooftree}
\AxiomC{$\neg\neg B \to B$}
\AxiomC{$\neg \forall_x \neg A$}
\AxiomC{$u_2 \colon \neg B$}
\AxiomC{$\forall_x (A \to B)$}
\AxiomC{x}
\BinaryInfC{$A \to B$}
\AxiomC{$u_1 \colon A$}
\BinaryInfC{B}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_1$}
\UnaryInfC{$\neg A$}
\UnaryInfC{$\forall_x \neg A$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_2$}
\UnaryInfC{$\neg\neg B$}
\BinaryInfC{B}
\end{prooftree}
$$

(3) Writing B_0 for $B[x{:=}c]$ we have

$$
\begin{prooftree}
\AxiomC{$\forall_x \neg (A \to B)$}
\AxiomC{c}
\BinaryInfC{$\neg (A \to B_0)$}
\AxiomC{$\bot \to B_0$}
\AxiomC{$A \to \tilde{\exists}_x B$}
\AxiomC{$u_2 \colon A$}
\BinaryInfC{$\tilde{\exists}_x B$}
\AxiomC{$\forall_x \neg (A \to B)$}
\AxiomC{x}
\BinaryInfC{$\neg (A \to B)$}
\AxiomC{$u_1 \colon B$}
\UnaryInfC{$A \to B$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_1$}
\UnaryInfC{$\neg B$}
\UnaryInfC{$\forall_x \neg B$}
\BinaryInfC{$\bot$}
\BinaryInfC{B_0}
\RightLabel{$\to^+ u_2$}
\UnaryInfC{$A \to B_0$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

(4)

$$\dfrac{\tilde{\exists}_x(A\to B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg B \quad x}{\neg B} \qquad \dfrac{u_1\colon A\to B \quad A}{B}}{\bot}}{\neg(A\to B)}\to^+ u_1}{\forall_x\neg(A\to B)}}{\bot}$$

(5)

$$\dfrac{\neg\neg B\to B \qquad \dfrac{\dfrac{\tilde{\exists}_x(A\to B) \qquad \dfrac{\dfrac{\dfrac{u_2\colon\neg B \qquad \dfrac{u_1\colon A\to B \qquad \dfrac{\forall_x A \quad x}{A}}{B}}{\bot}}{\neg(A\to B)}\to^+u_1}{\forall_x\neg(A\to B)}}{\bot}}{\neg\neg B}\to^+u_2}{B}$$

(6) We derive $\forall_x(\bot\to A)\to(\forall_x A\to B)\to\forall_x\neg(A\to B)\to\neg\neg A$. Writing Ax, Ay for $A(x), A(y)$ we have

$$\dfrac{\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)} \qquad \dfrac{\dfrac{\forall_xAx\to B \qquad \dfrac{\dfrac{\dfrac{\forall_y(\bot\to Ay)\quad y}{\bot\to Ay} \qquad \dfrac{u_1\colon\neg Ax \quad u_2\colon Ax}{\bot}}{Ay}}{\forall_y Ay}}{B}}{Ax\to B}\to^+u_2}{\bot}}{\neg\neg Ax}\to^+u_1$$

Using this derivation M we obtain

$$\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)} \qquad \dfrac{\dfrac{\forall_xAx\to B \qquad \dfrac{\dfrac{\dfrac{\forall_x(\neg\neg Ax\to Ax)\quad x}{\neg\neg Ax\to Ax} \qquad \begin{array}{c}\mid M\\ \neg\neg Ax\end{array}}{Ax}}{\forall_xAx}}{B}}{Ax\to B}}{\bot}$$

Since clearly $\vdash(\neg\neg A\to A)\to\bot\to A$ the claim follows. $\square$

REMARK. An immediate consequence of (6) is the classical derivability of the "drinker formula" $\tilde{\exists}_x(Px\to\forall_x Px)$, to be read "in every non-empty bar there is a person such that, if this person drinks, then everybody drinks". To see this let $A := Px$ and $B := \forall_x Px$ in (6).

Corollary.

$$\begin{aligned}
&\vdash_c (\tilde{\exists}_x A \to B) \leftrightarrow \forall_x (A \to B) \quad \textit{if } x \notin \mathrm{FV}(B),\\
&\vdash_i (A \to \tilde{\exists}_x B) \leftrightarrow \tilde{\exists}_x (A \to B) \quad \textit{if } x \notin \mathrm{FV}(A),\\
&\vdash_c \tilde{\exists}_x (A \to B) \leftrightarrow (\forall_x A \to B) \quad \textit{if } x \notin \mathrm{FV}(B).
\end{aligned}$$

There is a similar lemma on weak disjunction:

Lemma. *The following are derivable.*

$$\begin{aligned}
&(A \mathbin{\tilde{\vee}} B \to C) \to (A \to C),\\
&(A \mathbin{\tilde{\vee}} B \to C) \to (B \to C),
\end{aligned} \tag{7}$$

$$(\neg\neg C \to C) \to (A \to C) \to (B \to C) \to A \mathbin{\tilde{\vee}} B \to C, \tag{8}$$

$$(\bot \to B) \to (A \to B \mathbin{\tilde{\vee}} C) \to (A \to B) \mathbin{\tilde{\vee}} (A \to C), \tag{9}$$

$$(A \to B) \mathbin{\tilde{\vee}} (A \to C) \to A \to B \mathbin{\tilde{\vee}} C, \tag{10}$$

$$(\neg\neg C \to C) \to (A \to C) \mathbin{\tilde{\vee}} (B \to C) \to A \to B \to C, \tag{11}$$

$$(\bot \to C) \to (A \to B \to C) \to (A \to C) \mathbin{\tilde{\vee}} (B \to C). \tag{12}$$

Proof. We only consider (8) and (12); the rest is left as an exercise.
(8)

$$\dfrac{\neg\neg C \to C \qquad \dfrac{\dfrac{\dfrac{\neg A \to \neg B \to \bot \qquad \dfrac{\dfrac{u_1\colon \neg C \qquad \dfrac{A \to C \quad u_2\colon A}{C}}{\bot}}{\neg A}\,\to^+ u_2}{\neg B \to \bot} \qquad \dfrac{\dfrac{u_1\colon \neg C \qquad \dfrac{B \to C \quad u_3\colon B}{C}}{\bot}}{\neg B}\,\to^+ u_3}{\bot}}{\neg\neg C}\,\to^+ u_1}{C}$$

(12)

$$\dfrac{\neg(B \to C) \qquad \dfrac{\dfrac{\bot \to C \qquad \dfrac{\neg(A \to C) \qquad \dfrac{\dfrac{\dfrac{A \to B \to C \quad u_1\colon A}{B \to C} \qquad u_2\colon B}{C}}{A \to C}\,\to^+ u_1}{\bot}}{C}}{B \to C}\,\to^+ u_2}{\bot}$$

The general idea here is to view $\tilde{\vee}$ as a finitary version of $\tilde{\exists}$. $\square$

COROLLARY.

$$\vdash (A \tilde{\vee} B \to C) \to (A \to C), \quad \vdash (A \tilde{\vee} B \to C) \to (B \to C),$$
$$\vdash_c (A \to C) \to (B \to C) \to A \tilde{\vee} B \to C,$$
$$\vdash_i (A \to B \tilde{\vee} C) \leftrightarrow (A \to B) \tilde{\vee} (A \to C),$$
$$\vdash_c (A \to C) \tilde{\vee} (B \to C) \leftrightarrow (A \to B \to C).$$

1.2. Derivation terms, normalization

A derivation in normal form does not make "detours", or more precisely, it cannot occur that an elimination rule immediately follows an introduction rule. We use "conversions" to remove such "local maxima" of complexity, thus reducing any given derivation to normal form.

1.2.1. The Curry-Howard correspondence. Since natural deduction derivations can be notationally cumbersome, it will be convenient to represent them as typed "derivation terms", where the derived formula is the "type" of the term (and displayed as a superscript). This representation goes under the name of *Curry-Howard correspondence*. It dates back to Curry (1930) and somewhat later Howard, published only in (1980), who noted that the types of the combinators used in combinatory logic are exactly the Hilbert style axioms for minimal propositional logic. Subsequently Martin-Löf (1972) transferred these ideas to a natural deduction setting where natural deduction proofs of formulas A now correspond exactly to lambda terms with type A. This representation of natural deduction proofs will henceforth be used consistently.

We give an inductive definition of such derivation terms for the $\to, \forall$-rules in Table 1 where for clarity we have written the corresponding derivations to the left. This can be extended to the rules for $\exists$, $\vee$ and $\wedge$, but we will not do this here. The reason is that these connectives will be viewed as inductively defined (nullary) predicates with parameters.

Every derivation term carries a formula as its type. However, we shall usually leave these formulas implicit and write derivation terms without them.

Every derivation term can be written uniquely in one of the forms

$$u\vec{M} \mid \lambda_v M \mid (\lambda_v M)N\vec{L},$$

where u is an assumption variable or constant, v is an assumption or object variable, and M, N, L are derivation or object terms.

1.2.2. Normalization. A *conversion* removes an elimination immediately following an introduction. We consider the following conversions, for

Derivation	Term
$u\colon A$	u^A
$\dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{A\to B}\to^+u$	$(\lambda_{u^A} M^B)^{A\to B}$
$\dfrac{\begin{matrix}\mid M\\ A\to B\end{matrix}\qquad\begin{matrix}\mid N\\ A\end{matrix}}{B}\to^-$	$(M^{A\to B}N^A)^B$
$\dfrac{\begin{matrix}\mid M\\ A\end{matrix}}{\forall_x A}\forall^+x$ (with var.cond.)	$(\lambda_x M^A)^{\forall_x A}$ (with var.cond.)
$\dfrac{\begin{matrix}\mid M\\ \forall_x A(x)\end{matrix}\qquad t}{A(t)}\forall^-$	$(M^{\forall_x A(x)}t)^{A(t)}$

TABLE 1. Derivation terms for $\to$ and $\forall$

derivations written in tree notation and also as derivation terms. A *conversion* removes an elimination immediately following an introduction. We consider the following conversions, for derivations written in tree notation and also as derivation terms.

$\to$-conversion.

$$\dfrac{\dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{A\to B}\to^+u \qquad \begin{matrix}\mid N\\ A\end{matrix}}{B}\to^- \quad\mapsto_\beta\quad \begin{matrix}\mid N\\ A\\ \mid M\\ B\end{matrix}$$

or written as derivation terms

$$(\lambda_u M(u^A)^B)^{A \to B} N^A \mapsto_\beta M(N^A)^B.$$

The reader familiar with λ-calculus should note that this is nothing other than β-conversion.

$\forall$-*conversion.*

$$\dfrac{\dfrac{\begin{matrix} \mid M \\ A(x) \end{matrix}}{\forall_x A(x)}\,\forall^+ x \qquad t}{A(t)}\,\forall^- \qquad \mapsto_\beta \qquad \begin{matrix} \mid M' \\ A(t) \end{matrix}$$

or written as derivation terms

$$(\lambda_x M(x)^{A(x)})^{\forall_x A(x)} t \mapsto_\beta M(t).$$

The *closure* of the conversion relation $\mapsto_\beta$ is defined by

(a) If $M \mapsto_\beta M'$, then $M \mapsto M'$.
(b) If $M \mapsto M'$, then also $MN \mapsto M'N$, $NM \mapsto NM'$, $\lambda_v M \mapsto \lambda_v M'$ (*inner reductions*).

Therefore $M \mapsto N$ means that M *reduces in one step to* N, i.e., N is obtained from M by replacement of (an occurrence of) a redex M' of M by a conversum M'' of M', i.e., by a single conversion.

A term M is *in normal form*, or M is *normal*, if M does not contain a redex. A *reduction sequence* is a (finite or infinite) sequence $M_0 \mapsto M_1 \mapsto M_2 \dots$ such that $M_i \mapsto M_{i+1}$, for all i. Finite reduction sequences are partially ordered under the initial part relation; the collection of finite reduction sequences starting from a term M forms a tree, the *reduction tree* of M. The branches of this tree may be identified with the collection of all infinite and all terminating finite reduction sequences. A term is *strongly normalizing* if its reduction tree is finite.

THEOREM 1.2.1. *Every derivation term is strongly normalizing, and the final element of each reduction sequence is uniquely determined.*

For time reasons the proof is omitted. It can be found for instance in Troelstra and van Dalen (1988).