CHAPTER 4

Computational content of proofs

Proofs have two aspects: they provide insight into why an argument is correct, and they can also have computational content. The Brouwer-Heyting-Kolmogorov interpretation (BHK-interpretation for short) gives a good analysis of the latter. Here we take the point of view that computational content *only* arises from inductive and coinductive predicates, and modify the original BHK-interpretation accordingly. For simplicity in this introductory remarks we only consider inductive predicates.

A formula can be seen as a problem, and its proof as providing a solution to this problem. The clauses of the BHK-interpretation are:

(i) p proves $A \to B$ if and only if p is a construction transforming any proof q of A into a proof $p(q)$ of B.
(ii) $\bot$ is a proposition without proof.
(iii) p proves $\forall_x A$ if and only if p is a construction such that p proves A, irrespective of what x is.

The problem with the BHK-interpretation clearly is its reliance on some unexplained notions, in particular

what is a "construction"?

what is a proof of a prime formula?

Here we propose to take

construction := computable functional,

proof of a prime formula $I\vec{r}$:= a "generation tree" for $I\vec{r}$.

Consider Even defined by $0 \in \text{Even}$ and $\forall_n(n \in \text{Even} \to S(Sn) \in \text{Even})$. Then a generation tree for $6 \in \text{Even}$ should consist of a single branch with nodes $0 \in \text{Even}$, $2 \in \text{Even}$, $4 \in \text{Even}$ and $6 \in \text{Even}$. More formally, such a generation tree can seen as an ideal in a certain algebra ι_I associated naturally with I.

In addition to such prime formulas with generation trees we will also provide "non-computational" (n.c.) variants of them, which state the same thing but do not give access to the generation tree. We will do this by "decorating" the least-fixed-point operator μ used to introduce the inductive predicate and write μ^{nc} instead. Accordingly we keep the previous development of the theory by viewing the inductive (and coinductive) predicates involved as n.c. variants.

The next thing to do is to properly accomodate the BHK-interpretation and define what it means that a term t "realizes" the formula A, written $t\ \mathbf{r}\ A$. In the prime formula case $I\vec{r}$ this will involve a predicate "t realizes $I\vec{r}$", which will be defined inductively as well, following the clauses of I. But since this is a "meta" statement already containing the term t representing a generation tree, we are not interested in the generation tree for such realizing formulas and consider them as non-computational.

Finally we will define in Section 4.2.3 the "extracted term" $\mathrm{et}(M)$ of a proof M of a formula A. This is a term in T^+ incorporating the computational content of the proof M. In Section 4.3 we will then prove the important soundness theorem $\mathrm{et}(M)\ \mathbf{r}\ A$.

4.1. Decoration

4.1.1. Decorated predicates and formulas. We introduce decorated least-fixed-point operators μ^{c}, μ^{nc} (and greatest-fixed-point operators ν^{c}, ν^{nc}). Moreover we distinguish two sorts of predicate variables, computationally relevant ones written $X, Y, Z \dots$ and non-computational ones written X^{nc}, Y^{nc}, Z^{nc} .... Then we can define *decorated predicates and formulas* by essentially the same definition as in Section 3.1, provided we take both X and X^{nc} as initial decorated predicate forms. For readability we usually write μ, ν for μ^{c}, ν^{c}.

For simplicity we postpone the treatment of ν. Therefore for every c.r. inductive predicate I defined as $\mu_X \vec{K}$ we have a non-computational variant I^{nc} defined as $\mu^{\mathrm{nc}}_X \vec{K}$.

Examples. For the even numbers we have two variants:

$$\mathrm{Even} := \mu_X(0 \in X, \forall_n(n \in X \to S(Sn) \in X)),$$
$$\mathrm{Even}^{\mathrm{nc}} := \mu^{\mathrm{nc}}_X(0 \in X, \forall_n(n \in X \to S(Sn) \in X)).$$

Similarly for Ex_Y, $\mathrm{Cap}_{Y,Z}$ and $\mathrm{Cup}_{Y,Z}$ we have two variants, for instance

$$\mathrm{Ex}_Y := \mu_X(\forall_x(x \in Y \to X)),$$
$$\mathrm{Ex}^{\mathrm{nc}}_Y := \mu^{\mathrm{nc}}_X(\forall_x(x \in Y \to X)).$$

To every predicate or formula C we assign its *final predicate* $\mathrm{fp}(C)$ by

$$\mathrm{fp}(X) := X, \quad \mathrm{fp}(X^{\mathrm{nc}}) := X^{\mathrm{nc}} \qquad \mathrm{fp}(P\vec{r}) := \mathrm{fp}(P)$$
$$\mathrm{fp}(\{\, \vec{x} \mid A \,\}) := \mathrm{fp}(A) \qquad \mathrm{fp}(A \to B) := \mathrm{fp}(B)$$
$$\mathrm{fp}(I) := I, \quad \mathrm{fp}(I^{\mathrm{nc}}) := I^{\mathrm{nc}} \qquad \mathrm{fp}(\forall_x A) := \mathrm{fp}(A)$$

We call a predicate or formula C *non-computational* (n.c., or *Harrop*) if its final predicate $\mathrm{fp}(C)$ is of the form X^{nc} or I^{nc}. The other predicates and formulas are called *computationally relevant* (c.r.).

Similarly we assign to every predicate or formula C its *non-computational variant* C^{nc}: we already have X^{nc} and I^{nc}, and in the other cases let

$$\begin{aligned}
\{\, \vec{x} \mid A \,\}^{\mathrm{nc}} &:= \{\, \vec{x} \mid A^{\mathrm{nc}} \,\} \\
(P\vec{r})^{\mathrm{nc}} &:= P^{\mathrm{nc}}\vec{r} \\
(A \to B)^{\mathrm{nc}} &:= A \to B^{\mathrm{nc}} \\
(\forall_x A)^{\mathrm{nc}} &:= \forall_x A^{\mathrm{nc}}
\end{aligned}$$

Clearly each C^{nc} is non-computational in the sense above.

Since both μ and parameter predicate variables can be chosen as either n.c. or c.r. we obtain variants of inductive predicates. For intersection we only consider the nullary case (i.e., conjunction). Then

$$\begin{aligned}
\mathrm{CapD}_{Y,Z} &:= \mu_X(Y \to Z \to X), \\
\mathrm{CapL}_{Y,Z^{\mathrm{nc}}} &:= \mu_X(Y \to Z^{\mathrm{nc}} \to X), \\
\mathrm{CapR}_{Y^{\mathrm{nc}},Z} &:= \mu_X(Y^{\mathrm{nc}} \to Z \to X), \\
\mathrm{CapNc}_{Y,Z} &:= \mu_X^{\mathrm{nc}}(Y \to Z \to X).
\end{aligned}$$

Here D is for “double”, L for “left”, R for “right”. We use the abbreviations

$$\begin{aligned}
A \wedge^{\mathrm{d}} B &:= \mathrm{CapD}_{\{|A\},\{|B\}}, \\
A \wedge^{\mathrm{l}} B &:= \mathrm{CapL}_{\{|A\},\{|B\}}, \\
A \wedge^{\mathrm{r}} B &:= \mathrm{CapR}_{\{|A\},\{|B\}}, \\
A \wedge^{\mathrm{nc}} B &:= \mathrm{CapNc}_{\{|A\},\{|B\}}.
\end{aligned}$$

Since the decoration is determined by the c.r./n.c. status of its two parameter predicate variables we usually leave out the decoration and just write $\wedge$.

For union again we only consider the nullary case (i.e., disjunction). Then

$$\begin{aligned}
\mathrm{CupD}_{Y,Z} &:= \mu_X(Y \to X,\ Z \to X), \\
\mathrm{CupL}_{Y,Z^{\mathrm{nc}}} &:= \mu_X(Y \to X,\ Z^{\mathrm{nc}} \to X), \\
\mathrm{CupR}_{Y^{\mathrm{nc}},Z} &:= \mu_X(Y^{\mathrm{nc}} \to X,\ Z \to X), \\
\mathrm{CupU}_{Y^{\mathrm{nc}},Z^{\mathrm{nc}}} &:= \mu_X(Y^{\mathrm{nc}} \to X,\ Z^{\mathrm{nc}} \to X), \\
\mathrm{CupNc}_{Y,Z} &:= \mu_X^{\mathrm{nc}}(Y \to X,\ Z \to X).
\end{aligned}$$

Here U stands for “uniform”. We use the abbreviations

$$\begin{aligned}
A \vee^{\mathrm{d}} B &:= \mathrm{CupD}_{\{|A\},\{|B\}}, \\
A \vee^{\mathrm{l}} B &:= \mathrm{CupL}_{\{|A\},\{|B\}},
\end{aligned}$$

$$A \vee^{\mathrm{r}} B \;:= \mathrm{CupR}_{\{|A\},\{|B\}},$$
$$A \vee^{\mathrm{u}} B \;:= \mathrm{CupU}_{\{|A\},\{|B\}},$$
$$A \vee^{\mathrm{nc}} B := \mathrm{CupNc}_{\{|A\},\{|B\}}.$$

Again we usually leave out the decoration in $\vee^{\mathrm{d}}, \vee^{\mathrm{l}}, \vee^{\mathrm{r}}, \vee^{\mathrm{u}}$ defined with μ. However in the final nc-variant we suppress even the information which clause has been used, and hence must keep the notation $\vee^{\mathrm{nc}}$.

For Leibniz equality we from now on take the definition

$$\mathrm{EqD} := \mu_X^{\mathrm{nc}}(\forall_x Xxx).$$

4.1.2. Logic with decorations. For an n.c. inductive predicate I^{nc} the introduction axioms $(I^{\mathrm{nc}})_i^+$ and the elimination axiom $(I^{\mathrm{nc}})^-$ are formed similar to I_i^+ and I^-, but with an essential restriction: the elimination axiom $(I^{\mathrm{nc}})^-$ can only be used with X substituted by a *non-computational* competitor predicate. This is needed in the proof of the soundness theorem.

However, there is an important exception: in the special case of a *one-clause-nc* definition I^{nc} (i.e., defined with μ^{nc} and involving one clause) there are *no* restrictions on the elimination axiom. This is the case for Leibniz equality $\equiv$, and the non-computational variants of the existential quantifier and of conjunction.

Using the convention above on suppressing decorations clear from the context we can write the introduction and elimination axioms for $\exists$, $\wedge$ and $\vee$ as before. An exception is $(\vee^{\mathrm{nc}})^-$, where we need to restrict the conclusion C to n.c. formulas:

$$(\vee^{\mathrm{nc}})^- \colon A \vee^{\mathrm{nc}} B \to (A \to C) \to (B \to C) \to C \quad \text{for } C \text{ n.c.}$$

4.2. Realizers

We now view a formula A as a "computational problem", as done by Kolmogorov (1932). Then what should be the solution to the problem posed by the formula $I\vec{r}$, where I is inductively defined? The obvious idea here is to take the generation tree described before, witnessing how the arguments $\vec{r}$ were put into I.

When we want to generally define this concept of a generation tree, it seems natural to let the clauses of I determine the algebra to which such trees belong. Hence we will define ι_I to be the type $\mu_\xi(\kappa_0, \dots, \kappa_{k-1})$ generated from constructor types $\kappa_i := \tau(K_i)$, where K_i is the i-th clause of the inductive definition of I as $\mu_X(K_0, \dots K_{k-1})$, and $\tau(K_i)$ is the type of the clause K_i, relative to $\tau(X\vec{r}) := \xi$.

We begin with the definition of the type $\tau(A)$ of a formula A, the type of the solution to the problem posed by this formula. Then we define what it means for an x of type $\tau(A)$ to be a "realizer" (i.e., a solution) of the formula

A. Next we assign to any derivation M of a formula A its "extracted term" $\mathrm{et}(M)$, which should be the realizer of A provided by the proof M. Finally we prove the soundness theorem, saying that this indeed is the case.

4.2.1. Types of predicates and formulas. We refine the distinction between computationally relevant (c.r.) and non-computational (n.c.) predicates and formulas given in Section 4.1.1 by providing a type in the former case.

Definition (Type $\tau(C)$ of a c.r. predicate or formula C). Assume a global injective assignment of type variables ξ to c.r. predicate variables X.

$$\tau(X) := \xi,$$
$$\tau(\{\,\vec{x} \mid A\,\}) := \tau(A),$$
$$\tau(\underbrace{\mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k}}_{I}) := \underbrace{\mu_\xi((\tau(A_{i\nu})_{\nu<n_i}) \to \xi)_{i<k}}_{\iota_I}.$$
$$\tau(P\vec{r}\,) := \tau(P),$$
$$\tau(A \to B) := \begin{cases} \tau(A) \to \tau(B) & \text{if } A \text{ is c.r.} \\ \tau(B) & \text{if } A \text{ is n.c.} \end{cases}$$
$$\tau(\forall_x A) := \tau(A),$$

where $\tau(A_{i\nu})$ is missing in case $A_{i\nu}$ is n.c. We call ι_I the *algebra associated with* I. We call a predicate or formula C *finitary* if its type $\tau(C)$ is a finitary algebra.

4.2.2. Realizability. Assume that we have a global assignment giving for every (c.r.) predicate variable X of arity $\vec{\rho}$ an n.c. predicate variable $X^{\mathbf{r}}$ of arity $(\tau(X), \vec{\rho}\,)$. To ensure that realizing objects are always total (more precisely, in E^{nc}) we require as an axiom

$$\forall_{z,x}(X^{\mathbf{r}}zx \to z \in E^{\mathrm{nc}}).$$

Definition ($C^{\mathbf{r}}$ for predicates and formulas C). For every predicate or formula C we define an n.c. predicate or formula $C^{\mathbf{r}}$. For n.c. C let $C^{\mathbf{r}} := C$. In case C is c.r. $C^{\mathbf{r}}$ is a predicate of arity $(\tau(C), \vec{\sigma}\,)$ with $\vec{\sigma}$ the arity of C. We write $z \mathrel{\mathbf{r}} C$ for $C^{\mathbf{r}}z$ in case C is a c.r. formula. For c.r. *predicates* let $X^{\mathbf{r}}$ be the n.c. predicate variable provided, and

$$\{\,\vec{x} \mid A\,\}^{\mathbf{r}} := \{\, z, \vec{x} \mid z \mathrel{\mathbf{r}} A\,\}.$$

For a c.r. inductive predicate

$$I := \mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k}$$

we define the witnessing predicate $I^{\mathbf{r}}$ by

$$I^{\mathbf{r}} := \mu^{\mathrm{nc}}_{X^{\mathbf{r}}}(\forall_{\vec{x}_i,\vec{z}_i}((z_{i\nu} \mathrel{\mathbf{r}} A_{i\nu})_{\nu<n_i} \to \mathrm{C}_i\vec{z}_i \mathrel{\mathbf{r}} X\vec{r}_i))_{i<k}$$

where $z_{i\nu}$ is missing and we have $A_{i\nu}$ instead of $z_{i\nu}$ $\mathbf{r}$ $A_{i\nu}$ in case $A_{i\nu}$ is n.c. Here C_i is the i-th constructor of the algebra ι_I generated from the constructor types $\tau(K_i)$ with K_i the i-th clause of I. If I has a c.r. parameter predicate Y, then both Y and $Y^{\mathbf{r}}$ may appear in $I^{\mathbf{r}}$. For c.r. *formulas* let

$$z \mathbf{r}\ P\vec{r} := P^{\mathbf{r}}(z, \vec{r}),$$

$$z \mathbf{r}\ (A \to B) := z \in E^{\mathrm{nc}} \wedge \begin{cases} \forall_w (w \mathbf{r}\ A \to zw \mathbf{r}\ B) & \text{if } A \text{ is c.r.} \\ A \to z \mathbf{r}\ B & \text{if } A \text{ is n.c.} \end{cases}$$

$$z \mathbf{r}\ \forall_x A := \forall_x (z \mathbf{r}\ A).$$

Example. For the even numbers

$$\mathrm{Even} := \mu_X(0 \in X, \forall_n (n \in X \to S(Sn) \in X))$$

we obtain the associated algebra $\iota_{\mathrm{Even}} = \mathbf{N}$ and

$$\mathrm{Even}^{\mathbf{r}} := \mu^{\mathrm{nc}}_{X^{\mathbf{r}}}(X^{\mathbf{r}}00, \forall_{n,m}(X^{\mathbf{r}}mn \to X^{\mathbf{r}}(Sm, S(Sn)))).$$

The introduction axioms are

$$(\mathrm{Even}^{\mathbf{r}})^+_0 \colon 0 \mathbf{r}\ (0 \in \mathrm{Even}),$$

$$(\mathrm{Even}^{\mathbf{r}})^+_1 \colon \forall_{n,m}(m \mathbf{r}\ (n \in \mathrm{Even}) \to Sm \mathbf{r}\ (S(Sn) \in \mathrm{Even})).$$

Recall that the elimination axiom for an n.c. inductive predicate I^{nc} can only be used with non-computational competitor predicates (except one-clause-nc inductive predicates). Hence the elimination axiom is

$$(\mathrm{Even}^{\mathbf{r}})^- \colon \forall_{n,m}(m \mathbf{r}\ (n \in \mathrm{Even}) \to X^{\mathrm{nc}}00 \to \forall_{n,m}(m \mathbf{r}\ (n \in \mathrm{Even}) \to X^{\mathrm{nc}}mn \to X^{\mathrm{nc}}(Sm, S(Sn))) \to X^{\mathrm{nc}}mn).$$

For inductive predicates – for instance Even – we can prove that its realizers are total:

Lemma.

$$\forall_m (m \mathbf{r}\ (n \in \mathrm{Even}) \to m \in E^{\mathrm{nc}}_{\mathbf{N}}).$$

Proof. Exercise. Use $(\mathrm{Even}^{\mathbf{r}})^-$ with $X^{\mathrm{nc}} := \{\, m, n \mid m \in E^{\mathrm{nc}}_{\mathbf{N}} \,\}$ as competitor predicate. □

As a consequence we can show that quite generally realizers are total.

Lemma (Totality of realizers). *For c.r. formulas A we have*

$$\forall_z (z \mathbf{r}\ A \to z \in E^{\mathrm{nc}}_{\tau(A)}).$$

Proof. By induction on A. In case of a c.r. predicate variable X use the axiom $\forall_{z,x}(X^{\mathbf{r}}zx \to z \in E^{\mathrm{nc}})$ above, and in case of an inductive predicate the last lemma. The rest is left as an exercise. □

Remark. Similarly to the lemma on Even above we can prove

$$\forall_m (m \mathbf{r} (n \in E_{\mathbf{N}}) \to m \doteq n).$$

Next we study what our general definition says about realizers for the c.r. inductively defined decorated connectives.

Lemma (Realizers for $\wedge$).

$$\begin{aligned} \langle z, w \rangle \mathbf{r} (A \wedge B) &\leftrightarrow (z \mathbf{r} A) \wedge (w \mathbf{r} B) && \textit{for } A \textit{ c.r. and } B \textit{ c.r.} \\ z \mathbf{r} (A \wedge B) &\leftrightarrow (z \mathbf{r} A) \wedge B && \textit{for } A \textit{ c.r. and } B \textit{ n.c.} \\ w \mathbf{r} (A \wedge B) &\leftrightarrow A \wedge (w \mathbf{r} B) && \textit{for } A \textit{ n.c. and } B \textit{ c.r.} \end{aligned}$$

Proof. Exercise. □

Recall that for the sum type $\rho + \sigma$ we had the constructors $(\mathrm{Inl}_{\rho\sigma})^{\rho \to \rho + \sigma}$ and $(\mathrm{Inr}_{\rho\sigma})^{\sigma \to \rho + \sigma}$. In the special situation that one of the two parameter types is the unit type $\mathbf{U}$ it is common to view the sum type $\mathbf{U} + \sigma$ as a unary algebra form, with constructors DummyL of type $\mathbf{U} + \sigma$ and InrUysum of type $\sigma \to \mathbf{U} + \sigma$. Similarly $\rho + \mathbf{U}$ is viewed as a unary algebra form, with constructors InlYsumu of type $\rho \to \rho + \mathbf{U}$ and DummyR of type $\rho + \mathbf{U}$.

Lemma (Realizers for $\vee$).

$$\left.\begin{aligned} \mathrm{Inl}(z) \mathbf{r} (A \vee B) &\leftrightarrow z \mathbf{r} A \\ \mathrm{Inr}(w) \mathbf{r} (A \vee B) &\leftrightarrow w \mathbf{r} B \end{aligned}\right\} \quad \textit{for } A \textit{ c.r. and } B \textit{ c.r.}$$

$$\left.\begin{aligned} \mathrm{InlYsumu}(z) \mathbf{r} (A \vee B) &\leftrightarrow z \mathbf{r} A \\ \mathrm{DummyR} \mathbf{r} (A \vee B) &\leftrightarrow B \end{aligned}\right\} \quad \textit{for } A \textit{ c.r. and } B \textit{ n.c.}$$

$$\left.\begin{aligned} \mathrm{DummyL} \mathbf{r} (A \vee B) &\leftrightarrow A \\ \mathrm{InrUysum}(w) \mathbf{r} (A \vee B) &\leftrightarrow w \mathbf{r} B \end{aligned}\right\} \quad \textit{for } A \textit{ n.c. and } B \textit{ c.r.}$$

$$\left.\begin{aligned} \mathtt{tt} \mathbf{r} (A \vee B) &\leftrightarrow A \\ \mathtt{ff} \mathbf{r} (A \vee B) &\leftrightarrow B \end{aligned}\right\} \quad \textit{for } A \textit{ n.c. and } B \textit{ n.c.}$$

Proof. Exercise. Use the elimination axioms, for C n.c.:

$$\begin{aligned} &u \mathbf{r} (A \vee B) \to \forall_z (z \mathbf{r} A \to C) \to \forall_w (w \mathbf{r} B \to C) \to C \quad \text{for } A, B \text{ c.r.} \\ &u \mathbf{r} (A \vee B) \to \forall_z (z \mathbf{r} A \to C) \to (B \to C) \to C \quad \text{for } A \text{ c.r. and } B \text{ n.c.} \\ &u \mathbf{r} (A \vee B) \to (A \to C) \to \forall_w (w \mathbf{r} B \to C) \to C \quad \text{for } A \text{ n.c. and } B \text{ c.r.} \\ &u \mathbf{r} (A \vee B) \to (A \to C) \to (B \to C) \to C \quad \text{for } A \text{ n.c. and } B \text{ n.c.} \end{aligned}$$

□

We can now state the *invariance axioms* Inv_A and prove that identities realize them.

Axiom (Invariance under realizability). *For c.r. formulas A with $\tau(A)$ a finitary algebra – say $\mathbf{N}$ – we provide as an axiom*

$$\text{Inv}_A\colon A \leftrightarrow \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A). \tag{16}$$

Lemma. *For c.r. formulas A with $\tau(A)$ a finitary algebra – say $\mathbf{N}$ – we have*

$$(\lambda_w w)\ \mathbf{r}\ (A \to \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A),$$
$$(\lambda_w w)\ \mathbf{r}\ (\exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A) \to A).$$

Proof. Unfolding the definitions we obtain

$$(\lambda_w w)\ \mathbf{r}\ (A \to \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A))$$
$$\forall_w(w\ \mathbf{r}\ A \to w\ \mathbf{r}\ \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A))$$

and similarly

$$(\lambda_w w)\ \mathbf{r}\ (\exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A) \to A)$$
$$\forall_w(w\ \mathbf{r}\ \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A) \to w\ \mathbf{r}\ A).$$

But $w\ \mathbf{r}\ \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A)$ is $\exists_z(w\ \mathbf{r}\ (z \in E_{\mathbf{N}}) \wedge z\ \mathbf{r}\ A)$. Since by a remark above $w\ \mathbf{r}\ (z \in E_{\mathbf{N}})$ implies $w \doteq z$, and by a remark in Section 3.2 $\doteq$, $\equiv$ and $=$ are equivalent on $E_{\mathbf{N}}$, the latter formula is equivalent to $w\ \mathbf{r}\ A$. □

4.2.3. Extracted terms. For a derivation M of a c.r. formula A we define its *extracted term* $\text{et}(M)$, of type $\tau(A)$. It will be a term in our term language of Section 2.2. This definition is relative to a fixed assignment of object variables to assumption variables: to every assumption variable u^A for a c.r. formula A we assign an object variable z_u of type $\tau(A)$.

Definition (Extracted term $\text{et}(M)$ of a derivation M^A with A c.r.).

$$\begin{aligned}
\text{et}(u^A) &:= z_u^{\tau(A)} \quad (z_u^{\tau(A)} \text{ uniquely associated to } u^A),\\
\text{et}((\lambda_{u^A} M^B)^{A\to B}) &:= \begin{cases} \lambda_{z_u}^{\tau(A)}\text{et}(M) & \text{if } A \text{ is c.r.}\\ \text{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\text{et}((M^{A\to B}N^A)^B) &:= \begin{cases} \text{et}(M)\text{et}(N) & \text{if } A \text{ is c.r.}\\ \text{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\text{et}((\lambda_x M^A)^{\forall_x A}) &:= \text{et}(M),\\
\text{et}((M^{\forall_x A(x)} r)^{A(r)}) &:= \text{et}(M).
\end{aligned}$$

It remains to define extracted terms for the axioms. Consider a (c.r.) inductively defined predicate I. For its introduction and elimination axioms define $\text{et}(I_i^+) := \text{C}_i$ and $\text{et}(I^-) := \mathcal{R}$, where both the constructor C_i and the recursion operator $\mathcal{R}$ refer to the algebra ι_I associated with I.

In the next section we will prove the soundness theorem, which states that the term $\mathrm{et}(M)$ extracted from a proof M of a formula A is a realizer of A. Let us first see that $\mathrm{et}(M)$ is always total, provided the object variables z_u assigend to assumption variables u are total.

LEMMA (Totality of extracted terms). *For every term t with free variables and constants in E^{nc} we have $t \in E^{\mathrm{nc}}$.*

PROOF. We first show that for every term $t(\vec{x}\,)$ with free variables among $\vec{x}$ we have

$$\forall_{\vec{x},\vec{y}}(\vec{x} \doteq \vec{y} \to t(\vec{x}\,) \doteq t(\vec{y}\,)). \tag{17}$$

This is proved by induction on t. *Case* x. Immediate. *Case* $\lambda_x t(x, \vec{x}\,)$. Let $\vec{x} \doteq \vec{y}$. The goal is $\lambda_x t(x, \vec{x}\,) \doteq \lambda_x t(x, \vec{y}\,)$, which by definition means

$$\lambda_x t(x, \vec{x}\,), \lambda_x t(x, \vec{y}\,) \in E^{\mathrm{nc}},$$
$$\forall_x (x \in E^{\mathrm{nc}} \to t(x, \vec{x}\,) \doteq t(x, \vec{y}\,)).$$

For the first part we only show $\lambda_x t(x, \vec{x}\,) \in E^{\mathrm{nc}}$. By a lemma in Section 3.2 it suffices to prove $x \doteq y \to t(x, \vec{x}\,) \doteq t(y, \vec{x}\,)$. Since from $\vec{x} \doteq \vec{y}$ we can infer $\vec{x} \doteq \vec{x}$ the claim follows from the induction hypothesis. For the second part assume $x \in E^{\mathrm{nc}}$. The goal is $t(x, \vec{x}\,) \doteq t(x, \vec{y}\,)$. Since from $x \in E^{\mathrm{nc}}$ we can infer $x \doteq x$ the claim again follows from the induction hypothesis. *Case* ts. Use compatibility of application with $\doteq$ and the induction hypothesis.

Now from (17) we obtain the lemma as follows. Let t with free variables and constants in E^{nc} be given. The goal is $t \in E^{\mathrm{nc}}$. By (17) applied to the free variables and constants of t we obtain $t \doteq t$. Hence $t \in E^{\mathrm{nc}}$ by a lemma in Section 3.2. □

4.3. Soundness

We prove that the term extracted from a proof in $\mathrm{TCF} + \mathrm{Inv} + \mathrm{Ax}^{\mathrm{nc}}$ is a solution of the problem posed by the proven formula. Here $\mathrm{Ax}^{\mathrm{nc}}$ is an arbitrary set of non-computational formulas viewed as axioms.

Before doing anything general let us first look at an example. Totality for $\mathbf{N}$ has been inductively defined by the clauses

$$0 \in E_{\mathbf{N}}, \qquad \forall_n (n \in E_{\mathbf{N}} \to Sn \in E_{\mathbf{N}}).$$

Its elimination axiom is

$$\forall_n (n \in E_{\mathbf{N}} \to 0 \in X \to \forall_n (n \in E_{\mathbf{N}} \to n \in X \to Sn \in X) \to n \in X).$$

We show that their extracted terms 0, S and $\mathcal{R}$ are realizers. First we prove $\mathcal{R}_{\mathbf{N}} \in E^{\mathrm{nc}}$.

LEMMA. *$\mathcal{R}^{\tau}_{\mathbf{N}}$ of type $\mathbf{N} \to \tau \to (\mathbf{N} \to \tau \to \tau) \to \tau$ is in E^{nc}.*

PROOF. It suffices to prove

$$\forall_{n,n'}(n \doteq n' \to \mathcal{R}n \doteq \mathcal{R}n').$$

Since from $n \doteq n'$ we can infer $n \equiv n'$ the goal can be written as $\mathcal{R}n \doteq \mathcal{R}n$. By reflexivity of $\doteq$ it suffices to prove $\mathcal{R}n \in E^{\mathrm{nc}}$. Since from $n \doteq n'$ we can also infer $n \in E^{\mathrm{nc}}$, this proof can be done by induction on n.

Base. Again it suffices to prove

$$\forall_{x,x'}(x \doteq x' \to \mathcal{R}0x \doteq \mathcal{R}0x').$$

Assume $x \doteq x'$. By definition the goal is equivalent to (i) $\mathcal{R}0x, \mathcal{R}0x' \in E^{\mathrm{nc}}$ and (ii) $\forall_f(f \in E^{\mathrm{nc}} \to \mathcal{R}0xf \doteq \mathcal{R}0x'f)$. The latter follows from $x \doteq x'$ by the conversion rules for $\mathcal{R}$. For (i), by a lemma in Section 3.2 it suffices to prove $\forall_{f,f'}(f \doteq f' \to \mathcal{R}0xf \doteq \mathcal{R}0xf')$. But this again follows from the conversion rules for $\mathcal{R}$, since $x \doteq x$ is a consequence of the assumed $x \doteq x'$.

Step. Assume $\mathcal{R}n \in E^{\mathrm{nc}}$ or equivalently

$$\begin{aligned}
&\mathcal{R}n \doteq \mathcal{R}n\\
&\forall_{x,x'}(x \doteq x' \to \mathcal{R}nx \doteq \mathcal{R}nx')\\
&\forall_{x,x',f,f'}(x \doteq x' \to f \doteq f' \to \mathcal{R}nxf \doteq \mathcal{R}nx'f').
\end{aligned}$$

Similarly the goal $\mathcal{R}(Sn) \in E^{\mathrm{nc}}$ is equivalent to

$$\begin{aligned}
&\mathcal{R}(Sn) \doteq \mathcal{R}(Sn)\\
&\forall_{x,x'}(x \doteq x' \to \mathcal{R}(Sn)x \doteq \mathcal{R}(Sn)x')\\
&\forall_{x,x',f,f'}(x \doteq x' \to f \doteq f' \to \mathcal{R}(Sn)xf \doteq \mathcal{R}(Sn)x'f').\\
&\forall_{x,x',f,f'}(x \doteq x' \to f \doteq f' \to fn(\mathcal{R}nxf) \doteq f'n(\mathcal{R}nx'f')).
\end{aligned}$$

Using compatibility of application with $\doteq$ this follows from the assumption. □

We now show that $\mathcal{R}_{\mathbf{N}}$ is a realizer of the elimination axiom $E^-_{\mathbf{N}}$:

$$\forall_n(n \in E_{\mathbf{N}} \to 0 \in X \to \forall_n(n \in E_{\mathbf{N}} \to n \in X \to Sn \in X) \to n \in X).$$

For the proof recall from the examples in Section 4.2.2 that the witnessing predicate $E^{\mathbf{r}}_{\mathbf{N}}$ is defined by the clauses

$$E^{\mathbf{r}}_{\mathbf{N}}00, \qquad \forall_{n,m}(E^{\mathbf{r}}_{\mathbf{N}}mn \to E^{\mathbf{r}}_{\mathbf{N}}(Sm, Sn)),$$

and it has as its elimination axiom

$$\begin{aligned}
\forall_{n,m}(E^{\mathbf{r}}_{\mathbf{N}}mn \to X^{\mathrm{nc}}00 \to\\
&\forall_{n,m}(E^{\mathbf{r}}_{\mathbf{N}}mn \to X^{\mathrm{nc}}mn \to X^{\mathrm{nc}}(Sm, Sn)) \to\\
&X^{\mathrm{nc}}mn).
\end{aligned}$$

LEMMA.

$$0 \mathbf{r} (0 \in E_{\mathbf{N}}) \quad \textit{and} \quad S \mathbf{r} \forall_n (n \in E_{\mathbf{N}} \to Sn \in E_{\mathbf{N}}).$$

$$\mathcal{R} \mathbf{r} \forall_n (n \in E_{\mathbf{N}} \to 0 \in X \to \forall_n (n \in E_{\mathbf{N}} \to n \in X \to Sn \in X) \to n \in X).$$

PROOF. $0 \mathbf{r} (0 \in E_{\mathbf{N}})$ is defined to be $E^{\mathbf{r}}_{\mathbf{N}}00$. Moreover, by definition $S \mathbf{r} \forall_n (n \in E_{\mathbf{N}} \to Sn \in E_{\mathbf{N}})$ unfolds into $\forall_{n,m}(E^{\mathbf{r}}_{\mathbf{N}}mn \to E^{\mathbf{r}}_{\mathbf{N}}(Sm, Sn))$.

Now let n, m be given and assume $m \mathbf{r} (n \in E_{\mathbf{N}})$. Let further $w_0, w_1 \in E^{\mathrm{nc}}$ be such that $w_0 \mathbf{r} (0 \in X)$ and $w_1 \mathbf{r} \forall_n (n \in E_{\mathbf{N}} \to n \in X \to Sn \in X)$, i.e.,

$$\forall_{n,m}(E^{\mathbf{r}}_{\mathbf{N}}mn \to \forall_z (z \mathbf{r} (n \in X) \to w_1 m z \mathbf{r} (Sn \in X))).$$

Our goal is

$$\mathcal{R} m w_0 w_1 \mathbf{r} (n \in X) =: Qmn.$$

To this end we use the elimination axiom for $E^{\mathbf{r}}_{\mathbf{N}}$ above. Hence it suffices to prove its premises $Q00$ and $\forall_{n,m}(E^{\mathbf{r}}_{\mathbf{N}}mn \to Qmn \to Q(Sm, Sn))$. By a conversion rule for $\mathcal{R}$ the former is the same as $w_0 \mathbf{r} (0 \in X)$, which we have. For the latter assume n, m and its premises. We show $Q(Sm, Sn)$, i.e., $\mathcal{R}(Sm) w_0 w_1 \mathbf{r} (Sn \in X)$. By a conversion rule for $\mathcal{R}$ this is the same as

$$w_1 m (\mathcal{R} m w_0 w_1) \mathbf{r} (Sn \in X).$$

But with $z := \mathcal{R} m w_0 w_1$ this follows from what we have. □

By an n.c. part of a derivation we mean a subderivation with an n.c. end formula. Such n.c. parts do not contribute to the computational content.

THEOREM (Soundness). *Let M be a derivation of a formula A from assumptions $u_i \colon C_i$ $(i < n)$. Assume that n.c. parts of M contains n.c. formulas only. Then we can derive*

$$\begin{cases} \mathrm{et}(M) \mathbf{r} A & \textit{if } A \textit{ is c.r.} \\ A & \textit{if } A \textit{ is n.c.} \end{cases}$$

from assumptions

$$\begin{cases} z_{u_i} \mathbf{r} C_i & \textit{if } C_i \textit{ is c.r.} \\ C_i & \textit{if } C_i \textit{ is n.c.} \end{cases}$$

If not stated otherwise, all derivations are in $\mathrm{TCF} + \mathrm{Inv} + \mathrm{Ax}^{\mathrm{nc}}$. The proof is by induction on M.

REMARK. The assumption in the Soundness Theorem can be somewhat weakened by allowing finitary c.r. formulas in n.c. parts. We will prove the theorem in this stronger form, using the invariance axioms.

Proof. *Case* $u\colon A$. *Subcase* A c.r. Then $\mathrm{et}(u) = z_u$. *Subcase* A n.c. Immediate.

Case $c\colon A$ for a c.r. axiom c. We postpone the treatment of this case to the end of the proof.

Case $(\lambda_{u^A} M^B)^{A\to B}$ with B c.r. We must derive $\mathrm{et}(\lambda_u M)\ \mathbf{r}\ (A \to B)$. By the lemmata on totality of realizers and of extracted terms we already know $\mathrm{et}(\lambda_u M) \in E^{\mathrm{nc}}$.

Subcase A c.r. Then the remaining goal is

$$\forall_z (z\ \mathbf{r}\ A \to \mathrm{et}(\lambda_u M) z\ \mathbf{r}\ B).$$

Recall that $\mathrm{et}(\lambda_u M) = \lambda_{z_u} \mathrm{et}(M)$. Renaming z into z_u, our goal is to prove

$$\forall_{z_u}(z_u\ \mathbf{r}\ A \ \to\ \mathrm{et}(M)\ \mathbf{r}\ B),$$

since we identify terms with the same β-normal form. But by induction hypothesis we have a derivation of $\mathrm{et}(M)\ \mathbf{r}\ B$ from $z_u\ \mathbf{r}\ A$. An $\to$ and $\forall$ introduction give the desired result.

Subcase A n.c. Then the remaining goal is

$$A \to \mathrm{et}(\lambda_u M)\ \mathbf{r}\ B.$$

Recall that $\mathrm{et}(\lambda_u M) = \mathrm{et}(M)$. By induction hypothesis we have a derivation of $\mathrm{et}(M)\ \mathbf{r}\ B$ from A, which is what we want.

Case $(\lambda_{u^A} M^B)^{A\to B}$ with B n.c. We need a derivation of $A \to B$. *Subcase* A c.r. Recall that by our assumption $\tau(A)$ is finitary, say $\mathbf{N}$. By induction hypothesis we have a derivation of B from $z\ \mathbf{r}\ A$. Using the invariance axiom $A \to \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A)$ we obtain the required derivation of B from A by

$$\dfrac{\dfrac{A \to \exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A) \qquad A}{\exists_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[z \in E_{\mathbf{N}} \wedge z\ \mathbf{r}\ A]}{z\ \mathbf{r}\ A}\wedge^- \qquad \dfrac{\begin{matrix}[z\ \mathbf{r}\ A]\\ \mid \mathrm{IH}\\ B\end{matrix}}{z\ \mathbf{r}\ A \to B}}{B}}{z \in E_{\mathbf{N}} \wedge z\ \mathbf{r}\ A \to B}}{\forall_z(z \in E_{\mathbf{N}} \wedge z\ \mathbf{r}\ A \to B)}}{B}\,\exists^-$$

Subcase A n.c. By induction hypothesis we have a derivation of B from A, which is what we want.

Case $(M^{A\to B} N^A)^B$ with B c.r. We need a derivation of $\mathrm{et}(MN)\ \mathbf{r}\ B$. *Subcase* A c.r. Then $\mathrm{et}(MN) = \mathrm{et}(M)\mathrm{et}(N)$. By induction hypothesis we have derivations of $\mathrm{et}(M)\ \mathbf{r}\ (A \to B)$ and hence of

$$\forall_z(z\ \mathbf{r}\ A \ \to\ \mathrm{et}(M) z\ \mathbf{r}\ B)$$

and of $\mathrm{et}(N)\ \mathbf{r}\ A$. Hence the claim follows. *Subcase* A n.c. Then $\mathrm{et}(MN) = \mathrm{et}(M)$. By induction hypothesis we have derivations of $\mathrm{et}(M)\ \mathbf{r}\ (A \to B)$

and hence of

$$A \to \mathrm{et}(M)\ \mathbf{r}\ B$$

and of A. Applying the former to the latter gives $\mathrm{et}(M)\ \mathbf{r}\ B$.

Case $(M^{A\to B}N^{A})^{B}$ with B n.c. We must find a derivation of B. *Subcase* A c.r. Recall again that by our assumption $\tau(A)$ is finitary, say $\mathbf{N}$. By induction hypothesis we have derivations of $A \to B$ and of $\mathrm{et}(N)\ \mathbf{r}\ A$. We also know $\mathrm{et}(N) \in E_{\mathbf{N}}$ by totality of extracted terms. Using the invariance axiom $\forall_{z\in E_{\mathbf{N}}}(z\ \mathbf{r}\ A \to A)$ we obtain the required derivation of B by

$$\dfrac{\begin{array}{c}\mid \text{IH}\\ A\to B\end{array} \qquad \dfrac{\dfrac{\dfrac{\forall_{z\in E_{\mathbf N}}(z\ \mathbf{r}\ A\to A)\qquad \mathrm{et}(N)}{\mathrm{et}(N)\in E_{\mathbf N}\to \mathrm{et}(N)\ \mathbf{r}\ A\to A}\qquad \mathrm{et}(N)\in E_{\mathbf N}}{\mathrm{et}(N)\ \mathbf{r}\ A\to A}\qquad \begin{array}{c}\mid \text{IH}\\ \mathrm{et}(N)\ \mathbf{r}\ A\end{array}}{A}}{B}$$

Subcase A n.c. By induction hypothesis we have derivations of $A \to B$ and of A, hence also a derivation of B.

Case $(\lambda_x M^{A})^{\forall_x A}$ with $\forall_x A$ c.r. We need a derivation of $\mathrm{et}(\lambda_x M)\ \mathbf{r}\ \forall_x A$. By definition $\mathrm{et}(\lambda_x M) = \mathrm{et}(M)$. Hence we must derive

$$\mathrm{et}(M)\ \mathbf{r}\ \forall_x A, \quad \text{which is} \quad \forall_x(\mathrm{et}(M)\ \mathbf{r}\ A).$$

This follows from the induction hypothesis.

Case $(\lambda_x M^{A})^{\forall_x A}$ with $\forall_x A$ n.c. By induction hypothesis we have a derivation of A. Apply $\forall^{+}$.

Case $(M^{\forall_x A(x)}t)^{A(t)}$ with $A(t)$ c.r. We must derive $\mathrm{et}(Mt)\ \mathbf{r}\ A(t)$. By definition $\mathrm{et}(Mt) = \mathrm{et}(M)$, and by induction hypothesis can derive

$$\mathrm{et}(M)\ \mathbf{r}\ \forall_x A(x), \quad \text{which is} \quad \forall_x(\mathrm{et}(M)\ \mathbf{r}\ A(x)).$$

Case $(M^{\forall_x A(x)}t)^{A(t)}$ with $A(t)$ n.c. By induction hypothesis we have a derivation of $\forall_x A(x)$. Apply $\forall^{-}$.

It remains to prove the soundness theorem for the axioms, i.e., that their extracted terms are realizers.

We first prove soundness for introduction and elimination axioms of c.r. inductively defined predicates, and show that the extracted terms defined above are realizers. The proof uses the definition of $I^{\mathbf{r}}$ in Section 4.2.2.

By the clauses for $I^{\mathbf{r}}$ we clearly have $\mathrm{C}_i\ \mathbf{r}\ I_i^{+}$. For the elimination axiom we have to prove $\mathcal{R}\ \mathbf{r}\ I^{-}$, that is,

$$\mathcal{R}\ \mathbf{r}\ \forall_{\vec{x}}(I\vec{x} \to (K_i(I,P))_{i<k} \to P\vec{x}\,).$$

Let $\vec{x}, w$ be given and assume $w\ \mathbf{r}\ I\vec{x}$. Let further $w_0, \dots, w_{k-1}$ be such that $w_i\ \mathbf{r}\ K_i(I,P)$. For ease of notation we assume that $K_i(I,P)$ has the form

$$\forall_{\vec{x}}(\vec{A} \to (\forall_{\vec{y}_\nu}(\vec{B}_\nu \to I\vec{s}_\nu))_{\nu<n} \to (\forall_{\vec{y}_\nu}(\vec{B}_\nu \to P\vec{s}_\nu))_{\nu<n} \to P\vec{t}\,).$$

Then $w_i \mathbf{r} K_i(I, P)$ is

$$\begin{aligned}(18)\qquad \forall_{\vec{x},\vec{u},\vec{f},\vec{g}}(\vec{u}\ \mathbf{r}\ \vec{A} \to (\forall_{\vec{y}_\nu,\vec{v}_\nu}(\vec{v}_\nu\ \mathbf{r}\ \vec{B}_\nu \to f_\nu\vec{v}_\nu\ \mathbf{r}\ I\vec{s}_\nu))_{\nu<n} \to \\ (\forall_{\vec{y}_\nu,\vec{v}_\nu}(\vec{v}_\nu\ \mathbf{r}\ \vec{B}_\nu \to g_\nu\vec{v}_\nu\ \mathbf{r}\ P\vec{s}_\nu))_{\nu<n} \to \\ w_i\vec{u}\vec{f}\vec{g}\ \mathbf{r}\ P\vec{t}\,).\end{aligned}$$

Our goal is

$$\mathcal{R}w\vec{w}\ \mathbf{r}\ P\vec{x} =: Qw\vec{x}.$$

We use the elimination axiom for $I^{\mathbf{r}}$ with the n.c. $Q(w, \vec{x}\,)$, i.e.,

$$\forall_{\vec{x},w}(I^{\mathbf{r}}w\vec{x} \to (K_i^{\mathbf{r}}(I^{\mathbf{r}}, Q))_{i<k} \to Qw\vec{x}\,).$$

Hence it suffices to prove $K_i^{\mathbf{r}}(I^{\mathbf{r}}, Q)$ for every constructor formula K_i, i.e.,

$$\begin{aligned}(19)\qquad \forall_{\vec{x},\vec{u},\vec{f}}(\vec{u}\ \mathbf{r}\ \vec{A} \to (\forall_{\vec{y}_\nu,\vec{v}_\nu}(\vec{v}_\nu\ \mathbf{r}\ \vec{B}_\nu \to I^{\mathbf{r}}(f_\nu\vec{v}_\nu, \vec{s}_\nu)))_{\nu<n} \to \\ (\forall_{\vec{y}_\nu,\vec{v}_\nu}(\vec{v}_\nu\ \mathbf{r}\ \vec{B}_\nu \to Q(f_\nu\vec{v}_\nu, \vec{s}_\nu)))_{\nu<n} \to \\ Q(\mathrm{C}_i\vec{u}\vec{f}, \vec{t}\,)).\end{aligned}$$

Assume $\vec{x}, \vec{u}, \vec{f}$ and the premises of (19). We show $Q(\mathrm{C}_i\vec{u}\vec{f}, \vec{t}\,)$, i.e.,

$$\mathcal{R}(\mathrm{C}_i\vec{u}\vec{f}\,)\vec{w}\ \mathbf{r}\ P(\vec{t}\,).$$

By the conversion rules for $\mathcal{R}$ this is the same as

$$w_i\vec{u}\vec{f}(\lambda_{\vec{v}_\nu}\mathcal{R}(f_\nu\vec{v}_\nu)\vec{w})_{\nu<n}\ \mathbf{r}\ P(\vec{t}\,).$$

To this end we use (18) with $\vec{x}, \vec{u}, \vec{f}, (\lambda_{\vec{v}_\nu}\mathcal{R}(f_\nu\vec{v}_\nu)\vec{w})_{\nu<n}$. Its conclusion is what we want, and its premises follow from the premises of (19).

For the introduction and elimination axioms for n.c. inductive predicates there is nothing to show since these axioms are n.c. as well. The only exception are one-clause-nc inductive predicates where the competitor predicate in the elimination axiom can be c.r. (cf. Section 4.1.2). But for these the identity is a realizer, since for $\vec{A}$ n.c. we have

$$\begin{aligned}&(\lambda_z z)\ \mathbf{r}\ \forall_{\vec{x}}(I^{\mathbf{r}}\vec{x} \to \forall_{\vec{y}}(\vec{A} \to X\vec{r}\,) \to X\vec{x}\,)\\ &\forall_{\vec{x}}(I^{\mathbf{r}}\vec{x} \to (\lambda_z z)\ \mathbf{r}\ (\forall_{\vec{y}}(\vec{A} \to X\vec{r}\,) \to X\vec{x}\,))\\ &\forall_{\vec{x}}(I^{\mathbf{r}}\vec{x} \to \forall_z(z\ \mathbf{r}\ \forall_{\vec{y}}(\vec{A} \to X\vec{r}\,) \to z\ \mathbf{r}\ X\vec{x}\,))\\ &\forall_{\vec{x}}(I^{\mathbf{r}}\vec{x} \to \forall_z(\forall_{\vec{y}}(\vec{A} \to z\ \mathbf{r}\ X\vec{r}\,) \to z\ \mathbf{r}\ X\vec{x}\,))\end{aligned}$$

which is an instance of the same elimination axiom. □

4.4. Soundness for coinductive predicates

We now extend the development to coinductive predicates; recall that their treatment was postponed.

First we add another axiom, which could have been done in Section 3.4 already. Recall that $\approx_{\mathbf{N}}$ was defined by the closure axiom $\approx_{\mathbf{N}}^{-}$:

$$\forall_{n,m}(n \approx_{\mathbf{N}} m \to (n \equiv 0 \land m \equiv 0) \lor \\ \exists_{n',m'}(n' \approx_{\mathbf{N}} m' \land n \equiv Sn' \land m \equiv Sm')).$$

Therefore from $n \approx_{\mathbf{N}} m$ we can infer cototality $n, m \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}}$; this has already been proved in Section 3.4. Moreover, in our model of partial continuous functionals validity of $n \approx_{\mathbf{N}} m$ implies that n and m denote the same cototal object. We take this fact as an axiom in our theory TCF.

AXIOM (Bisimilarity). *For finitary algebras – say* $\mathbf{N}$ *– we provide as an axiom*

$$\mathrm{Bisim}_{\mathbf{N}} \colon \forall_{n,m}(n \approx_{\mathbf{N}} m \to n \equiv m). \tag{20}$$

We use the bisimilarity axiom to transfer a useful lemma from Section 3.2 to the coinductive setting.

LEMMA. (a) *For finitary* ρ *from* $\forall_x(x \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\rho} \to fx \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma})$ *we can infer* $\forall_{x,y}(x \approx_{\rho} y \to fx \approx_{\sigma} fy)$.

(b) *The converse holds generally:*

$$\forall_{x,y}(x \approx_{\rho} y \to fx \approx_{\sigma} fy) \to \forall_x(x \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\rho} \to fx \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma}).$$

PROOF. (a). Assume $\forall_x(x \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\rho} \to fx \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma})$ and $x \approx_{\rho} y$. The goal is $fx \approx_{\sigma} fy$. From $x \approx_{\rho} y$ we obtain $x \equiv y$ by the bisimilarity axiom, and also $x \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\rho}$ (by a lemma), hence $fx \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma}$ (by assumption), hence $fx \approx_{\sigma} fx$ (by reflexivity of $\approx$ on ${}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma}$), hence $fx \approx_{\sigma} fy$ (since $x \equiv y$).

(b). Assume $\forall_{x,y}(x \approx_{\rho} y \to fx \approx_{\sigma} fy)$ and $x \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\rho}$. The goal is $fx \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma}$. Then $x \approx_{\rho} x$ (by reflexivity of $\approx$ on ${}^{\mathrm{co}}E^{\mathrm{nc}}_{\rho}$), hence $fx \approx_{\sigma} fx$ (by assumption), hence $fx \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\sigma}$ (by a lemma). □

In Section 3.3 coinductive predicates were defined by

$${}^{\mathrm{co}}I(\vec{\alpha}, \vec{Y}) := \nu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k}$$

with closure and greatest-fixed-point axioms:

$${}^{\mathrm{co}}I^{-} \colon \forall_{\vec{x}}({}^{\mathrm{co}}I\vec{x} \to \bigvee_{i<k} \exists_{\vec{x}_i}(\bigwedge_{\nu<n_i} A_{i\nu}({}^{\mathrm{co}}I) \land \vec{x} \equiv \vec{r}_i)),$$

$${}^{\mathrm{co}}I^{+} \colon \forall_{\vec{x}}(X\vec{x} \to \forall_{\vec{x}}(X\vec{x} \to \bigvee_{i<k} \exists_{\vec{x}_i}(\bigwedge_{\nu<n_i} A_{i\nu}({}^{\mathrm{co}}I \lor X) \land \vec{x} \equiv \vec{r}_i)) \to {}^{\mathrm{co}}I\vec{x}).$$

The type of a coinductive predicate is the same as the type of the corresponding inductive one.

The definition of $C^{\mathbf{r}}$ is extended by adding coinductive definitions. Since then we cannot have that realizers are total (we need ${}^{\mathrm{co}}\mathcal{R}$ to realize coinduction axioms), we can simplify the definition in case of an implication accordingly:

$$z \mathbf{\ r\ } (A \to B) := \begin{cases} \forall_w (w \mathbf{\ r\ } A \to zw \mathbf{\ r\ } B) & \text{if } A \text{ is c.r.} \\ A \to z \mathbf{\ r\ } B & \text{if } A \text{ is n.c.} \end{cases}$$

Examples. (i). The closure axiom ${}^{\mathrm{co}}E^-$ is

$$\forall_n (n \in {}^{\mathrm{co}}E \to n \equiv 0 \lor \exists_m (m \in {}^{\mathrm{co}}E \land n \equiv Sm)).$$

By definition the closure axiom $({}^{\mathrm{co}}E^{\mathbf{r}})^-$ is

$$\forall_{z,n}(z \mathbf{\ r\ } (n \in {}^{\mathrm{co}}E) \to (0 \mathbf{\ r\ } (0 \in {}^{\mathrm{co}}E) \land z \equiv 0 \land n \equiv 0) \lor \\ \exists_{z',n'}(z' \mathbf{\ r\ } (n' \in {}^{\mathrm{co}}E) \land z \equiv Sz' \land n \equiv Sn')).$$

Hence from $z \mathbf{\ r\ } (n \in {}^{\mathrm{co}}E)$ we can infer that $z, n \in {}^{\mathrm{co}}E$ and $z \approx n$. Therefore by the bisimilarity axiom $z \equiv n$.

(ii). Recall from Section 3.3 the closure axiom ${}^{\mathrm{co}}\mathrm{Even}^-$:

$$\forall_n (n \in {}^{\mathrm{co}}\mathrm{Even} \to n \equiv 0 \lor \exists_m (m \in {}^{\mathrm{co}}\mathrm{Even} \land n \equiv S(Sm))).$$

Then $({}^{\mathrm{co}}\mathrm{Even}^{\mathbf{r}})^-$ is

$$\forall_{z,n}(z \mathbf{\ r\ } (n \in {}^{\mathrm{co}}\mathrm{Even}) \to (0 \mathbf{\ r\ } (0 \in {}^{\mathrm{co}}\mathrm{Even}) \land z \equiv 0 \land n \equiv 0) \lor \\ \exists_{z',n'}(z' \mathbf{\ r\ } (n' \in {}^{\mathrm{co}}\mathrm{Even}) \land z \equiv Sz' \land n \equiv S(Sn'))).$$

Again from $z \mathbf{\ r\ } (n \in {}^{\mathrm{co}}\mathrm{Even})$ we can infer that $z \in {}^{\mathrm{co}}E$.

(iii). Recall from Section 3.3 the (simplified) closure axiom ${}^{\mathrm{co}}I^-$:

$$\forall_x (x \in {}^{\mathrm{co}}I \to \exists_{d,x'}(d \in \mathrm{Sd} \land x' \in {}^{\mathrm{co}}I \land x = \frac{x'+d}{2})).$$

Then $({}^{\mathrm{co}}I^{\mathbf{r}})^-$ is

$$\forall_{v,x}(v \mathbf{\ r\ } (x \in {}^{\mathrm{co}}I) \to \\ \exists_{v',d,x'}(d \in \mathrm{Sd} \land v' \mathbf{\ r\ } (x' \in {}^{\mathrm{co}}I) \land x = \frac{x'+d}{2} \land v \equiv \mathrm{C}_d(v'))).$$

Here again from $v \mathbf{\ r\ } (x \in {}^{\mathrm{co}}I)$ we can infer that $v \in {}^{\mathrm{co}}I$.

Recall that in example (i) above we observed that from $z \mathbf{\ r\ } (n \in {}^{\mathrm{co}}E)$ we can infer $z \approx n$, and therefore by the bisimilarity axiom $z \equiv n$. Using this fact we can generalize the *invariance axioms* Inv_A as follows, and prove that identities realize them.

In Section 4.1 we have assigned to every predicate or formula C its final predicate $\mathrm{fp}(C)$. We extend this concept by also taking parameter predicates into account. The set $\mathrm{fps}(C)$ of final predicates of a predicate or formula C is defined by

$$\begin{aligned}
\mathrm{fps}(X) &:= \{X\}, \quad \mathrm{fps}(X^{\mathrm{nc}}) := \{X^{\mathrm{nc}}\},\\
\mathrm{fps}(\{\, \vec{x} \mid A \,\}) &:= \mathrm{fps}(A),\\
\mathrm{fps}(I(\vec{P}\,)) &:= \{I(\vec{P}\,)\} \cup \bigcup_{P \in \vec{P}} \mathrm{fps}(P),\\
\mathrm{fps}(I^{\mathrm{nc}}(\vec{P}\,)) &:= \{I^{\mathrm{nc}}(\vec{P}\,)\} \cup \bigcup_{P \in \vec{P}} \mathrm{fps}(P),\\
\mathrm{fps}(P\vec{r}\,) &:= \mathrm{fps}(P),\\
\mathrm{fps}(A \to B) &:= \mathrm{fps}(B),\\
\mathrm{fps}(\forall_x A) &:= \mathrm{fps}(A).
\end{aligned}$$

Axiom (Invariance under realizability). *For c.r. formulas A with $\tau(A)$ a finitary algebra we provide the following axioms.*

(a) *If all predicates in* $\mathrm{fps}(A)$ *are inductive or coinductive predicate constants we take as axioms*

$$\mathrm{Inv}_A \colon A \leftrightarrow \exists_{z \in {}^{\mathrm{co}}E_{\tau(A)}}(z \mathbf{~r~} A).$$

(b) *If all predicates in* $\mathrm{fps}(A)$ *are inductive predicate constants we take*

$$\mathrm{Inv}_A \colon A \leftrightarrow \exists_{z \in E_{\tau(A)}}(z \mathbf{~r~} A).$$

The justification of these (c.r.) axioms is that they are realized by identities. However, to prove this we need another axiom.

Axiom (Totality of realizers). *For c.r. formulas A with $\tau(A)$ a finitary algebra we provide the following axioms.*

(a) *If all predicates in* $\mathrm{fps}(A)$ *are inductive or coinductive predicate constants we take as axioms*

$$\mathrm{EReal}_A \colon \forall_z(z \mathbf{~r~} A \to z \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\tau(A)}).$$

(b) *If all predicates in* $\mathrm{fps}(A)$ *are inductive predicate constants we take*

$$\mathrm{EReal}_A \colon \forall_z(z \mathbf{~r~} A \to z \in E^{\mathrm{nc}}_{\tau(A)}).$$

Lemma. *For c.r. formulas A with $\tau(A)$ a finitary algebra we have*

(a) *If all predicates in* $\mathrm{fps}(A)$ *are inductive or coinductive predicate constants then*

$$\begin{aligned}
&(\lambda_w w) \mathbf{~r~} (A \to \exists_{z \in {}^{\mathrm{co}}E_{\tau(A)}}(z \mathbf{~r~} A)),\\
&(\lambda_w w) \mathbf{~r~} (\exists_{z \in {}^{\mathrm{co}}E_{\tau(A)}}(z \mathbf{~r~} A) \to A).
\end{aligned}$$

(b) *If all predicates in* fps(A) *are inductive predicate constants then*

$$(\lambda_w w) \mathbf{r} (A \to \exists_{z \in E_{\tau(A)}}(z \mathbf{r} A),$$
$$(\lambda_w w) \mathbf{r} (\exists_{z \in E_{\tau(A)}}(z \mathbf{r} A) \to A).$$

Proof. (a). Assume that $\tau(A)$ is $\mathbf{N}$. Unfolding the definitions we obtain

$$(\lambda_w w) \mathbf{r} (A \to \exists_{z \in {}^{\mathrm{co}}E_{\mathbf{N}}}(z \mathbf{r} A))$$
$$\forall_w (w \mathbf{r} A \to w \mathbf{r} \exists_{z \in {}^{\mathrm{co}}E_{\mathbf{N}}}(z \mathbf{r} A))$$

and similarly

$$(\lambda_w w) \mathbf{r} (\exists_{z \in {}^{\mathrm{co}}E_{\mathbf{N}}}(z \mathbf{r} A) \to A)$$
$$\forall_w (w \mathbf{r} \exists_{z \in {}^{\mathrm{co}}E_{\mathbf{N}}}(z \mathbf{r} A) \to w \mathbf{r} A).$$

But $w \mathbf{r} \exists_{z \in {}^{\mathrm{co}}E_{\mathbf{N}}}(z \mathbf{r} A)$ is defined to mean $\exists_z (w \mathbf{r} (z \in {}^{\mathrm{co}}E_{\mathbf{N}}) \wedge z \mathbf{r} A)$. Since by a remark above $w \mathbf{r} (z \in {}^{\mathrm{co}}E_{\mathbf{N}})$ implies $w \approx z$ and hence $w \equiv z$ by the bisimilarity axiom, the latter formula is equivalent to $w \mathbf{r} A$. Here we need the axioms on totality of realizers.

(b) is proved similarly. □

The definition of extracted terms is extended to the closure and greatest-fixed-point axioms of general (c.r.) coinductively defined predicate ${}^{\mathrm{co}}I$. Recall the *destructor* D_ι, disassembling a constructor-built object into its parts. For the closure and greatest-fixed-point axioms of ${}^{\mathrm{co}}I$ define $\mathrm{et}({}^{\mathrm{co}}I_i^+) := {}^{\mathrm{co}}\mathcal{R}$ and $\mathrm{et}(I^-) := \mathrm{D}$, where both the corecursion operator ${}^{\mathrm{co}}\mathcal{R}$ and the destructor D refer to the algebra ι_I associated with I.

Example. The destructor $\mathrm{D}_{\mathbf{N}}$ has type $\mathbf{N} \to \mathbf{U} + \mathbf{N}$. It is defined by the computation rules

$$\mathrm{D}_{\mathbf{N}} 0 = \mathrm{DummyL}_{\mathbf{U}+\mathbf{N}}, \qquad \mathrm{D}_{\mathbf{N}}(Sn) = \mathrm{InrUysum}_{\mathbf{N} \to \mathbf{U}+\mathbf{N}}(n).$$

We still need to extend the soundness theorem, which states that the term $\mathrm{et}(M)$ extracted from a proof M of a formula A is a realizer of A. Let us first see that $\mathrm{et}(M)$ is always cototal, provided the object variables z_u assigned to assumption variables u are cototal.

Lemma (Cototality of extracted terms). *For every term t with free variables and constants in ${}^{\mathrm{co}}E^{\mathrm{nc}}$ we have $t \in {}^{\mathrm{co}}E^{\mathrm{nc}}$.*

Proof. We first show that for every term $t(\vec{x}\,)$ without constants and with free variables among $\vec{x}$ we have

$$\forall_{\vec{x},\vec{y}}(\vec{x} \approx \vec{y} \to t(\vec{x}\,) \approx t(\vec{y}\,)). \tag{21}$$

This is proved by induction on t. *Case* x. Immediate. *Case* $\lambda_x t(x, \vec{x}\,)$. Let $\vec{x} \approx \vec{y}$. The goal is $\lambda_x t(x, \vec{x}\,) \approx \lambda_x t(x, \vec{y}\,)$, which by definition means

$$\lambda_x t(x, \vec{x}\,), \lambda_x t(x, \vec{y}\,) \in {}^{\mathrm{co}}E^{\mathrm{nc}},$$
$$\forall_x (x \in {}^{\mathrm{co}}E^{\mathrm{nc}} \to t(x, \vec{x}\,) \approx t(x, \vec{y}\,)).$$

For the first part we only show $\lambda_x t(x, \vec{x}\,) \in {}^{\mathrm{co}}E^{\mathrm{nc}}$. By a lemma in Section 3.4 it suffices to prove $x \approx y \to t(x, \vec{x}\,) \approx t(y, \vec{x}\,)$. Since from $\vec{x} \approx \vec{y}$ we can infer $\vec{x} \approx \vec{x}$ the claim follows from the induction hypothesis. For the second part assume $x \in {}^{\mathrm{co}}E^{\mathrm{nc}}$. The goal is $t(x, \vec{x}\,) \approx t(x, \vec{y}\,)$. Since from $x \in {}^{\mathrm{co}}E^{\mathrm{nc}}$ we can infer $x \approx x$ the claim again follows from the induction hypothesis. *Case* ts. Use compatibility of application with $\approx$ and the induction hypothesis.

Now from (21) we obtain the lemma as follows. Let t with free variables and constants in ${}^{\mathrm{co}}E^{\mathrm{nc}}$ be given. The goal is $t \in {}^{\mathrm{co}}E^{\mathrm{nc}}$. By (21) applied to the free variables and constants of t we obtain $t \approx t$. Hence $t \in {}^{\mathrm{co}}E^{\mathrm{nc}}$ by a lemma in Section 3.4. □

We now aim at extending the soundness theorem. To this end we extend and/or adapt the material in Section 4.3. We restrict ourselves to look at an example. Cototality for $\mathbf{N}$ has been coinductively defined by the closure axiom ${}^{\mathrm{co}}E^{-}$:

$$\forall_n (n \in {}^{\mathrm{co}}E \to n \equiv 0 \vee \exists_m (m \in {}^{\mathrm{co}}E \wedge n \equiv Sm)).$$

Its greatest-fixed-point axiom is ${}^{\mathrm{co}}E^{+}_{\mathbf{N}}$:

$$\forall_n (n \in X \to \forall_n (n \in X \to n \equiv 0 \vee \exists_m ((m \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee m \in X) \wedge n \equiv Sm)) \to$$
$$n \in {}^{\mathrm{co}}E_{\mathbf{N}}).$$

We show that their extracted terms $\mathrm{D}_{\mathbf{N}}$ and ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbf{N}}$ are realizers, provided τ is finitary. Recall that ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbf{N}}$ has type $\tau \to (\tau \to \mathbf{U} + (\mathbf{N} + \tau)) \to \mathbf{N}$.

LEMMA. (a) $\mathrm{D}_{\mathbf{N}}$ *is in* ${}^{\mathrm{co}}E^{\mathrm{nc}}$.
(b) ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbf{N}}$ *is in* ${}^{\mathrm{co}}E^{\mathrm{nc}}$*, provided* τ *is finitary.*

PROOF. (a). By part (a) of the first lemma in the present section it suffices to prove

$$\forall_n (n \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}} \to \mathrm{D}_{\mathbf{N}} n \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{U}+\mathbf{N}}).$$

Fix n and assume $n \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}}$. To prove the goal $\mathrm{D}_{\mathbf{N}} n \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{U}+\mathbf{N}}$ we use coinduction, or more precisely $({}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{U}+\mathbf{N}})^{+}$:

$$\forall_x (x \in X^{\mathrm{nc}} \to$$
$$\forall_x (x \in X^{\mathrm{nc}} \to x \equiv \mathrm{DummyL} \vee^{\mathrm{nc}} \exists_m (m \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}} \wedge x \equiv \mathrm{InrUysum}(m))) \to$$
$$x \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{U}+\mathbf{N}}).$$

with competitor predicate

$$X^{\mathrm{nc}} := \{\, x \mid \exists_m (m \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}} \wedge x \equiv \mathrm{D}_{\mathbf{N}} m) \,\}$$

and applied to $\mathrm{D}_{\mathbf{N}} n$. It suffices to prove the step formula

$$\forall_x (x \in X^{\mathrm{nc}} \to x \equiv \mathrm{DummyL} \vee^{\mathrm{nc}} \exists_m (m \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}} \wedge x \equiv \mathrm{InrUysum}(m))).$$

Let $m \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}}$ and $x \equiv \mathrm{D}_{\mathbf{N}} m$. By the closure axiom $({}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}})^-$ from $m \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}}$ we obtain

$$m \equiv 0 \vee^{\mathrm{nc}} \exists_{m'} (m' \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}} \wedge m \equiv Sm').$$

Case 1. Then $\mathrm{D}_{\mathbf{N}} m$ is DummyL by definition.

Case 2. Then $\mathrm{D}_{\mathbf{N}} m$ is $\mathrm{D}_{\mathbf{N}}(Sm')$ with $m' \in {}^{\mathrm{co}}E^{\mathrm{nc}}_{\mathbf{N}}$, which by definition of $\mathrm{D}_{\mathbf{N}}$ is m': we have that $\mathrm{D}_{\mathbf{N}} m$ is $\mathrm{InrUysum}(m')$.

(b). By part (b) of the first lemma in the present section it suffices to prove

$$\forall_{x,x',f,f'} (x \approx_\tau x' \to f \approx f' \to {}^{\mathrm{co}}\mathcal{R} x f \approx_{\mathbf{N}} {}^{\mathrm{co}}\mathcal{R} x' f').$$

Assume $x \approx_\tau x'$ and $f \approx f'$. Let $n := {}^{\mathrm{co}}\mathcal{R} x f$ and $n' := {}^{\mathrm{co}}\mathcal{R} x' f'$. To prove the goal $n \approx_{\mathbf{N}} n'$ we use coinduction, or more precisely $\approx^+_{\mathbf{N}}$ with competitor predicate

$$X^{\mathrm{nc}} := \{\, n, n' \mid \exists_{y,y'} (n \equiv {}^{\mathrm{co}}\mathcal{R} y f \wedge n' \equiv {}^{\mathrm{co}}\mathcal{R} y' f') \,\}$$

This means that we have to show

$$\begin{aligned} &(n \equiv 0 \wedge n' \equiv 0) \vee^{\mathrm{nc}} \\ &\exists_{m,m'} ((m \approx m' \vee^{\mathrm{nc}} \exists_{y,y'} (y \approx y' \wedge m \equiv {}^{\mathrm{co}}\mathcal{R} y f \wedge m' \equiv {}^{\mathrm{co}}\mathcal{R} y' f')) \wedge \\ &\qquad n \equiv Sm \wedge n' \equiv Sm'). \end{aligned} \tag{22}$$

From $x \approx_\tau x'$ and $f \approx f'$ we obtain $fx \approx f'x'$, of type $\mathbf{U} + (\mathbf{N} + \tau)$, which by assumption is finitary. By the closure axiom $\approx^-_{\mathbf{U}+(\mathbf{N}+\tau)}$ from $fx \approx f'x'$ we obtain

$$\begin{aligned} &(fx \equiv \mathrm{DummyL} \wedge f'x' \equiv \mathrm{DummyL}) \vee^{\mathrm{nc}} \\ &\exists_{u,u'} (u \approx_{\mathbf{N}+\tau} u' \wedge fx \equiv \mathrm{InrUysum}(u) \wedge f'x' \equiv \mathrm{InrUysum}(u')). \end{aligned}$$

Recall the conversion

$${}^{\mathrm{co}}\mathcal{R} y f = [\lambda\_0, \lambda_x (S([\mathrm{id}^{\mathbf{N}\to\mathbf{N}}, \lambda_z ({}^{\mathrm{co}}\mathcal{R} z f)] x))](fy).$$

Case 1. Then by the conversion rules for ${}^{\mathrm{co}}\mathcal{R} x f$ and ${}^{\mathrm{co}}\mathcal{R} x' f'$ we have ${}^{\mathrm{co}}\mathcal{R} x f \equiv 0$ and ${}^{\mathrm{co}}\mathcal{R} x' f' \equiv 0$, hence $n \equiv 0 \wedge n' \equiv 0$ and therefore (22).

Case 2. Similary by the closure axiom $\approx^-_{\mathbf{N}+\tau}$ from $u \approx_{\mathbf{N}+\tau} u'$ we obtain

$$\begin{aligned} &\exists_{m,m'} (m \approx_{\mathbf{N}} m' \wedge u \equiv \mathrm{Inl}(m) \wedge u' \equiv \mathrm{Inl}(m')) \vee^{\mathrm{nc}} \\ &\exists_{y,y'} (y \approx_\tau y' \wedge u \equiv \mathrm{Inr}(y) \wedge u' \equiv \mathrm{Inr}(y')) \end{aligned}$$

Case 2.1. Then $fx \equiv \text{InrUysum}(\text{Inl}(m))$ and $f'x' \equiv \text{InrUysum}(\text{Inl}(m'))$. By the conversion rule for ${}^{\text{co}}\mathcal{R}$ we have

$$ {}^{\text{co}}\mathcal{R}fx \equiv Sm \quad \text{and} \quad {}^{\text{co}}\mathcal{R}f'x' \equiv Sm'. $$

Hence $n \equiv Sm$ and $n' \equiv Sm'$, and with $m \approx_{\mathbf{N}} m'$ we get (22).

Case 2.2. Then $fx \equiv \text{InrUysum}(\text{Inr}(y))$ and $f'x' \equiv \text{InrUysum}(\text{Inr}(y'))$. By the conversion rule for ${}^{\text{co}}\mathcal{R}$ we have

$$ {}^{\text{co}}\mathcal{R}fx \equiv S({}^{\text{co}}\mathcal{R}yf) \quad \text{and} \quad {}^{\text{co}}\mathcal{R}f'x' \equiv S({}^{\text{co}}\mathcal{R}y'f'). $$

With $m := {}^{\text{co}}\mathcal{R}yf$ and $m' := {}^{\text{co}}\mathcal{R}y'f'$ and y, y' as given in the subcase we again obtain (22). □

LEMMA. (a) $\text{D}_{\mathbf{N}}$ *realizes the closure axiom* ${}^{\text{co}}E_{\mathbf{N}}^{-}$*:*

$$ \forall_n(n \in {}^{\text{co}}E_{\mathbf{N}} \to n \equiv 0 \lor \exists_m(m \in {}^{\text{co}}E_{\mathbf{N}} \land n \equiv Sm)). $$

(b) ${}^{\text{co}}\mathcal{R}_{\mathbf{N}}^{\tau}$ *realizes the greatest-fixed-point axiom* ${}^{\text{co}}E_{\mathbf{N}}^{+}$*:*

$$ \forall_n(n \in X \to \forall_n(n \in X \to n \equiv 0 \lor \exists_{n'}((n' \in {}^{\text{co}}E_{\mathbf{N}} \lor n' \in X) \land n \equiv Sn')) \to n \in {}^{\text{co}}E_{\mathbf{N}}). $$

PROOF. (a). The goal is

$$ \forall_{n,z}(z \mathbf{\ r\ } (n \in {}^{\text{co}}E_{\mathbf{N}}) \to \text{D}_{\mathbf{N}}z \mathbf{\ r\ } (n \equiv 0 \lor \exists_m(m \in {}^{\text{co}}E_{\mathbf{N}} \land n \equiv Sm))). $$

Assume $z \mathbf{\ r\ } (n \in {}^{\text{co}}E_{\mathbf{N}})$. Recall that from $z \mathbf{\ r\ } (n \in {}^{\text{co}}E_{\mathbf{N}})$ we can infer that $z, n \in {}^{\text{co}}E_{\mathbf{N}}$ and $z \approx n$. Therefore by the bisimilarity axiom $z \equiv n$, and we must show

$$ \text{D}_{\mathbf{N}}n \mathbf{\ r\ } (n \equiv 0 \lor \exists_m(m \in {}^{\text{co}}E_{\mathbf{N}} \land n \equiv Sm)). \tag{23} $$

To this end we apply the closure axiom ${}^{\text{co}}E_{\mathbf{N}}^{-}$:

$$ \forall_n(n \in {}^{\text{co}}E_{\mathbf{N}} \to n \equiv 0 \lor \exists_m(m \in {}^{\text{co}}E_{\mathbf{N}} \land n \equiv Sm)) $$

to $n \in {}^{\text{co}}E_{\mathbf{N}}$.

Case $n \equiv 0$. Then by definition $\text{D}_{\mathbf{N}}n \equiv \text{DummyL}$, and (23) follows from the lemma on realizers for $\lor$ in Section 4.2.2.

Case $n \equiv Sm$ with $m \in {}^{\text{co}}E_{\mathbf{N}}$. Then $\text{D}_{\mathbf{N}}n \equiv \text{InrUysum}(m)$ and (again by the lemma on realizers for $\lor$) (23) is equivalent to

$$ m \mathbf{\ r\ } \exists_m(m \in {}^{\text{co}}E_{\mathbf{N}} \land n \equiv Sm) $$

which holds in the present case.

(b). For the proof recall that the witnessing predicate ${}^{\text{co}}E^{\mathbf{r}}$ has the closure axiom $({}^{\text{co}}E^{\mathbf{r}})^{-}$:

$$ \forall_{z,n}(z \mathbf{\ r\ } (n \in {}^{\text{co}}E_{\mathbf{N}}) \to (0 \mathbf{\ r\ } (0 \in {}^{\text{co}}E_{\mathbf{N}}) \land z \equiv 0 \land n \equiv 0) \lor^{\text{nc}} \exists_{z',n'}(z' \mathbf{\ r\ } (n' \in {}^{\text{co}}E_{\mathbf{N}}) \land z \equiv Sz' \land n \equiv Sn')). $$

The greatest-fixed-point axiom is $({}^{\mathrm{co}}E^{\mathbf{r}}_{\mathbf{N}})^+$:

$$\begin{aligned}
&\forall_{n,z}(X^{\mathrm{nc}}zn \to \\
&\forall_{n,z}(X^{\mathrm{nc}}zn \to ((0 \mathbin{\mathbf{r}} (0 \in {}^{\mathrm{co}}E_{\mathbf{N}}) \vee^{\mathrm{nc}} X^{\mathrm{nc}}00) \wedge z \equiv 0 \wedge n \equiv 0) \vee^{\mathrm{nc}} \\
&\qquad\qquad \exists_{z',n'}((z' \mathbin{\mathbf{r}} (n' \in {}^{\mathrm{co}}E_{\mathbf{N}}) \vee^{\mathrm{nc}} X^{\mathrm{nc}}z'n') \wedge \\
&\qquad\qquad\qquad z \equiv Sz' \wedge n \equiv Sn')) \to \\
&z \mathbin{\mathbf{r}} (n \in {}^{\mathrm{co}}E_{\mathbf{N}})).
\end{aligned}$$

Assume we have n, z with $z \mathbin{\mathbf{r}} (n \in X)$ and f with

$$\begin{aligned}
&f \mathbin{\mathbf{r}} \forall_n(n \in X \to n \equiv 0 \vee \exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X) \wedge n \equiv Sn')), \quad \text{i.e.,} \\
&\forall_{n,z}(z \mathbin{\mathbf{r}} (n \in X) \to fz \mathbin{\mathbf{r}} (n \equiv 0 \vee \exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X) \wedge n \equiv Sn')).
\end{aligned}$$

The goal is ${}^{\mathrm{co}}\mathcal{R}zf \mathbin{\mathbf{r}} (n \in {}^{\mathrm{co}}E_{\mathbf{N}})$. We use $({}^{\mathrm{co}}E^{\mathbf{r}}_{\mathbf{N}})^+$ with competitor predicate

$$X^{\mathrm{nc}} := \{\, z, n \mid \exists_{z_1}(z \equiv {}^{\mathrm{co}}\mathcal{R}z_1 f \wedge z_1 \mathbin{\mathbf{r}} (n \in X)) \,\}$$

We need to prove $\exists_{z_1}({}^{\mathrm{co}}\mathcal{R}zf \equiv {}^{\mathrm{co}}\mathcal{R}z_1 f \wedge z_1 \mathbin{\mathbf{r}} (n \in X))$ (take $z_1 := z$) and for this X^{nc} the step formula above:

$$\begin{aligned}
&\forall_{n,z}(\exists_{z_1}(z \equiv {}^{\mathrm{co}}\mathcal{R}z_1 f \wedge z_1 \mathbin{\mathbf{r}} (n \in X)) \to \\
&((0 \mathbin{\mathbf{r}} (0 \in {}^{\mathrm{co}}E_{\mathbf{N}}) \vee^{\mathrm{nc}} \exists_{z_1}(0 \equiv {}^{\mathrm{co}}\mathcal{R}z_1 f \wedge z_1 \mathbin{\mathbf{r}} (0 \in X))) \wedge z \equiv 0 \wedge n \equiv 0) \vee^{\mathrm{nc}} \\
&\exists_{z',n'}((z' \mathbin{\mathbf{r}} (n' \in {}^{\mathrm{co}}E_{\mathbf{N}}) \vee^{\mathrm{nc}} \exists_{z_1}(z' \equiv {}^{\mathrm{co}}\mathcal{R}z_1 f \wedge z_1 \mathbin{\mathbf{r}} (n' \in X))) \wedge \\
&z \equiv Sz' \wedge n \equiv Sn'))
\end{aligned}$$

Again fix n, z and z_1 such that $z \equiv {}^{\mathrm{co}}\mathcal{R}z_1 f$ and $z_1 \mathbin{\mathbf{r}} (n \in X)$. We need to show the (n.c.) disjunction C. Since $z_1 \mathbin{\mathbf{r}} (n \in X)$ we know

$$\begin{aligned}
&fz_1 \mathbin{\mathbf{r}} (n \equiv 0 \vee \exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X) \wedge n \equiv Sn')) \quad \text{i.e.,} \\
&fz_1 \mathbin{\mathbf{r}} (n \equiv 0 \vee \exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X))) \wedge n \equiv Sn'
\end{aligned}$$

We now use the elimination axiom

$$u \mathbin{\mathbf{r}} (A \vee B) \to (A \to C) \to \forall_w(w \mathbin{\mathbf{r}} B \to C) \to C$$

with $n \equiv 0$ for A and $\exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X) \wedge n \equiv Sn')$ for B.

For $A \to C$. Then we have $n \equiv 0$. Go for the l.h.s. of the disjunction. We have $0 \mathbin{\mathbf{r}} (0 \in {}^{\mathrm{co}}E_{\mathbf{N}})$ (by definition), $z \equiv 0$ (since ${}^{\mathrm{co}}\mathcal{R}z_1 f$ is 0), and $n \equiv 0$.

For $\forall_w(w \mathbin{\mathbf{r}} \exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X) \wedge n \equiv Sn') \to C)$. Let w be given such that $w \mathbin{\mathbf{r}} \exists_{n'}((n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X) \wedge n \equiv Sn')$. Then we have n' such that $n \equiv Sn'$ and $w \mathbin{\mathbf{r}} (n' \in {}^{\mathrm{co}}E_{\mathbf{N}} \vee n' \in X)$. Again by elimination it suffices to prove (i) $m \mathbin{\mathbf{r}} (n' \in {}^{\mathrm{co}}E_{\mathbf{N}}) \to C$ and (ii) $y \mathbin{\mathbf{r}} (n' \in X) \to C$.

For (i). Assume $m \mathbin{\mathbf{r}} (n' \in {}^{\mathrm{co}}E_{\mathbf{N}})$. Go for the r.h.s. of the disjunction with $z' := m$ and n'. Then

$$z' \mathbin{\mathbf{r}} (n' \in {}^{\mathrm{co}}E_{\mathbf{N}}) \quad \text{(since } z' \text{ is } m\text{)},$$

$$z \equiv Sz' \qquad \text{(since } z \equiv {}^{\mathrm{co}}\mathcal{R} z_1 f\text{, and } {}^{\mathrm{co}}\mathcal{R} z_1 f \text{ is } Sm\text{).}$$

For (ii). Assume $y \;\mathbf{r}\; (n' \in X)$. Go for the r.h.s. of the disjunction with $z' := {}^{\mathrm{co}}\mathcal{R} y f$ and n'. Then

$$\exists_{z_1}(z' \equiv {}^{\mathrm{co}}\mathcal{R} z_1 f \wedge z_1 \;\mathbf{r}\; (n' \in X)) \quad \text{(take } y\text{),}$$
$$z \equiv Sz' \qquad \text{(since } {}^{\mathrm{co}}\mathcal{R} z_1 f \text{ is } S({}^{\mathrm{co}}\mathcal{R} y f)\text{).} \quad \square$$