Prof. Dr. Bachmann | PARTIAL DIFFERENTIAL EQUATIONS I | WS 2016/17
A. Dietlein, R. Schulte | HOMEWORK SHEET 12 | January 16, 2017

Exercise 1 (Energy methods for the wave equation; 5 Points). Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with $\partial\Omega \in C^1$ and let $u_0 \in C^2(\overline{\Omega})$ and $u_1 \in C^1(\overline{\Omega})$. Let $u \in C^2([0,\infty) \times \overline{\Omega})$ be a solution of the problem

$$\begin{cases} u_{tt} - \Delta u + u^3 = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 1 & \text{on } [0,\infty) \times \partial\Omega, \\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1 & \text{on } \Omega. \end{cases}$$

Show that $\|u(t,\cdot)\|_{L^4(\Omega)} \leq C$ for all $t \geq 0$ and a suitable constant $C > 0$.

Hint: Consider, for a suitable $\alpha \in \mathbb{R}$, the function $e(t) := \int_\Omega (u_t^2 + |\nabla u|^2 + \alpha u^4)(t,x)\,\mathrm{d}x$.

Exercise 2 (A generalized Duhamel's principle; 5 Points). Let $n \in \mathbb{N}$ and let $L = L(\partial_t, \partial_x)$ be a differential operator of second order with $n+1$ variables $t, x_1, \ldots .x_n$ and constant coefficients $a_{\alpha,k} \in \mathbb{R}$ such that

$$L(\partial_t, \partial_x) := \sum_{k=0}^{1} \sum_{\substack{\alpha \in \mathbb{N}_0^n, \\ |\alpha|+k \leq 2}} a_{k,\alpha} \partial_x^\alpha \partial_t^k.$$

For any $f \in C^1([0,\infty) \times \mathbb{R}^n)$ consider the inhomogeneous partial differential equation

$$\begin{cases} u_{tt}(t,x) + L(\partial_t,\partial_x)u(t,x) = f(t,x) & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = 0,\ u_t(0,x) = 0 & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{1}$$

Suppose that for all $0 \leq \sigma$ there exists a solution $\omega(\cdot,\sigma) \in C^2([0,\infty) \times \mathbb{R}^n)$ to the initial value problem

$$\begin{cases} \omega_{tt}(t,x,\sigma) + L(\partial_t,\partial_x)\omega(t,x,\sigma) = 0 & \text{for } (t,x) \in (\sigma,\infty) \times \mathbb{R}^n, \\ \omega(t,x,\sigma) = 0 & \text{for } (t,x) \in \{\sigma\} \times \mathbb{R}^n, \\ \omega_t(t,x,\sigma) = f(\sigma,x) & \text{for } (t,x) \in \{\sigma\} \times \mathbb{R}^n. \end{cases}$$

Further suppose that $\omega(t,x,\cdot), \omega_t(t,x,\cdot) \in C((0,\infty))$.
Prove that

$$u(t,x) := \int_0^t \omega(t,x,\sigma)\,\mathrm{d}\sigma$$

is a solution of (1).

Exercise 3 (Fourier transformation of the wave equation; 5 Points)**.** For $u \in \mathcal{S}(\mathbb{R}^n)$ let $\hat{u} := \mathcal{F}[u]$ denote the Fourier transform of u.
Let $g, h \in \mathcal{S}(\mathbb{R}^n)$. Let u be a solution of

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = g(x),\ u_t(0,x) = h(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{2}$$

Prove that

$$u(t,x) = \mathcal{F}^{-1}\left[\hat{g}(\cdot)\cos(t|\cdot|) + \frac{\hat{h}(\cdot)}{|\cdot|}\sin(t|\cdot|)\right].$$

Hint: You may use the results of Homework Sheet 8 Exercise 1.

Exercise 4 (asymptotic equipartition of energy; 5 Points)**.**

(i) Let $f \in C_c^\infty(\mathbb{R}^n)$, then

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} \sin(t|x|) f(x)\,\mathrm{d}x = 0,$$
$$\lim_{t\to\infty} \int_{\mathbb{R}^n} \cos(t|x|) f(x)\,\mathrm{d}x = 0.$$

(ii) Consider the initial value problem (2) with $\hat{g}, \hat{h} \in \mathcal{S}(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$. Prove that assymptotically the total energy splits equally into its potential and kinetic part, i.e.

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} k(t) = \frac{1}{2}e(0).$$

You can drop your homework solutions until **Monday, January 23** at **16 o'clock** into the appropriate letterbox on the first floor near the library.