LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Partial Differential Equations I
Homework Sheet 11

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Exercise 1. Assume that for some attenuation function $\alpha = \alpha(r)$ and delay function $\beta = \beta(r) \geq 0$, there exist for all profiles ϕ solutions of the wave equation in $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ having the form

$$u(t,x) = \alpha(r)\phi(t - \beta(r)).$$

Here $r = |x|$ and we assume $\beta(0) = 0$. Show that this is possible only if $n = 1$ or $n = 3$, and compute the form of the functions α, β.

Exercise 2. Let $g = \mathrm{diag}(-1,1,1,1) \in \mathbb{R}^{4\times 4}$. A real 4×4 matrix $\Lambda \in \mathbb{R}^{4\times 4}$ is called a Lorentz transformation if and only if $\Lambda^T g \Lambda = g$, where Λ^T denotes the transpose of Λ.

(i) Show that the product of two Lorentz transformations is also a Lorentz transformation.

(ii) Show that every Lorentz transformation is invertible, and that its inverse is also a Lorentz transformation. (Hence the set of all Lorentz transformations is a group.)

(iii) Define the quadratic form $\langle x, y\rangle_g := x^T g y$, for x, $y \in \mathbb{R}^4$. Show that for every Lorentz transformation Λ we have $\langle \Lambda x, \Lambda y\rangle_g = \langle x, y\rangle_g$.

(iv) Show that the following are Lorentz transformations (where $(t,x) \in \mathbb{R}^4$, with $t \in \mathbb{R}$, $x \in \mathbb{R}^3$):

(a) $(t,x) \mapsto (t, Ox)$, where O is an orthogonal transformation of $\mathbb{R}^3$,

(b) $(t,x) \mapsto (-t, x)$,

(c) $(t,x) \mapsto \left(\frac{t-ax_1}{\sqrt{1-a^2}}, \frac{x_1-at}{\sqrt{1-a^2}}, x_2, x_3\right)$, where $0 < a < 1$.
(This transformation is called a Lorentz boost.)

(v) Show that the wave equation is Lorentz-covariant. That is, if u is a solution to the wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^4$ (allowing negative times), then for every Lorentz transformation Λ, the function $v(t,x) := u(\Lambda(t,x))$ is also a solution.

Exercise 3. Let $S := \{(\phi(x), x) : x \in \mathbb{R}^3\} \subset \mathbb{R}^4$ be a smooth hypersurface (i.e. $\phi \in C^\infty(\mathbb{R}^3)$). The Cauchy problem for the wave equation with initial surface S is:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^4 \\ u = g \quad u_t = h & \text{on } S\,. \end{cases}$$

We say that S is space-like if $1 - |\nabla\phi|^2 > 0$ on $\mathbb{R}^3$.
Show that the Cauchy problem for the wave equation with the space-like initial surface $S = \{(t, x) \in \mathbb{R}^4 : t = ax_1\}$, $0 < a < 1$, is equivalent to the initial-value problem (i.e. when $S = \{(t, x) \in \mathbb{R}^4 : t = 0\}$).
Hint: Use a Lorentz transformation.

Exercise 4. Let $n = 3$ or $n = 2$, and let $u \in C^2([0, \infty) \times \mathbb{R}^n)$ be the solution of

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } (0, \infty) \times \mathbb{R}^n, \\ u = 0, \quad u_t = h & \text{in } \{t = 0\} \times \mathbb{R}^n, \end{cases}$$

with $h \in C_c^2(\mathbb{R}^n)$, given by Kirchhoff's formula or Poisson's formula, respectively.

(i) Let $\alpha \in (0, 1)$. Prove that there exist constants $C_3, C_2 > 0$, depending on the support of h (and on α in the second case) such that for all $t > 0$

$$\sup_{x\in\mathbb{R}^3} |u(t, x)| \le \frac{C_3}{t} \sup_{\mathbb{R}^3} |h| \quad (n = 3), \qquad \sup_{x\in B_{\alpha t}(0)} |u(t, x)| \le \frac{C_2}{t} \sup_{\mathbb{R}^2} |h| \quad (n = 2).$$

(ii) For $n = 3$, prove that

$$\sup_{x\in\mathbb{R}^3} |u(t, x)| \le \frac{1}{4\pi} \min\left(\frac{1}{t}\|Dh\|_{L^1(\mathbb{R}^3)}, \|D^2h\|_{L^1(\mathbb{R}^3)}\right), \quad t > 0.$$

Hint: For h with compact support, use that $h(x + tz) = -\int_t^\infty \partial_s h(x + sz)\, ds$.

You can drop your homework solutions until **Monday, January 16** at **16 o'clock** into the appropriate letterbox on the first floor near the library.