LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK SHEET 10

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Exercise 1 (Semigroup properties; 5 points)**.** For $t > 0$ we define $T_t : C_c^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ via

$$C_c^\infty(\mathbb{R}^n) \ni u_0 \mapsto T_t(u_0)(x) := u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) u_0(y)\,\mathrm{d}y,$$

where $u(t,x)$ is the exponentially decaying solution of the Heat Equation on $\mathbb{R}^n$ with initial data u_0 (which is unique, see lecture) and Φ is the fundamental solution of the Heat Equation. Prove the follwing:

(a) $T_t(u_0) \in L^\infty(\mathbb{R}^n)$ and hence T_t is well defined.

(b) $T_{t+s} = T_t \circ T_s$ for $s, t > 0$.

(c) For $u_0 \in C_c^\infty(\mathbb{R}^n)$ we have $\|T_t(u_0) - u_0\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $t \searrow 0$.

Exercise 2 (Energy methods, part 1; 5 points)**.** Let $U \subset \mathbb{R}^n$ be open, bounded and with C^1-boundary, $c \in \mathbb{R}$ and $a \geq 0$. Let moreover $f \in C^0((0,\infty) \times U)$ and $g \in C^0(U)$. Prove, via energy methods, the uniqueness of the classical solution $u \in C^{2,1}([0,\infty) \times \overline{U})$ of the initial value problem

$$\begin{cases} u_t - \Delta u + c^2 u = f & \text{in } (0,\infty) \times U \\ au + \frac{\partial u}{\partial \nu} = 0 & \text{on } [0,\infty) \times \partial U \\ u(0,\cdot) = g & \text{on } U \end{cases}, \tag{1}$$

where $\frac{\partial u}{\partial \nu}$ is the outer normal derivative of u with respect to U.
Hint: Let u_1, u_2 be two solutions of (1). Then define $w := u_1 - u_2$ and consider the function $\mathbb{R}_{\geq 0} \ni t \mapsto e(t) := \int_U w^2(t,x)\,\mathrm{d}x$

Exercise 3 (Energy methods, part 2; 5 points)**.**

(a) Prove the Poincaré-inequality: Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then there exists a constant $C_P > 0$ such that for $u \in C^1(\overline{\Omega})$ subject to $u = 0$ on $\partial\Omega$ the following inequality holds:

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)}.$$

Hint: Let $R > 0$ with $\Omega \subset B_R(0)$ and $x := (x', x_n) := (x_1, ..., x_n) \in \Omega$ with $x' := (x_1, ..., x_{n-1})$. Then there exists $a < x_n$ such that $y := (x', a) \in \partial\Omega$ and $(x', s) \in \Omega$ for all $s \in (a, x_n]$. Now, prove the inequality

$$|u(x)| \leq \sqrt{x_n - a} \left(\int_a^{x_n} (u_{x_n})^2 (x', s)\,\mathrm{d}s \right)^{1/2}. \tag{2}$$

Then use $(a, x_n] \subset I_{x'} := \{s \in \mathbb{R} : \ (x', s) \in \Omega\} \subset (-R, R)$ to estimate the right hand side of (2). Integrate the inequality you obtained this way to obtain the Poincaré-inequality with constant $C_P \stackrel{1}{=} 2R$.

(b) Let $f \in C^0(\mathbb{R})$ and $M > 0$ such that $|f(z)| \le M|z|$ holds for all $z \in \mathbb{R}$. Moreover let $\Omega \subset \mathbb{R}^n$ be open, bounded and with C^1 boundary. Prove the following: There exists $M_0 > 0$ such that for $M < M_0$ and solutions $u \in C^{2,1}([0,\infty) \times \overline{\Omega})$ of the boundary value problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } (0,\infty) \times \Omega \\ u = 0 & \text{on } [0,\infty) \times \partial\Omega \end{cases}$$

for all $t \ge 0$ the estimate

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} \le e^{-\delta_M t} \|u(0,\cdot)\|^2_{L^2(\Omega)}$$

holds, where the constant $\delta_M > 0$ only depends on M and C_P.
Hint: Consider the function $\mathbb{R}_{\ge 0} \ni t \mapsto e(t) := \int_\Omega u^2(t,x)\,\mathrm{d}x$. *Use the Poincaré-inequality to prove that* $\dot{e}(t) \le -\delta_M e(t)$ *for* $t > 0$ *and a suitable constant* δ_M.

Exercise 4 (Black-Scholes Equation; 5 Points)**.** Let $\sigma, r, E, T \in \mathbb{R}$ with $\sigma, r, T > 0$. We consider the following boundary value problem:

$$\begin{cases} V_t(t,s) + \frac{1}{2}\sigma^2 s^2 V_{ss}(t,s) + rsV_s(t,s) - rV(t,s) = 0 & \text{for } (t,s) \in (0,T) \times (0,\infty) \\ V(T,s) - \max\{s-E, 0\} = 0 & \text{for } s \in (0,\infty) \\ V(t,0) = 0 & \text{for } t \in [0,T] \\ \lim_{s\to\infty} V(t,s)/s = 1 & \text{for } t \in [0,T] \end{cases} \tag{3}$$

Find a solution V for the above boundary value problem. To do so you can for instance proceed along the following points.

- Do a change of variables $(x,\tau) := \left(\log(\frac{s}{E}), \frac{1}{2}\sigma^2(T-t)\right)$ to transform the differential equation in (3). For the function

  $$v(\tau,x) := \frac{V(t,s)}{E} = \frac{V(T - \frac{2\tau}{\sigma^2}, Ee^x)}{E}$$

  the PDE obtained this way should read

  $$v_\tau(\tau,x) = v_{xx}(\tau,x) + (k_1 - 1)v_x(\tau,x) - k_1 v(\tau,x) \qquad \text{for } (\tau,x) \in (0, \sigma^2 T/2) \times \mathbb{R} \tag{4}$$

  for $k_1 := 2r/\sigma^2$. Also transform the boundary conditions in (3).

- Make the Ansatz $u(\tau,x) := e^{-\alpha x + \beta\tau} v(\tau,x)$ to transform, for suitable choices of α, β, the equation (4) into the Heat equation. Also rewrite the boundary conditions in terms of the function u.

- Use the solution formula from the lecture to solve the so obtained problem for the Heat Equation.

You can drop your homework solutions until **Monday, January 9** at **16 o'clock** into the appropriate letterbox on the first floor near the library.