Prof. Dr. Bachmann | PARTIAL DIFFERENTIAL EQUATIONS I | WS 2016/17
A. Dietlein, R. Schulte | HOMEWORK SHEET 9 | December 12, 2016

For the last exercise we'll need the following fact which you can use without proof. For $p > 1$ we define the smooth function

$$f(t) := \begin{cases} \exp(-t^{-p}) & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{1}$$

i.e. $f \in C^\infty(\mathbb{R})$. One can then prove that for fixed $p > 0$ there exists $\theta := \theta(p) \in (0,1)$ such that

$$|f^{(k)}(t)| \leq \frac{k!}{(\theta t)^k} \exp\left(-\frac{1}{2t^p}\right), \qquad t > 0.$$

Exercise 1 (Non-uniqueness of the initial value problem for the Heat Equation; 5 Points)**.** Let $p > 1$ and f the function defined in (1) above. We then define the function $g : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ via

$$g_p(t,x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(t)}{(2k)!} x^{2k}, \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2}$$

(a) Prove that g_p is well defined and that the series in (2) converges absolutely, $g_p \in C^\infty((0,\infty) \times \mathbb{R})$ holds and $\lim_{t\to 0} g_p(t,x) = 0$ for $x \in \mathbb{R}^n$.
Hint: For suitable $\theta(p)$ prove an estimate such as

$$\sum_{k=0}^{\infty} \left| \frac{f^{(k)}(t)}{(2k)!} x^{2k} \right| \leq \exp\left(\frac{|x|^2}{\theta t} - \frac{1}{2t^p}\right).$$

(b) Prove that there exist infinitely many solutions of the initial value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,\infty) \times \mathbb{R}, \\ u = 0 & \text{on } \{t = 0\} \times \mathbb{R}. \end{cases}$$

Exercise 2 (Weierstrauss' approximation Theorem; 5 Points)**.** Let $n = 1$ and Φ be the fundamental solution of the heat equation. Use properties of the convolution

$$u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y) f(y)\,\mathrm{d}y$$

to prove Weierstrauss' approximation theorem: A function $f \in C([a,b])$ can be approximated uniformly by polynomials. That is, there exists a sequence of polynomials p_j such that

$$\max_{x\in[a,b]} |f(x) - p_j(x)| \to 0 \quad \text{as } j \to \infty.$$

Hint: Define $f(x) = f(b)$ for $x > b$ and $f(x) = f(a)$ for $x < a$. You may use that in this case $u(t,x) \to f(x)$ as $t \to 0$ uniformly for $a \leq x \leq b$. Approximate $\Phi(t, x-y)$ by its truncated power series with respect to $x - y$.

Exercise 3 (Subsolution to the Heat Equation; 5 Points)**.** Let $U \subset \mathbb{R}^n$ be open, bounded, $T > 0$ and $U_T := (0, T] \times U$. We say $v \in \mathcal{C}^2(U_T) \cap C^0(\overline{U_T})$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T.$$

(i) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \int \int_{E(t,x;r)} v(s,y) \frac{|x-y|^2}{(t-s)^2} dyds$$

for all $E(t, x; r) \subset U_T$.

(ii) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove v is a subsolution.

(iii) Prove $v := |Du|^2 + u_t^2$ is a subsolution whenever $u \in C^3(U_T)$ solves the heat equation.

Exercise 4 (Comparison Principle; 5 Points)**.** Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary $\partial U \in C^1$, $T > 0$ and U_T defined as before.
Assume $u_1, u_2 \in C^2(U_T) \cap C^0(\overline{U_T})$ are solutions of the (nonlinear) initial/boundary value problem

$$\begin{aligned} \partial_t u_i(t,x) - \triangle u_i(t,x) &= f(t,x,u_i(t,x)) && \text{for all } (t,x) \in U_T \\ u_i|_{\partial' U_T} &= g_i, \end{aligned}$$

where $f \in C^0(U_T \times \mathbb{R})$ and $g_i \in C^0(\partial' U_T)$ for $i \in \{1, 2\}$.
Show that if $f(t, x, u_1(t, x)) \leq f(t, x, u_2(t, x))$ holds for all $(t, x) \in U_T$ and $g_1 \leq g_2$ holds on $\partial' U_T$, the solution satisfy $u_1 \leq u_2$.

Hint: Use the previous exercise to derive a maximum principle.

You can drop your homework solutions until **Monday, December 19** at **16 o'clock** into the appropriate letterbox on the first floor near the library.