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PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK SHEET 8

WS 2016/17
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The aim of the following two exercises is to apply the Fourier Transform – which is a powerful tool for analyzing linear constant coefficient differential equations on the whole space (i.e. on $\mathbb{R}^n$) – to a concrete problem: The Heat Equation. The first exercise establishes basic properties of the Fourier transform and is of preparatory nature. In the subsequent exercise we discuss an alternative approach towards finding the solution of the initial value problem for the Heat Equation on $\mathbb{R}^n$. For $u \in L^1(\mathbb{R}^n)$ we define the Fourier Transform $\mathcal{F}(u)$ and its inverse $\mathcal{F}^{-1}(u)$ as

$$\mathcal{F}(u)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(y)\,\mathrm{d}y, \qquad (x \in \mathbb{R}^n),$$
$$\mathcal{F}^{-1}(u)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(y)\,\mathrm{d}y, \qquad (x \in \mathbb{R}^n).$$

You can apply without proof Plancharel's formula: For $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ one has $\mathcal{F}(u), \mathcal{F}^{-1}(u) \in L^2(\mathbb{R}^n)$ and $\|u\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(u)\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}^{-1}u\|_{L^2(\mathbb{R}^n)}$.

Exercise 1 (Properties of the Fourier Transform; 8 points)**.** Let $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

(a) Assume that $\mathcal{F}(u), \mathcal{F}(v) \in L^2(\mathbb{R}^n)$. Show $\int_{\mathbb{R}^n} \overline{v(x)}u(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \overline{\mathcal{F}(v)(x)}\mathcal{F}(u)(x)\,\mathrm{d}x$. Here $\overline{\cdot}$ denotes complex conjugation.
Tipp: Apply Plancharel's formula for $u + \alpha v$ for suitable values of $\alpha \in \mathbb{C}$.

(b) Assume moreover that $u \in C^2(\mathbb{R}^n)$. Show that if for a multiindex $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2$ $D^\alpha u \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ holds, then $\mathcal{F}(D^\alpha u)(x) = (ix)^\alpha \mathcal{F}(u)(x)$ holds for all $x \in \mathbb{R}^n$.

(c) Show that the formula $\mathcal{F}(u * v) = (2\pi)^{n/2}\mathcal{F}(u)\mathcal{F}(v)$ holds.

(d) Show that if $\mathcal{F}(u) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $(\mathcal{F}^{-1} \circ \mathcal{F})(u) = u$.
Tipp: Prove for suitable functions v that $\int_{\mathbb{R}^n} (\mathcal{F}^{-1} \circ \mathcal{F})(u)(x)v(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} u(x)v(x)\,\mathrm{d}x$ holds.

Exercise 2 (Alternative approach for the Heat Equation on $\mathbb{R}^n$; 8 points)**.** Let $g \in C_c^2(\mathbb{R}^n)$. Find a classical solution $u \in C^{1,2}((0,\infty) \times \mathbb{R}^n) \cap C^0([0,\infty) \times \mathbb{R}^n)$ for the initial value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty) \times \mathbb{R}^n \\ u(0,\cdot) = g & \text{on } \{t = 0\} \times \mathbb{R}^n \end{cases} \tag{1}$$

by proceeding as follows:

(a) Assume there exists a classical solution u for (1) such that $u(t,\cdot) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ holds for all $t \geq 0$. Then u satisfies the equation

$$\begin{cases} \partial_t \mathcal{F}(u)(t,x) + |x|^2 \mathcal{F}(u)(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n \\ \mathcal{F}(u)(0,x) = \mathcal{F}(g)(x) & \text{for } (t,x) \in \{t=0\} \times \mathbb{R}^n \end{cases} \tag{2}$$

where $\mathcal{F}(u)$ is the Fourier Transform of $u(t,\cdot)$ on $\mathbb{R}^n$.

(b) Show that

$$u(t,\cdot) := \mathcal{F}^{-1}\left(e^{-t|\cdot|^2}\mathcal{F}(g)\right) = \frac{g * \mathcal{F}^{-1}(e^{-t|\cdot|^2})}{(2\pi)^{n/2}}$$

is a solution of (2).

(c) Prove that

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,\mathrm{d}y$$

holds for $(t,x) \in (0,\infty) \times \mathbb{R}^n$.
Hint: You can apply the following formula without proof:

$$\int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^2}\,\mathrm{d}y = \left(\frac{\pi}{t}\right)^{n/2} e^{-\frac{|x|^2}{4t}} \qquad x \in \mathbb{R}^n, t > 0.$$

Exercise 3 (Heat Equation with convection respectively source/sink; 4 points)**.**

(a) Solve the intial value problem for the Heat Equation with convection:

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u = 0 & \text{in } (0,\infty) \times \mathbb{R}^n \\ u(0,\cdot) = g(\cdot) & \text{on } \{t=0\} \times \mathbb{R}^n\,, \end{cases}$$

where $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $b \in \mathbb{R}^n$ is a constant. Write your solution in terms of an integral involving the Heat Kernel $\Phi(\cdot,\cdot)$.

(b) Similarly, solve the intial value problem for the Heat Equation with source/sink:

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } (0,\infty) \times \mathbb{R}^n \\ u(0,\cdot) = g(\cdot) & \text{on } \{t=0\} \times \mathbb{R}^n\,, \end{cases}$$

where $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $c \in \mathbb{R}$ is a constant.

Hint: Recall methods used previously for the Transport Equation

You can drop your homework solutions until **Monday, December 12** at **16 o'clock** into the appropriate letterbox on the first floor near the library.