Prof. Dr. Bachmann
A. Dietlein, R. Schulte

PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK SHEET 7

WS 2016/17
November 28, 2016

Exercise 1 (Variational problem for the p-Laplacian; 5 Points)**.** Let $p > 2$, $\Omega \subset \mathbb{R}^n$ be open, bounded and such that $\partial\Omega \in C^1$ and let $g \in C^0(\partial\Omega)$. A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is called a classical solution of the p-Laplace Dirichlet problem iff

$$\begin{cases} \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1}$$

holds. We define the associated variational integral

$$F(v) := \frac{1}{p}\int_\Omega |\nabla v(x)|^p \,\mathrm{d}x,$$

on the set $\mathcal{D}_F := \{v \in C^2(\overline{\Omega}) : v = g \text{ auf } \partial\Omega\}$. Prove:

(a) If $u \in \mathcal{D}_F$ is a minimizer of F, i.e. $F(u) = \min_{v\in\mathcal{D}_F} F(v)$, then u is a classical solution of (1).

(b) If $u \in C^2(\overline{\Omega})$ is a classical solution of (1) then $F(u) = \min_{v\in\mathcal{D}_F} F(v)$.

Exercise 2 (An energy estimate, Helmholtz equation; 5 Punkte)**.**

(a) Let $U \subseteq \mathbb{R}^n$ be open and $u, v \in L^2(U)$. Prove that for all $\varepsilon > 0$

$$\|uv\|_{L^1(U)} \leq \frac{1}{2}\left(\frac{1}{\varepsilon}\|u\|^2_{L^2(U)} + \varepsilon\|v\|^2_{L^2(U)}\right).$$

(b) Let $U \subset \mathbb{R}^n$ be open, bounded and such that $\partial U \in C^1$, and let $\lambda > 0$ and $f \in C^0(\overline{U})$ be given. Assume that $u \in C^2(\overline{U})$ satisfies the boundary value problem

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Prove that u then satisfies the inequality

$$\int_U |\nabla u(x)|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_U |u(x)|^2 \,\mathrm{d}x \leq \frac{1}{2\lambda}\int_U |f(x)|^2 \,\mathrm{d}x$$

For the followng exercise we need some auxiliary notation. Let $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$ *and* $X := \{u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta\}$, $\alpha, \beta \in \mathbb{R}$. *For a function* $u \in X$ *we consider* $I(u) := \int_a^b f(x, u(x), u'(x))\,dx$ *and the minimization problem*

$$m := \inf_{u \in X} I(u) \tag{2}$$

of the functional I *over the set* X. *We say that* (2) *has a minimizer* u *iff there exists* $u \in X$ *such that* $I(u) = m$.

Exercise 3 (The Euler-Lagrange equation; 5+5 Points)**.**

(a) If (2) admits a minimizer $\hat{u} \in X \cap C^2([a,b])$ then

$$\frac{d}{dx}\left(f_\xi(x, \hat{u}(x), \hat{u}'(x))\right) = f_u(x, \hat{u}(x), \hat{u}'(x)) \qquad (x \in (a,b)). \tag{3}$$

Here, for $f = f(x,u,\xi)$, we denoted by $f_u = \partial_u f$ and $f_\xi = \partial_\xi f$ the partial derivatives of f with respect to the second respectively third variable.

(b) Assume in addition that, for every fixed $x \in [a,b]$, the function $(u,\xi) \mapsto f(x,u,\xi)$ is convex. Under this additional assumption, if a function $\hat{u} \in X \cap C^2([a,b])$ satisfies the equation (3) then $\hat{u}$ is a minimizer of (2), i.e. $I(\hat{u}) = m$.
Hint: Assume $\hat{u}$ *is as above. You may use without proof that, because* $(u,\xi) \mapsto f(x,u,\xi)$ *is convex for every* $x \in [a,b]$, *the inequality*

$$f(x,u,u') \geq f(x,\hat{u},\hat{u}') + f_u(x,\hat{u},\hat{u}')(u - \hat{u}) + f_\xi(x,\hat{u},\hat{u}')(u' - \hat{u}')$$

holds for every $u \in X$, *where we wrote* u *for* $u(x)$ *and* u' *for* $u'(x)$ *(and used similar shortcuts for* $\hat{u}$*).*

(c) Now we even assume that, for every fixed $x \in [a,b]$, the function $(u,\xi) \mapsto f(x,u,\xi)$ is strictly convex. Then there exists at most one minimizer of (2), i.e. the minimizer is unique if it exists.
Hint: Let u, v *be two minimizers of* (2). *Consider* $w := 1/2(u+v)$ *to prove that* $u = v$ *holds.*

You can drop your homework solutions until **Monday, December 5** at **16 o'clock** into the appropriate letterbox on the first floor near the library.