Prof. Dr. Bachmann | Partial Differential Equations I | WS 2016/17
A. Dietlein, R. Schulte | Homework Sheet 6 | November 28, 2016

For the following exercise we recall the definition of a regular boundary (of some set in, say, $\mathbb{R}^n$) via local coordinates:

Definition. The boundary $\partial\Omega$ of an open and bounded set $\Omega \subset \mathbb{R}^n$ is C^k, $k \in \mathbb{N}$, if for every $y_0 \in \partial\Omega$ there exists $r > 0$ and a function $\gamma_{y_0} \in C^k(\mathbb{R}^{n-1})$ such that after suitable relabelling of all coordinates we have

$$\Omega \cap B_r(y_0) = \{x \in B_r(y_0) : \; x_n > \gamma_{y_0}(x_1, ..., x_{n-1})\}.$$

Exercise 1 (Dirichlet-regular sets; 5+5 Points)**.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Prove that $\partial\Omega$ is Dirichlet regular in case (at least) one of the following conditions is met:

(a) The set Ω is convex

(b) The boundary $\partial\Omega$ ist C^2

Hint: In both cases try to prove that $x_0 \in \partial\Omega$ meets the exterior ball condition. In part (a) w.l.o.g. you can assume that $x_0 = 0$ and $\mathbb{R}^n_+ \cap \Omega = \emptyset$ (why?). For part (b) similar reasoning yields that w.l.o.g. you can assume $\nu(x_0) = -e_n$, where $\nu(x_0)$ is the outer normal vector of Ω at x_0 and e_n denotes the unit vector in n-th direction. Then you can conclude $\nabla\gamma_{x_0}(x_0) = 0$, where γ_{x_0} denotes the local parametrization of $\partial\Omega$ around x_0. Finally, the exterior ball condition can be verified via a Taylor expansion of γ_{x_0}.

The aim of the following two exercises is to prove the counterexample for general solvability of the Dirichlet problem mentioned in the lecture.

Exercise 2 (Couterexample - A preparatory Lemma; 5 Punkte)**.** Let $\Omega \subset \mathbb{R}^n$ be non-empty, open and bounded and $T \subset \overline{\Omega}$ such that $\Omega \setminus T$ is open. Assume there exists a function u which satisfies the following properties:

- u is harmonic in $\Omega \setminus T$
- For all $x_0 \in \partial\Omega \setminus T$ we have
$$\lim_{\substack{x \in \Omega \setminus T \\ x \to x_0}} u(x) = 0$$
- There exists a harmonic function $w_T : \Omega \setminus T \to (0, \infty)$ such that for all $\xi \in \partial T \cap \Omega$ we have
$$\lim_{\substack{x \in \Omega \setminus T \\ x \to \xi}} \frac{|u(x)|}{w_T(x)} = 0$$

Prove that then $u \equiv 0$.

Exercise 3 (Counterexample; 5 Points).

(a) Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $\xi \in \overline{\Omega}$. Assume that the function $u \in C^0(\overline{\Omega} \setminus \{\xi\})$ is harmonic on $\Omega \setminus \{\xi\}$ and bounded such that $u|_{\partial\Omega\setminus\{\xi\}} \equiv 0$. Prove that then $u \equiv 0$ readily holds.

(b) Use part (a) and the example on p.44 of the hand-written lecture notes (you may use without proof the properties (i)–(iv)) to prove the following: There exists a bounded and open set $\Omega \subset \mathbb{R}^n$ and a function $g \in C^0(\partial\Omega)$ such that there is no solution of the corresponding Dirichlet problem, i.e. there exists no harmonic function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $u|_{\partial\Omega} = g$.

You can drop your homework solutions until **Monday, November 28** at **16 o'clock** into the appropriate letterbox on the first floor near the library.