Prof. Dr. Bachmann
A. Dietlein, R. Schulte

Partial Differential Equations I
Homework Sheet 5

WS 2016/17
November 22, 2016

Exercise 1 (Mirror principle; 5 Points)**.** Let $U \subset \mathbb{R}^n$ be open and symmetric with respect to the hyperplane $\partial\mathbb{R}^n_+$ in $\mathbb{R}^n$, and let $u \in C^0(U \cap \overline{\mathbb{R}^n_+}) \cap C^2(U \cap \mathbb{R}^n_+)$ be harmonic in $U \cap \mathbb{R}^n_+$ with $u = 0$ at $U \cap \partial\mathbb{R}^n_+$. Show, that there is an extension v of u onto U such that v is harmonic.

Hint: You may use without proof: A function $v \in C^0(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, is harmonic if and only if for all $x \in \Omega$ there exists a $r_x > 0$ such that $\overline{B_r(x)} \subset \Omega$ with $⨍_{B_r(x)} v(y)dy = v(x)$ for all $r \in (0, r_x]$.

Exercise 2 (boundary minimum principle)**.** Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^0(\Omega)$ such that $u^{-1}(\inf_{x\in\Omega}\{u(x)\})$ is open. Let $c \in \mathbb{R}$. For every sequence $(x_n)_n \subset \Omega$ with either $x_n \to x \in \partial\Omega$ or $|x_n| \to \infty$ for $n \to \infty$ let be $\liminf_n u(x_n) \geq c > -\infty$. Show that $u(x) \geq c$ holds for all $x \in \Omega$.

Hint: Consider a minimizing sequence $(x_n)_n \in \Omega$, i.e. $u(x_n) \to \inf_{x\in\Omega} u(x)$, and discuss possible subsequences.

Exercise 3 (Uniform Equicontinuity of Harmonic Functions; 5 Points)**.** Let $\Omega \in \mathbb{R}^n$ be open and let U be a set of harmonic functions on Ω with are uniformly bounded, i.e. there exists a $M > 0$ such that for all $u \in U$ and all $x \in \Omega$ holds $|u(x)| \leq M$. Let $\Omega' \subset\subset \Omega$. Show, that all $u \in U$ restricted to Ω' are uniformly equicontinuous, i.e. for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in \Omega'$ with $|x - y| < \delta$ anf for all $u \in U$ holds $|u(x) - u(y)| < \varepsilon$.

Exercise 4 (Superharmonic Functions; 5 Points)**.** Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega)$. Show that the following statements are equivalent:

(i) $H_B(u) \leq u$ for any ball $B \subset\subset \Omega$,

(ii) $\Delta u \leq 0$.

Hint: Use the minium principle.

You can drop your homework solutions until **Monday, November 21** at **16 o'clock** into the appropriate letterbox on the first floor near the library.