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PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK SHEET 3

WS 2016/17
November 3, 2016

Exercise 1 (Mean value formula for boundary values; 5 Points)**.** Let $n \geq 3, r > 0$, $f \in C^0(\overline{B_r(0)})$. Let $g \in C^0(\partial B_r(0))$ and let $u \in C^2(\overline{B_r(0)})$ be a solution of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } B_r(0) \\ u = g & \text{on } \partial B_r(0)\,. \end{cases}$$

Show, that

$$u(0) = \fint_{\partial B_r(0)} g(x)\,\mathrm{d}S(x) + \frac{1}{n(n-2)\alpha_n}\int_{B_r(0)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f(x)\,\mathrm{d}x\,. \tag{1}$$

Hint: Use the proof of the mean value formula, to show that for all $\varepsilon \in (0, r)$:

$$\fint_{\partial B_r(0)} g(x)\,\mathrm{d}S(x) - \fint_{\partial B_\varepsilon(0)} u(x)\,\mathrm{d}S(x) = \frac{-1}{n\alpha_n}\int_\varepsilon^r \frac{1}{s^{n-1}}\left(\int_{B_s(0)} f(x)\,\mathrm{d}x\right)\mathrm{d}s$$

To get equation (1) *you can use partial integration.*

Exercise 2 (A-priori estimate for derivatives of a harmonic function; 5 Points)**.** Let $U \subseteq \mathbb{R}^n$ an open set and $-\infty \leq m \leq M \leq \infty$. Let u be a harmonic function on U with $m \leq u \leq M$. Show that for all $r > 0$ and $x \in U$ with $\overline{B_r(x)} \subset U$ and for all $i \in \{1, \cdots, n\}$:

$$|u_{x_i}(x)| \leq \frac{n(M-m)}{2r}.$$

Exercise 3 (Extension of bounded harmonic functions, part 1; 5 Points)**.** Let $n \geq 2$. Let $u \in L^\infty(B_1(0)) \cap C^2(B_1(0)\backslash\{0\})$ be harmonic on $B_1(0)\backslash\{0\}$.

(a) Let $n \geq 3$. Show that $u \in C^2(B_1(0))$ and u is harmonic on $B_1(0)$.

(b) Show that for all $r \in (0, 1)$ the integral $\fint_{\partial B_r(0)} u(x)\,\mathrm{d}S(x)$ is constant.

Hint: For (a) show that $0 = \int_{B_1(0)} u(x)\Delta\phi(x)\,\mathrm{d}x$ for all $\phi \in C_c^\infty(B_1(0))$. To achive this result, separate ϕ into $\phi = \phi\eta_\varepsilon + \phi(1-\eta_\varepsilon)$ for a suitable $\eta_\varepsilon \in C_c^\infty(B_{2\varepsilon}(0))$ such that $\eta_\varepsilon|_{B_\varepsilon(0)} = 1$. Apply the compactness theorem for harmonic functions onto functions of the form $\phi_\varepsilon * u$, where ϕ_ε is a smooth mollifier.
For (b) show that $\int_{\partial B_r(0)} \nabla u(x) \cdot \frac{x}{|x|}\,\mathrm{d}S(x) = const.$.

Exercise 4 (Extension of bounded harmonic functions, part 2; 5 Points)**.** Let $n = 2$. Let $u \in L^\infty(B_1(0)) \cap C^2(B_1(0)\backslash\{0\})$ be harmonic on $B_1(0)\backslash\{0\}$.

(a) Show, that there exists a constat $C \in \mathbb{R}$ such that for all $\phi \in C_0^\infty(B_1(0))$ and for all $r > 0$ is

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B_r(0)} \phi(\varepsilon x) u(\varepsilon x)\, \mathrm{d}S(x) = 2\pi r C \phi(0).$$

(b) Show that $u \in C^2(B_1(0))$ and u is harmonic on $B_1(0)$.

Hint: For (a) use Exercise 3 (b). You can assume w.l.o.g. that $u \geq 0$.
For (b) proceed as in Exercise 3 (a). This time, choose a radially symmetric cut-off function $\eta(\cdot) = f(|\cdot|)$. *You may use the results of Tutorial Sheet 2 Exercise 3. Use (a) to determine* $\lim_{\varepsilon\downarrow 0} \int \Delta\eta_\varepsilon(x)\phi(x)u(x)\,\mathrm{d}x$.

You can drop your homework solutions until **Monday, November 6** at **16 o'clock** into the appropriate letterbox on the first floor near the library.