Prof. Dr. Bachmann
A. Dietlein, R. Schulte

PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK SHEET 2

WS 2016/17
November 22, 2016

Exercise 1 (An explicitely solvable Initial Value Problem; 5 Points)**.** Let $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $g \in C^1(\mathbb{R}^n)$.
Determine an explicit formula for the solution for the initial value problem:

$$u_t(x,t) + b \cdot Du(x,t) + cu(x,t) = 0, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty),$$
$$u(x,0) = g(x), \quad x \in \mathbb{R}^n.$$

Exercise 2 (Weak Solution; 5 Points)**.** Let $\Omega := \mathbb{R}^n \times (0,\infty)$, $g \in L^\infty(\mathbb{R}^n)$ and $b \in \mathbb{R}^n$. Show that the function $u \colon \overline{\Omega} \to \mathbb{R}$, $u(x,t) := g(x - bt)$ is a weak solution of the Initial Value Problem

$$\begin{cases} u_t + b \cdot D_x u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}\,. \end{cases} \tag{1}$$

Here we say that u is a *weak solution* of (1) precisely when for all $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \Big(\phi_t(x,t) + b \cdot D_x\phi(x,t)\Big) u(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

and $u(x,0) = g(x)$ for almost all $x \in \mathbb{R}^n$.

Hint: Use the change-of-variables $(x,t) \mapsto (y,t) := (x - bt, t)$.

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space and $A \in \mathcal{A}$ with $\mu(A) < \infty$. For any $f \in L^1(X)$ the mean value integral is defined as

$$⨍_A f\,\mathrm{d}\mu := \frac{1}{\mu(A)} \int_A f\,\mathrm{d}\mu.$$

Exercise 3 (5 Points)**.** For $x \in \mathbb{R}^n$ and $R > 0$ let $u \in C(B_R(x))$. For $r \in (0,R)$ we define the functions

$$\phi(r) := ⨍_{\partial B_r(x)} u(y)\,\mathrm{d}S(y) = ⨍_{\partial B_1(0)} u(x + rz)\,\mathrm{d}S(z)$$
$$\psi(r) := \int_{B_r(x)} u(y)\,\mathrm{d}y\,.$$

(a) Show that $\phi \in C((0,R))$.

(b) Show that $\psi \in C^1((0,R))$ with derivative

$$\psi'(r) = \int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y) \quad \Big(= \omega_n r^{n-1}\phi(r)\Big).$$

(c) Show that under the additional assumption $u \in C^1(\overline{B_R(x)})$ we have $\phi \in C^1((0,R))$ with derivative

$$\phi'(r) = \fint_{\partial B_1(0)} \nabla u(x+rz)\cdot z\,\mathrm{d}S(z).$$

Hint: You may use that $\frac{d}{dr}(u(x+rz)) = \nabla u(x+rz)\cdot z$ together with the theorem of dominated convergence.

Exercise 4 (Picard-Lindelöf Theorem; 5 Points)**.** Let $y_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $a,\ r,\ M > 0$. Let $f : [t_0, t_0+a]\times\mathbb{R}^n \to \mathbb{R}^n$ be a continuous function on $R := [t_0, t_0+a]\times\{y : \ |y-y_0| \le r\}$, with the following properties:

1. $|f|$ is bounded by M on R
2. $f(t,\cdot)$ is uniformly Lipschitz continuous for all $t \in [t_0, t_0+a]$, i.e. there exists a constalt L>0 such that for all $(t,x),(t,y) \in R$ with $t \in [t_0,t_0+a]$ the estimate $|f(t,x)-f(t,y)| \le L|x-y|$ holds.

Define $\alpha := \min\{a, r/M\}$.

(a) We inductively define the sequence of functions $\{y_n\}_{n\in\mathbb{N}} \subset C^1([t_0, t_0+\alpha])$ via

$$y_n(t) = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,\mathrm{d}s,$$

where the function y_0 is defined as the constant function $y_0(\cdot) = y_0$. Show that for all $n \in \mathbb{N}$ and for all $t \in [t_0, t_0+\alpha]$ is

$$|y_{n-1}(t) - y_n(t)| \le M\frac{L^{n-1}(t-t_0)^n}{n!}.$$

(b) Show that $\{y_n\}_{n\in\mathbb{N}}$ converges uniformly to a function $y \in C^1([t_0, t_0+\alpha])$, with

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1}$$

(c) Show that the solution of the initial value problem (1) on $[t_0, t_0+\alpha]$ is unique. *Hint: Assume z is another solution. Find an estimate of the function $|z-y_n|$ similar to the one derived in (a).*

You can drop your homework solutions until **Monday, October 31** at **16 o'clock** into the appropriate letterbox on the first floor near the library.