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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 11

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T 1. Show that the solution of the initial value problem

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ u(0,x) = 0 & \text{for } x \in \mathbb{R}^3 \\ u_t(0,x) = x_1^2 + x_1 x_2 + x_3^2 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

is given by $u(t,x) := t(x_1^2 + x_1 x_2 + x_3^2) + \frac{2}{3}t^3$

T 2. Let $g \in C_c^2(\mathbb{R})$ and $h \in C_c^1(\mathbb{R})$ be two compactly supported functions. Let u be the solution of the initial value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } (0,\infty) \times \mathbb{R} \\ u(0,\cdot) = g & \text{on } \mathbb{R} \\ u_t(0,\cdot) = h & \text{on } \mathbb{R} \end{cases}$$

given by d'Alemberts formula. As in the lecture we define the *kinetic energy* as $k(t) := \frac{1}{2}\int_{\mathbb{R}} u_t(t,x)^2\,dx$ and the *potential energy* as $p(t) := \frac{1}{2}\int_{\mathbb{R}} u_x(t,x)^2\,dx$. Prove *Equipartition of energy*: $k(t) = p(t)$ for all large enough times (i.e. there exists t_0 such that for all $t \geq t_0$ we have $k(t) = p(t)$).

T 3. Let $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$ be two rotationally symmetric functions (i.e., for instance, $g(x) = g(y)$ whenever $|x| = |y|$). Directly derive (without applying the solution formula for $d = 3$ dimensions) a solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } (0,\infty) \times \mathbb{R}^3 \\ u(0,\cdot) = g & \text{on } \mathbb{R}^3 \\ u_t(0,\cdot) = h & \text{on } \mathbb{R}^3 \end{cases} \tag{1}$$

by arguing as follows: Assume that u is a rotationally symmetric solution of (1). Define $\tilde{u} : (0,\infty) \times (0,\infty) \to \mathbb{R}$ via $\tilde{u}(t,r) := u(t,x)$ for some $x \in \mathbb{R}^3$ such that $|x| = r$. Set up the boundary value problem which is solved by $\tilde{u}$.