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Partial Differential Equations I
Tutorial Sheet 8

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T 1. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $T > 0$, $g \in C^0(\partial'\Omega_T)$ with $0 \leq g \leq 1$ and let $u \in C^2((0,T) \times \Omega) \cap C^0([0,T] \times \bar{\Omega})$ be a solution of

$$\begin{cases} u_t - \Delta u = \sin^2(u) & \text{in } (0,T] \times \Omega \\ u = g & \text{auf } \partial' U_T. \end{cases}$$

Show, that $0 \leq u(t,x) \leq t + 1$ for all $(t,x) \in [0,T] \times \bar{\Omega}$.

T 2 (Heat Equation on $\mathbb{R}_+$)**.** Let $u_0 \in C^0([0,\infty)) \cap L^\infty([0,\infty))$ with $u_0(0) = 0$. Determine an explicite expresion for the solution $u \in C^2([0,\infty) \times [0,\infty))$ of the following initial/boundary valute problem

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0 & \forall (t,x) \in (0,\infty) \times (0,\infty), \\ u(t,0) = 0 & \forall t \in [0,\infty), \\ u(0,x) = u_0(x) & \forall x \in (0,\infty). \end{cases}$$

Hint: Extend u to $[0,\infty) \times \mathbb{R}$ by reflexion.

Definition 1. The Schwatz space $S(\mathbb{R}^n)$ or space of rapidly decreasing functions on $\mathbb{R}^n$ is defined as

$$S(\mathbb{R}^n) := \left\{ \phi \in C^\infty(\mathbb{R}^n) : \ \forall \alpha, \beta \in \mathbb{N}_0^n \text{ is } \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta \phi(x) \right| < \infty \right\}.$$

T 3. Let $f \in L^p(\mathbb{R}^n)$, $\phi \in S(\mathbb{R}^n)$ with $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x = 1$. For all $\epsilon > 0$ define $\phi_\epsilon(\cdot) := \epsilon^{-n}\phi(\epsilon^{-1}\cdot)$.
Show, that for

$$\|f * \phi_\epsilon - f\|_p := \left(\int_{\mathbb{R}^n} (f * \phi_\epsilon - f)^p \,\mathrm{d}x \right)^{1/p} \longrightarrow 0 \qquad \text{for } \epsilon \to 0.$$

Proceed as following

(i) Prove the claim for $f \in C^0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

(ii) Prove that $\|\phi_\epsilon * f\|_p \leq \|f\|_p$.

(iii) Use that $C^0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ lies dense in $L^p(\mathbb{R}^n)$.