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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 6

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T 1 (Robin boundary condition)**.** Let Ω be an open set such that the divergence theorem holds. Let a, h be functions on the boundary $\partial\Omega$ with $a > 0$ and let $f \in C^0(\Omega)$. $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is called a solution of the Robin Problem if

$$\begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ \frac{\partial}{\partial n} u(x) + a(x)u(x) = h(x) & x \in \partial\Omega. \end{cases}$$

Show uniquness for the Robin problem.

Definition 1 (Green's Function)**.** Let $\Omega \subset\subset \mathbb{R}^n$ open. A function $G : \overline{\Omega} \times \Omega \to \mathbb{R}$ is a Green's function for Ω if for any $y \in \Omega$

(i) $x \mapsto G(x, y) - \Phi(x, y)$ is continuous on $\overline{\Omega}$ and harmonic on Ω,

(ii) $G(x, y) = 0$ for all $x \in \partial\Omega$, $y \in \Omega$.

The fundamental solution of $-\Delta$ is given by

$$\Phi_n(x, y) := \begin{cases} \frac{1}{(n-2)\omega_n} |x - y|^{2-n} & \text{for } n \geq 3 \\ -\frac{1}{2\pi} \ln|x - y| & \text{for } n = 2 \end{cases}$$

T 2. Determine the Green's function for

(i) the half-space $\Omega = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : \; x_n > 0\}$,

(ii) the complement of a ball $\Omega = \overline{B_R(0)}^{\,c}$,

(iii) the half-ball $\Omega = B_R(0) \cap \mathbb{R}^n_+$.

Hints for the respective subtasks:

(i) Consider the mirroring-transformation $\tau_n : \mathbb{R}^n \to \mathbb{R}^n$, $\tau_n(x_1, \ldots, x_n) := (x_1, \ldots, x_{n-1}, -x_n)$.

(ii) Let $\Omega \subset \mathbb{R}^n \backslash \{0\}$ *be an open set and let* $\tilde{\Omega} := \tau_R(\Omega)$ *the image of* Ω *under the transformation* $\tau_R(x) := \left(\frac{R}{|x|}\right)^2$. *Consider the Kelvin-transformation* $K : C^2(\Omega) \to C^2(\tilde{\Omega})$ *with*

$$K[u](x) := \left(\frac{R}{|x|}\right)^{n-2} u(\tau_R(x)).$$

You may use without proof that $\Delta K[u](x) = \frac{1}{R^2} \left(\frac{R}{|x|}\right)^{n+2} (\Delta u)(\tau_R(x))$.

(iii) Combine the results of the two previous points.