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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 5

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We define $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ and, for fixed n, $|\partial B(0,1)|$ denotes the surface area of the unit sphere in $\mathbb{R}^n$.

T 1. Let $g \in C^0(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and define the function $u : \mathbb{R}^n \to \mathbb{R}$ via

$$u(x) := \frac{2x_n}{|\partial B(0,1)|} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} \mathrm{d}y \qquad (x \in \mathbb{R}^n_+).$$

Prove that u then is a solution to the Dirichlet problem on the half space $\mathbb{R}^n_+$ with boundary value g on $\partial\mathbb{R}^n_+$. More precisely, prove that

(a) $u \in C^2(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(b) $\Delta u = 0$ in $\mathbb{R}^n_+$,

(c) For each point $x_0 \in \partial\mathbb{R}^n_+$ we have

$$\lim_{\substack{x \in \mathbb{R}^n_+ \\ x \to x_0}} u(x) = g(x_0).$$

Hint: You may use, and prove at the very end if time permits, that

$$\frac{2x_n}{|\partial B(0,1)|} \int_{\partial \mathbb{R}^n_+} \frac{1}{|x-y|^n} \mathrm{d}y = 1 \qquad (x \in \mathbb{R}^n_+).$$

T 2 (classical Harnack inequality)**.** Let u be a non-negative harmonic function on $B_R(0)$, $R > 0$. Show, that for all $x \in B_R(0)$ the following inequality holds

$$\frac{R - |x|}{(R + |x|)^{n-1}} R^{n-2} u(0) \le u(x) \le \frac{R + |x|}{(R - |x|)^{n-1}} R^{n-2} u(0).$$

T 3. Let $\Omega \subseteq \mathbb{R}^n$ open and $u \in C^0(\Omega)$. Prove the equivalence of the following:

(i) u is superharmonic in Ω

(ii) $\int_{\partial B_r(x)} u(y)\, \mathrm{d}S(y) \le r^{n-1} \omega_n u(x)$ for all balls $B_r(x) \subset\subset \Omega$

(iii) $\int_{B_r(x)} (u(y) - u(x))\, \mathrm{d}y \le 0$ for all balls $B_r(x) \subset\subset \Omega$.