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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 4

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T 1. Let $u \in C^2(\mathbb{R}^n)$ be harmonic and such that

$$\int_{\mathbb{R}^n} |u(x)|^p < \infty.$$

(a) Prove that for any function $\psi \in C_c^\infty(\mathbb{R}^n)$ the function $\psi * u \in C^\infty(\mathbb{R}^n)$ is bounded.

(b) Conclude that $u \equiv 0$ (i.e. $u(x) = 0$ for all $x \in \mathbb{R}^n$).

T 2. (a) Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial\Omega)$. Prove that there exists a constant C, which only depends on Ω, such that

$$\max_{x\in\overline{\Omega}} |u(x)| \leq C\left(\max_{x\in\partial\Omega} |g(x)| + \max_{x\in\overline{\Omega}} |f(x)| \right)$$

Hint: As an intermediate step you can prove that $-\Delta\Big(u(x) + \frac{|x|^2}{2n}\lambda\Big) \leq 0$ *for* $\lambda := \max_{x\in\overline{\Omega}} |f(x)|$.

(b) Prove that the solution of (1) depends continuously on the 'data' f and g. More precisely, prove that there exists a constant C, which only depends on Ω, such that for solutions $u_i \in C^2(\Omega) \cap C^0(\overline{\Omega})$,$i = 1, 2$, of

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega \\ u_i = g_i & \text{on } \partial\Omega\,, \end{cases}$$

where $f_i \in C(\overline{\Omega})$ and $g_i \in C(\partial\Omega)$ the following estimate holds:

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \leq C\Big(\|g_1 - g_2\|_{L^\infty(\partial\Omega)} + \|f_1 - f_2\|_{L^\infty(\Omega)}\Big).$$

For the next problem we define $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ and, for fixed n, $|\partial B(0,1)|$ denotes the surface area of the unit sphere in $\mathbb{R}^n$.

T 3. Let $g \in C^0(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and define the function $u : \mathbb{R}^n \to \mathbb{R}$ via

$$u(x) := \frac{2x_n}{|\partial B(0,1)|} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} \mathrm{d}y \qquad (x \in \mathbb{R}^n_+).$$

Prove that u then is a solution to the Dirichlet problem on the half space $\mathbb{R}^n_+$ with boundary value g on $\partial\mathbb{R}^n_+$. More precisely, prove that

(a) $u \in C^2(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(b) $\Delta u = 0$ in $\mathbb{R}^n_+$,

(c) For each point $x_0 \in \partial\mathbb{R}^n_+$ we have $\lim_{\substack{x \in \mathbb{R}^n_+ \\ x \to x_0}} u(x) = g(x_0)$.

Hint: You may use, and prove at the very end if time permits, that

$$\frac{2x_n}{|\partial B(0,1)|} \int_{\partial \mathbb{R}^n_+} \frac{1}{|x-y|^n} \mathrm{d}y = 1 \qquad (x \in \mathbb{R}^n_+).$$