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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 3

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T 1. Let $\Omega \subset \mathbb{R}^n$ be an non-empty, bounded domain and let u be harmonic on Ω such that its gradient $\nabla u : \Omega \to \mathbb{R}^n$ can be extended continuously to $\overline{\Omega}$. Show that the function $|\nabla u|^2$ attains its maximum at the boundary $\partial\Omega$ of Ω.

T 2. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain, let $f_1,\ f_2 \in C(\Omega)$ be real-valued functions with $f_1 \leq f_2$ and let $g_1,\ g_2 \in C(\partial\Omega)$ be real-valued functions with $g_1 \geq g_2$ (i.e. $g_1(x) \geq g_2(x)$ for all $x \in \partial\Omega$). For $i \in \{1, 2\}$ let $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to the Dirichlet boundary problem

$$\begin{cases} \Delta u_i(x) = f_i(x), & x \in \Omega \\ u_i(x) = g_i(x), & x \in \partial\Omega. \end{cases}$$

Show that this implies $u_1 \geq u_2$.

T 3. For $n \geq 2$ define the functions $\Phi_n : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ by

$$\Phi_n(x) := \begin{cases} -\frac{1}{2\pi} \ln|x|, & n = 2 \\ \frac{1}{(n-2)\omega_n} \cdot \frac{1}{|x|^{n-2}}, & n > 2 \end{cases}$$

(a) Show that $\Delta\Phi_n(x) = 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$.

(b) Show that for $n \geq 2$ the derivatives of Φ_n satisfy the estimates

$$|\nabla\Phi_n(x)| \leq \frac{C_n}{|x|^{n-1}}, \qquad |\partial_i\partial_j\Phi_n(x)| \leq \frac{C_n}{|x|^n}$$

for all x in $\mathbb{R}^n \backslash \{0\}$, where the constant $0 < C_n$ is independent of x.

(c) Verify that $\Phi_n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

T 4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain on which the divergence theorem holds. Let us assume that for $u \in C^2(\overline{\Omega})$

$$\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x = \min \left\{ \int_\Omega |\nabla v(x)|^2 \, \mathrm{d}x : \ v \in C^1(\Omega) \cap C(\overline{\Omega}) \text{ with } \ v|_{\partial\Omega} = u|_{\partial\Omega} \right\}.$$

holds. Show that u then is a weak solution to $\Delta u = 0$, i.e. for all $\psi \in C_c^\infty(\Omega)$ we have $\int_\Omega \Delta\psi(x) u(x) \, \mathrm{d}x = 0$.

Hint: For any $\psi \in C_c^\infty(\Omega)$ consider the function

$$\mathbb{R} \ni t \mapsto g(t) := \int_\Omega |\nabla(u(x) + t\psi(x))|^2 \, \mathrm{d}x.$$