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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 2

WS 2016/17
October 28, 2016

Exercise 1. (a) Let $U \subseteq \mathbb{R}^n$ open and $u : U \to \mathbb{R}$ harmonic. Show that $\varphi : U \to \mathbb{R}, x \mapsto \phi(x) := x \cdot (\nabla u)(x)$ is harmonic as well.

(b) Prove that $D^\alpha \Delta u = \Delta D^\alpha u$ and $p(D)\Delta u = \Delta p(D)u$ holds for all $u \in C^\infty(\mathbb{R}^n)$. Here we used the notation $p(D)u := \sum_{|\alpha|\le m} c_\alpha D^\alpha u$ with coefficients $c_\alpha \in \mathbb{R}$. Conclude that for a harmonic function u all partial derivatives $D^\alpha u$ and the functions $p(D)u$ are harmonic.

(c) Prove for $u \in C^2(\mathbb{R}^d)$ the formulas $\Delta(uv) = (\Delta u)v + 2\nabla u \cdot \nabla v + u(\Delta v)$ (product rule) and $\frac{\mathrm{d}}{\mathrm{d}r}|_{r=1} u(rx) = x \cdot \nabla u(x)$ (Euler's relation).

(d) Let $s, t \in \mathbb{R}$ and $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ homogeneous of degree s (i.e. $u(\lambda x) = \lambda^s u(x)$ holds for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$). Prove that

$$\Delta(|x|^t u(x)) = t(n + t + 2s - 2)|x|^{t-2}u(x) + |x|^t \Delta u(x).$$

Hence a function u is harmonic and homogeneous of degree s exactly if $|\cdot|^{2-n-2s}u$ is harmonic and homogeneous of degree $2 - n - s$.

Exercise 2. Suppose $u \in C^\infty(\mathbb{R}^n)$ satisfies $\Delta u = 0$. Let O be an orthogonal $n \times n$ matrix and define

$$v(x) := u(Ox), \qquad (x \in \mathbb{R}^n).$$

Show that $\Delta v = 0$, i.e. Laplace's equation is rotation invariant.

Exercise 3. Suppose $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ has the form $u(x) = f(|x|)$, where $f \in C^2((0,\infty))$ (u is rotationally invariant). Show that

$$\Delta u(x) = f''(|x|) + \frac{n-1}{|x|} f'(|x|).$$

Use this to determine all rotationally invariant harmonic maps u on $\mathbb{R}^n \setminus \{0\}$.

Definition. Let $U \subseteq \mathbb{R}^n$ be an open set and let $v \in C^2(U)$. We call v subharmonic, if $-\Delta v(x) \le 0$ holds for all $x \in U$. Analogously we call v superharmonic, if $-\Delta v(x) \ge 0$ holds for all $x \in U$.

Exercise 4. (a) Let $v \in C^2(U)$ be in U. Show, that for all $\overline{B_r(x)} \subset U$ is

$$v(x) \le \fint_{B_r(x)} v(y)\ \mathrm{d}y.$$

(b) Let $\phi \in C^2(\mathbb{R})$ be convex and $u \in C^2(U)$ be harmonic on U. Show that the function $v := \phi \circ u$ defined on U is subharmonic.

(c) Let $u \in C^2(U)$ be harmonic. Show that the function $v := |Du|^2$ is subharmonic.