Homework Sheet 11

$\boxed{E1}$ Assume that fore some function $\alpha \in C^2(\mathbb{R}^+)$ and a delay fct. $\beta \in C^2(\mathbb{R}^+)$, $\beta \geq 0$, there ex. for all profiles $\phi \in C^2(\mathbb{R})$ solutions of the wave equation in $\mathbb{R} \times (\mathbb{R}^d \setminus \{0\})$ having the form

$$u(t,x) = \alpha(r)\,\phi(t-\beta(r))$$

Here $r = |x|$ and we assume $\beta(0)=0$. Show that this possible only if $d=1$ or $d=3$, and compute the form of the fct. α, β.

Proof. $\frac{d^2}{dt^2} u(t,x) = \alpha(r)\,\phi''(t-\beta(r))$

$$\Delta u(t,x) = \underbrace{\Delta(\alpha(r))}_{=\alpha''(r)+(d-1)\alpha'(r)\frac{1}{r}}\,\phi(t-\beta(r)) + 2\,\underbrace{\nabla(\alpha(r))}_{=\frac{x}{r}\alpha'(r)} \cdot \underbrace{\nabla(\phi(t-\beta(r)))}_{=-\phi'(t-\beta(r))\beta'(r)\frac{x}{r}} + \alpha(r)\,\underbrace{\Delta(\phi(t-\beta(r)))}_{=\phi''(t-\beta(r))(\beta'(r))^2-\phi'(t-\beta(r))\beta''(r)-(d-1)\phi'(t-\beta(r))\beta'(r)\frac{1}{r}}$$

$$\Rightarrow -\frac{d^2}{dt^2}u(t,x) + \Delta u(t,x) = \phi(t-\beta(r))\left(\alpha''(r) + (d-1)\alpha'(r)\tfrac{1}{r}\right)$$
$$+ \phi'(t-\beta(r))\left(2\alpha'(r)\beta'(r) - \alpha(r)\beta''(r) - \alpha(r)(d-1)\beta'(r)\tfrac{1}{r}\right)$$
$$+ \phi''(t-\beta(r))\left(\alpha(r)(\beta'(r))^2 - \alpha(r)\right) \stackrel{!}{=} 0$$

Choose $\phi \equiv$ const. $\Rightarrow$ $0 = \alpha''(r) + (d-1)\alpha'(r)\frac{1}{r}$ (I)

Choose $\phi' \equiv$ const. $\Rightarrow$ $0 = 2\alpha'(r)\beta'(r) + \alpha(r)\beta''(r) + (d-1)\alpha(r)\beta'(r)\frac{1}{r}$ (II)

Choose $\phi'' \equiv$ const. $\Rightarrow$ $0 = \alpha(r)(\beta'(r))^2 - \alpha(r)$ (III)

(III) $\Rightarrow$ $\beta'(r) = 1$ if $\alpha(r) \neq 0$.

(II) $\Rightarrow$ $0 = 2\alpha'(r) + (d-1)\alpha(r)\frac{1}{r}$

Let $0<a<b$ such that $\alpha|_{[a,b]} \neq 0 \Rightarrow \forall r \in [a,b]$: $\frac{d}{dr}\ln\alpha(r) = \frac{\alpha'(r)}{\alpha(r)} = \frac{1-d}{2}\frac{1}{r}$

$\Rightarrow \ln\alpha(r) - \ln\alpha(a) = \int_a^r \frac{d}{ds}\ln\alpha(s)\,ds = \int_a^r \frac{1-d}{2r} = \frac{1-d}{2}\ln r - \frac{1-d}{2}\ln a$

$\Rightarrow \alpha(r) = \text{Const.}\; r^{\frac{1-d}{2}}$

1. case $d=1$: $\Rightarrow$ $\alpha|_{[a,b]} = $ const., by iteration one gets $\alpha|_{\mathbb{R}^+} = $ Const, with $\beta'(r) = 1 \Rightarrow \beta(r) = r$ and all three equations are fulfilled.

2. case. $d \neq 1$: $\Rightarrow$ (I) $\quad 0 = \text{Const.}\left(\frac{1-d}{2}\left(\frac{1-d}{2}-1\right) + (d-1)\frac{1-d}{2}\right) r^{\frac{1-d}{2}-1}$ for $r \in [a,b]$

$\Rightarrow \quad 0 = \frac{1-d}{2}\left(\frac{1-d}{2}-1\right) + (d-1)\frac{1-d}{2} = \frac{-1}{4}(d-1)(d-3)$

$\Rightarrow$ only possible if $d=3$.

Choose $\alpha(r) :=$ const. $r^{\frac{1-d}{2}}$, $\beta(r) = r$ $\Rightarrow$ all three equations are fullfilled. $\square$

E2 Let $g = \mathrm{diag}(-1,1,1,1) \in \mathbb{R}^{4\times 4}$. A real 4×4 matrix $\Lambda \in \mathbb{R}^{4\times 4}$ is called a Lorentz transformation if and only if $\Lambda^T g \Lambda = g$, where Λ^T denotes the transpose of Λ.

(i) The product of two Lorentz transformations is also a Lorentz transformation.

Proof. $\Lambda_1, \Lambda_2 \in \mathbb{R}^{4\times 4}$, $\Lambda_i^T g \Lambda_i = g$ for $i \in \{1,2\}$ $\Rightarrow$ $(\Lambda_1\Lambda_2)^T g \Lambda_1 \Lambda_2 = \Lambda_2^T \underbrace{\Lambda_1^T g \Lambda_1}_{=g} \Lambda_2 = \Lambda_2^T g \Lambda_2 = g$

(ii) Every Lorentz. trans. is invertable and its inverse is a Lorentz trans., too.

Proof. Λ Lorentz trafo $\Rightarrow$ $g = \Lambda^T g \Lambda$ $\Rightarrow$ $\dim \ker \Lambda \leq \dim \ker \Lambda^T g \Lambda = \dim \ker g = 0$

$\Rightarrow \ker \Lambda = \{0\} \Rightarrow \Lambda$ invertible $\Rightarrow (\Lambda^{-1})^T g \Lambda^{-1} = (\Lambda^{-1})^T \Lambda^T g \Lambda \Lambda^{-1} = (\underbrace{\Lambda\Lambda^{-1}}_{=\mathbb{1}})^T g (\Lambda\Lambda^{-1}) = g$

$\Rightarrow \Lambda^{-1}$ is Lorentz transformation.

(iii) Define the quadratic form $\langle x,y\rangle_g := x^T g y$ for $x,y \in \mathbb{R}^4$. $\Rightarrow \forall \Lambda$ Lorentz tranf. $\langle \Lambda x, \Lambda y\rangle_g = \langle x,y\rangle_g$

Proof. $\forall x,y \in \mathbb{R}^4$: $\langle \Lambda x, \Lambda y\rangle_g = (\Lambda x)^T g \Lambda y = x^T \Lambda^T g \Lambda y = x^T g y = \langle x,y\rangle_g$

(iv) The following are Lorentz tranformations (where $(t,x) \in \mathbb{R}^4$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$)

(a) $(t,x) \mapsto (t, Qx)$ where Q is an orthogonal transformation of $\mathbb{R}^3$

(b) $(t,x) \mapsto (-t,x)$

(c) $(t,x) \mapsto \left(\frac{t - a x_1}{\sqrt{1-a^2}}, \frac{x_1 - at}{\sqrt{1-a^2}}, x_2, x_3\right)$, where $0 < a < 1$

Proof. (a) $\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Rightarrow \Lambda_1^T g \Lambda_1 = \begin{pmatrix} 1 & 000 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & Q^T \end{pmatrix} \begin{pmatrix} -1 & 000 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & \mathbb{1}_3 \end{pmatrix} \begin{pmatrix} 1 & 000 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & Q \end{pmatrix} = \begin{pmatrix} -1 & 000 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & Q^T \cdot Q \end{pmatrix} = \begin{pmatrix} -1 & 000 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & \mathbb{1}_3 \end{pmatrix} = g$

(b) $\Lambda_2 = \begin{pmatrix} -1 & 000 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & \mathbb{1}_3 \end{pmatrix} = g \Rightarrow \Lambda_2^T g \Lambda_2 = g^3 = \underbrace{g^2}_{=\mathbb{1}_4} \cdot g = g$

(c) $\Lambda_3 = \begin{pmatrix} \frac{1}{\sqrt{1-a^2}} & \frac{-a}{\sqrt{1-a^2}} & 0 & 0 \\ \frac{-a}{\sqrt{1-a^2}} & \frac{1}{\sqrt{1-a^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \Lambda_3^T g \Lambda_3 = \begin{pmatrix} \frac{1}{\sqrt{1-a^2}} & \frac{-a}{\sqrt{1-a^2}} & 0 & 0 \\ \frac{-a}{\sqrt{1-a^2}} & \frac{1}{\sqrt{1-a^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-a^2}} & \frac{-a}{\sqrt{1-a^2}} & & \\ \frac{-a}{\sqrt{1-a^2}} & \frac{1}{\sqrt{1-a^2}} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$= \frac{1}{1-a^2} \begin{pmatrix} \begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} & 0 \\ 0 & (1-a^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}\begin{pmatrix} -1 & a \\ -a & 1 \end{pmatrix} & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$

$= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

(v) The wave equation is Lorentz-covariant, i.e. if $u_{tt} - \Delta u = 0$ in $\mathbb{R}^4 \Rightarrow v(\cdot) := u(\Lambda \cdot)$ is also a solution to the wave eq.

Proof. $\frac{d^2}{dt^2} - \Delta = \begin{pmatrix} d_t \\ d_{x_1} \\ d_{x_2} \\ d_{x_3} \end{pmatrix}^T g \begin{pmatrix} d_t \\ d_{x_1} \\ d_{x_2} \\ d_{x_3} \end{pmatrix} = \nabla^T g \nabla \Rightarrow \left(\frac{d^2}{dt^2} - \Delta\right) v(x) = \nabla^T g \nabla (u(\Lambda x)) = \nabla^T (g \cdot \Lambda \nabla u(\Lambda x)) = \nabla^T \Lambda^T g \Lambda \nabla u(\Lambda x) = \underbrace{(\nabla^T g \nabla u)}_{=0}(\Lambda x) = 0$

$\square$

E3 Let $S := \{(\phi(x), x) : x \in \mathbb{R}^3\} \subset \mathbb{R}^4$ be a smooth hypersurface (i.e. $\phi \in C^\infty(\mathbb{R}^3)$) The Cauchy problem for the wave equation with initial surface S_a is: $\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^4 \\ u = g & \text{on } S_a \end{cases}$ (1)

We say, that S is space like if $1 - |\nabla\phi|^2 > 0$ on $\mathbb{R}^3$.

$\Rightarrow$ The Cauchy problem for the wave equation with the space-like initial surface $S_a = \{(t,x) \in \mathbb{R}^4 : t = a x_1\}$, $0 < a < 1$, is equivalent to the initial value problem (i.e. when $S_0 = \{(t,x) \in \mathbb{R}^4 : t = 0\}$

Proof. Note S_a is indeed space-like, since for $\phi(x) := a x_1$, $1 - |\nabla\phi|^2 = 1 - a^2 > 0$

Note further, that $\Lambda_3(a) S_a \subseteq S_0$ and $\Lambda_3(-a) S_0 \subseteq S_a$,

since $\forall (t,x) \in S_a : \Lambda_3(a)\binom{t}{x} = \left(\frac{t - a x_1}{\sqrt{1-a^2}}, \frac{x_1 - at}{\sqrt{1-a^2}}, x_2, x_3\right) = \left(0, \frac{x_1 - at}{\sqrt{1-a^2}}, x_2, x_3\right) \in S_0$

and $\forall (t,x) \in S_0 : \Lambda_3(-a)\binom{t}{x} = \left(\frac{0 + a x_1}{\sqrt{1-a^2}}, \frac{x_1 + a0}{\sqrt{1-a^2}}, x_2, x_3\right) = \left(a \frac{x_1}{\sqrt{1-a^2}}, \frac{x_1}{\sqrt{1-a^2}}, x_2, x_3\right) \in S_a$

$\Rightarrow S_0 \subseteq \Lambda_3(-a)^{-1} S_a = \Lambda_3(a) S_a \subseteq S_0 \Rightarrow S_0 = \Lambda_3(a) S_a$

Let be u a solution of (1) $\Rightarrow$ $v(t,x) := v(\Lambda_3(-a)(t,x)^T)$ is also a solution to the wave equation E2(u) to the initial values $v(0,x) = g(\underbrace{\Lambda_3(-a)(0,x)}_{\in S_a})$. $\square$

E4 Let $n=3$ or $n=2$, and let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ be the solution of $\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0, \; u_t = h & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases}$ with $h \in C_c^2(\mathbb{R}^n)$, given by Kirchhoff's formula or Poisson's formula, respectively.

(i) Let $\alpha \in (0,1)$. There ex. Constants $C_3, C_2 > 0$, depending on the support of h (and on α in the second case) such that for all $t > 0$

$$\sup_{x \in \mathbb{R}^3} |u(t,x)| \le \frac{C_3}{t} \sup_{\mathbb{R}^3} |h| \quad (n=3), \quad \sup_{x \in B_{\alpha t}(0)} |u(t,x)| \le \frac{C_2}{t} \sup_{x \in \mathbb{R}^2} |h| \quad (n=2)$$

Proof. $n=3$: $u(t,x) = \fint_{\partial B_t(x)} t h(y)\, dS(y)$ (Kirchhoff)

$n=2$: $u(t,x) = \frac{1}{2} \fint_{B_t(x)} \frac{t^2 h(y)}{(t^2 - |y-x|^2)^{1/2}} dy$

$n=2$: $\forall x \in B_{\alpha t}(0), \forall y \in \operatorname{supp} h \subseteq B_R(0)$: $|y-x| \le R + \alpha t$.

Choose $\varepsilon \in (0,1)$ such that $\alpha \le (1-\varepsilon^2)^{1/2}$ $\Rightarrow \exists t_0 > 0 \; \forall t \ge t_0$ $|y-x| \le R + \alpha t \le (1-\varepsilon^2)^{1/2} t$

$\Rightarrow (t^2 - |x-y|^2)^{1/2} \ge (t^2 - (1-\varepsilon^2)t^2)^{1/2} = \varepsilon t$

$\Rightarrow |u(t,x)| \le \frac{1}{2} \frac{1}{\pi t^2} \int_{B_t(x)} \frac{t^2 |h(y)|}{(t^2 - |y-x|^2)^{1/2}} dy \le \frac{1}{2\pi} \int_{B_t(x)} \frac{|h(y)|}{\varepsilon t} dy \le \frac{1}{2\pi} \underbrace{|\operatorname{supp} h|}_{\le \pi R^2} \frac{1}{\varepsilon t} \|h\|_\infty$ f.a. $x \in B_{\alpha t}(0)$, $t \ge t_0$

and $|u(t,x)| \le \frac{1}{2\pi} \int_{B_t(x)} \frac{|h(y)|}{(t^2 - |y-x|^2)^{1/2}} dy = \frac{1}{2\pi t} \int_{B_t(x)} \frac{|h(y)|}{(1 - |x-y|^2/t^2)^{1/2}} \le \frac{1}{2\pi t} \int_{B_t(0)} \frac{\|h\|_\infty}{(1 - |y|^2/t^2)^{1/2}} dy$

$= \frac{1}{2\pi t} \int_{B_1(0)} \frac{\|h\|_\infty}{(1-|y|^2)^{1/2}} t^2 dy = \frac{t}{\cancel{2\pi}} \int_0^1 \underbrace{\frac{r}{(1-r^2)^{1/2}}}_{= \frac{d}{dr}(-\sqrt{1-r^2})} dr \, \|h\|_\infty \le \frac{t_0^2 \|h\|_\infty}{t}$ f.a. $x \in B_{\alpha t}(0)$, $t \le t_0$

$n=3$: Let $R > 0$ be a constant with $\operatorname{supp} h \subseteq B_R(0)$ $\Rightarrow \forall x \in \mathbb{R}^3$, and $\forall r > 0$ $\lambda_2(\partial B_r(x) \cap \operatorname{supp} h) \le \lambda_2(\partial B_R(0)) = 4\pi R^2$

$\Rightarrow |u(t,x)| \le \left| \fint_{\partial B_t(x)} t h(y)\, dS(y) \right| \le \frac{t}{4\pi t^2} \int_{\partial B_t(x) \cap \operatorname{supp} h} \|h\|_\infty \, dS(y) \le \frac{1}{t} R^2 \|h\|_\infty$

(iii) $n=3$: $\sup |u(t,x)| \leq \frac{1}{4\pi} \min\left(\frac{1}{t}\|Dh\|_{L^1(\mathbb{R}^3)}, \|D^2 h\|_{L^1(\mathbb{R}^3)}\right)$, for $t>0$

Proof. $$|u(t,x)| \leq \left| \fint_{\partial B_t(x)} t\, h(y)\,dS(y)\right| = \left|\frac{1}{4\pi t^2} \int_{\partial B_t(x)} t\,h(y)\,dS(y)\right| = \left|\frac{1}{4\pi}\int_{\partial B_1(0)} t\,h(x+tz)\,dS(z)\right|$$

$$= \left|-\frac{1}{4\pi}\int_{\partial B_1(0)} t \int_t^\infty \partial_s h(x+sz)\,ds\,dz\right| = \frac{t}{4\pi}\left|\int_{\partial B_1(0)}\int_t^\infty Dh(x+sz)\cdot z\,ds\,dz\right|$$

$$\leq \int_t^\infty \fint_{\partial B_1(0)} \underbrace{t}_{\leq \frac{s^2}{t},\ \text{since } t\leq s} |Dh(x+sz)|\,dz\,ds \leq \frac{1}{t}\int_0^\infty \fint_{\partial B_1(0)} |Dh(x+sz)|\,dz\, s^2 ds \underset{\uparrow}{=} \frac{1}{t4\pi}\|Dh\|_{L^1}$$

Coarea-Formula

and $$|u(t,x)| \leq \frac{1}{4\pi} t \left|\int_{\partial B_1(0)} \int_t^\infty \partial_s h(x+sz)\,ds\,dz\right| = \frac{t}{4\pi}\left|\int_{\partial B_1(0)}\int_t^\infty\int_s^\infty \underbrace{\partial_u^2 h(x+uz)}_{=\langle z, D^2 h(x+uz) z\rangle}\,du\,ds\,dz\right|$$

$$\underset{\text{Tonelli}}{\leq} \frac{t}{4\pi}\left|\int_{\partial B_1(0)}\int_t^\infty\int_t^u |\partial_u^2 h(x+uz)|\,ds\,du\,dz\right| = \frac{1}{4\pi}\int_{\partial B_1(0)}\int_t^\infty \underbrace{t(u-t)}_{\leq u^2} \underbrace{|\partial_u^2 h(x+uz)|}_{\leq \|D^2 h(x+uz)\|}\,du\,dz \underset{\uparrow}{=} \frac{1}{4\pi}\|D^2 h\|_{L^1}$$

Coarea Formula