$\boxed{\text{Üb09}}$

$\boxed{\text{E1}}$ Let $p>1$, $f\in C^\infty(\mathbb{R})$ with $f(t) := \begin{cases} \exp(-t^{-p}) & t>0 \\ 0 & t\le 0 \end{cases}$, $g:(0,\infty)\times\mathbb{R}\to\mathbb{R}$ $g(t,x) := \sum_{k=0}^\infty \frac{f^{(k)}(t)}{(2k)!}x^{2k}$

(a) g is well-defined and $g\in C^\infty((0,\infty)\times\mathbb{R})$ and $\lim_{t\to 0} g(t,x)=0$ for $x\in\mathbb{R}^n$.

(b) There exists infinitely many solutions of the initial value problem $\begin{cases} u_t - u_{xx} = 0 \text{ in } (0,\infty)\times\mathbb{R} \\ u=0 \text{ on } \{t=0\}\times\mathbb{R}\end{cases}$

Proof. (a) For all $k\in\mathbb{N}_0$: $\left|\frac{f^{(k)}(t)}{(2k)!}x^{2k}\right| \le \underbrace{\frac{k!}{(2k)!}}_{\le \frac{1}{k!}} \frac{e^{-1/2t^p}}{(\theta t)^k} x^{2k} = e^{-1/2t^p}\cdot\left(\frac{|x|^2}{\theta t}\right)^k\frac{1}{k!}$

$\Rightarrow |g_p(t,x)| \le e^{-1/2t^p}\sum_{k=0}^\infty \left(\frac{|x|^2}{\theta t}\right)^k \frac{1}{k!} = e^{\frac{|x|^2}{\theta t}-\frac{1}{2t^p}} \Rightarrow g$ is well-def.

For all $\alpha,\beta\in\mathbb{N}_0$: $\left|\partial_t^\alpha\partial_x^\beta \frac{f^{(k)}(t)}{(2k)!}x^{2k}\right| = \left|\frac{f^{(k+\alpha)}(t)}{(2k-\beta)!}x^{2k-\beta}\right|$

$\le \underbrace{\frac{(k+\alpha)!}{(2k-\beta)!}}_{\underset{\alpha+\beta\le k}{\le} \frac{1}{(k-\alpha-\beta)!}} e^{-\frac{1}{2t^p}}\frac{1}{(\theta t)^{k+\alpha}}|x|^{2k-\beta} \le \frac{1}{(k-\alpha-\beta)!}\underbrace{\left(\frac{|x|^2}{\theta t}\right)^{k-\alpha-\beta}}_{\le C_{\varepsilon t_0 x_0}^{k-\alpha-\beta}} \cdot \underbrace{\frac{|x|^{\beta+\alpha}}{(\theta t)^{-2\alpha-\beta}} e^{-\frac{1}{2t^p}}}_{< C_{\varepsilon t_0 x_0} \le \infty \text{ for } (t,x)\in B_\varepsilon((t_0,x_0))}$ for $k\ge\alpha+\beta$

$\Rightarrow \left|\sum_{k=0}^\infty \left|\partial_t^\alpha\partial_x^\beta \frac{f^{(k)}(t)}{(2k)!}x^{2k}\right|\right| \le \sum_{k=0}^{\alpha+\beta-1}\partial_t^\alpha\partial_x^\beta\frac{f^{(k)}(t)}{(2k)!}x^{2k} + \frac{|x|^{\beta+\alpha}}{(\theta t)^{-2\alpha-\beta}}e^{-\frac{1}{2t^p}+\frac{|x|^2}{\theta t}} < \infty$ for $x\in B_\varepsilon((t_0,x_0))$

Theorem of parameter dependant fct.

$\Rightarrow \partial_t^\alpha\partial_x^\beta g_p(t,x) = \sum_{k=0}^\infty \partial_t^\alpha\partial_x^\beta\frac{f^{(k)}(t)}{(2k)!}x^{2k}$ and $g_p\in C^\infty((0,\infty)\times\mathbb{R})$

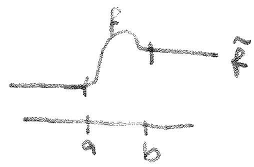
and $\lim_{t\to 0} g_p(t,x) = g_p(0,x) = \sum_{k=0}^\infty 0 = 0$

(b) $\partial_t g_p(t,x) - \partial_{xx}g_p(t,x) = \sum_{k=0}^\infty(\partial_t-\partial_{xx})\frac{f^{(k)}(t)}{(2k)!}x^{2k} = \sum_{k=0}^\infty \frac{f^{(k+1)}(t)}{(2k)!}x^{2k} - \sum_{k=1}^\infty\frac{f^{(k)}(t)}{(2(k-1))!}x^{2(k-1)}$

$= \sum_{k=0}^\infty \frac{f^{(k+1)}-f^{(k+1)}}{(2k)!}x^{2k} = 0$ and $g_p(0,x)=0$

This is true for all $p>0$ $\Rightarrow$ infinitely many solutions to the problem.

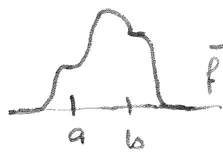
E2 Let $N=1$ and Φ be the heat kernel. Use properties of the convolution $u(t,x)=\int_{\mathbb{R}}\Phi(t,x-y)f(y)\,dy$ to prove Weierstraß' approximation theorem:
A function $f\in C([a,b])$ can be approximated uniformly by polynomials.



Proof. Extende f continuously by $\tilde f:\mathbb{R}\to\mathbb{R}$, $\tilde f|_{(-\infty,a)}=f(a)$, $\tilde f|_{(b,\infty)}=f(b)$.

Let χ be a smooth cutoff function $\chi|_{(0,\infty)}=1$, $\chi|_{(-\infty,-1)}=0$,

Define $\bar f(\cdot):=\chi(\cdot-a)\,\chi(b-\cdot)\,\tilde f(\cdot) \Rightarrow \operatorname{supp}\bar f\subseteq[a-1,b+1]$



Let be $\varepsilon>0$. $\Phi(t,\cdot)*\bar f \xrightarrow{t\to 0} f$ uniformly on $[a,b]$

$\Rightarrow \exists t_\varepsilon>0 : \max_{x\in[a,b]} |u(t_\varepsilon,x)-f(x)|<\varepsilon/2$

$$|\Phi(t_\varepsilon,x-y)| \le \frac{1}{(4\pi t_\varepsilon)^{1/2}}\sum_{k=0}^{\infty}\frac{(|x|+|y|)^{2k}/(4t_\varepsilon)^k}{k!} \le \frac{1}{(4\pi t_\varepsilon)}\sum_{k=0}^{\infty}\frac{\max(|a|+1,|b|+1)^{2k}/t_\varepsilon^k}{k!}$$ for all $x\in[a,b]$, $y\in\operatorname{supp}\bar f$

There exists $N_\varepsilon\in\mathbb{N}$ such that

$$\frac{1}{(4\pi t_\varepsilon)^{1/2}}\sum_{k=N_\varepsilon+1}^{\infty}\frac{\max(|a|+1,|b|+1)^{2k}}{t_\varepsilon^k\,k!} < \varepsilon/(2\|\bar f\|_\infty(b-a+2))$$

$$\Rightarrow \left|\int_{\mathbb{R}}\bar f(y)\underbrace{\frac{1}{(4\pi t_\varepsilon)^{1/2}}\sum_{k=N_\varepsilon+1}^{\infty}\frac{1}{k!(4t_\varepsilon)^k}(-|x-y|^2)^k}_{\le \varepsilon/2\cdot\frac{1}{\|\bar f\|_\infty(b-a+2)}}dy\right| \le \frac{\varepsilon}{2}\int_{a-1}^{b+1}\bar f(y)\,dy/(\|\bar f\|_\infty(b-a+2)) \le \frac{\varepsilon}{2}$$ f.a. $x\in[a,b]$

Furthermore is $$P_\varepsilon(x):=\int_{\mathbb{R}}\bar f(y)\frac{1}{(4\pi t_\varepsilon)^{1/2}}\underbrace{\sum_{k=0}^{N_\varepsilon}\frac{1}{k!(4t_\varepsilon)^k}(-(x-y)^2)^k}_{\text{polynomial in } x \text{ for fixed } y}dy$$ a polynomial

and $|P_\varepsilon(x)-f(x)| \le |P_\varepsilon(x)-u(t_\varepsilon,x)|+|u(t_\varepsilon,x)-f(x)|$

$$=\left|\int_{\mathbb{R}}\bar f(y)\frac{1}{(4\pi t_\varepsilon)^{1/2}}\sum_{k=N_\varepsilon+1}^{\infty}\frac{1}{k!(4t_\varepsilon)^k}(-|x-y|^2)^k dy\right| + |u(t_\varepsilon,x)-f(x)|$$

$$\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$ f.a. $x\in[a,b]$

$\Rightarrow \max_{x\in[a,b]}|P_\varepsilon(x)-f(x)|\le\varepsilon$

E3 $U \subseteq \mathbb{R}$ open, bounded, $T>0$, $U_T := (0,T)\times U$, $\Gamma_T = \partial U_T$.

We call $v \in C^2(U_T)\cap C^0(\overline{U_T})$ a subsolution of the heat equation if $v_t - \Delta v \le 0$ in U_T.

(i) v subsolution $\Rightarrow v(x,t) \le \frac{1}{4r^n}\iint_{E(t,x;r)} v(s,y)\,ds\,dy$ for all $E(t,x;r) \subseteq U_T$

(ii) Let $\phi: \mathbb{R}\to\mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$.
$\Rightarrow$ v is a subsolution

(iii) u solves the heat equation $\Rightarrow v := |Du|^2 + u_t^2$ is a subsolution.

Proof. (i) Define $\varphi(r) := \frac{1}{4r^n}\iint_{E(t,x;r)} u(s,y)\frac{|x-y|^2}{(t-s)^2}\,ds\,dy$

and $\psi(s,y) := \ln(r^n\phi(t-s,x-y))$ we get

$\varphi'(r) = -\frac{n}{r^{n+1}}\iint_{\{\psi(s,y)\ge 0\}} \underbrace{(v_s - \Delta_y v)(s,y)}_{\le 0}\underbrace{\psi(s,y)}_{\ge 0}\,dy\,ds \ge 0$

$\Rightarrow \varphi(r) \ge \lim_{\delta\downarrow 0}\varphi(\delta) \overset{S.78}{=} \Big(\lim_{\delta\downarrow 0}\iint_{\{\phi(z,\tau)\ge 1\}} v(x-\delta z, t-\delta^2 z)\frac{|z|^2}{4\tau^2}\,dz\,d\tau$

$= v(x,t)\underbrace{\lim_{\delta\downarrow 0}\iint_{\{\phi(z,\tau)\ge 1\}}\frac{|z|^2}{4\tau^2}\,dz\,d\tau}_{=1 \text{ Lecture}}\Big) = v(x,t)$

(ii) $\partial_t v - \Delta v = \phi'(v)\partial_t v - \nabla\cdot(\phi'(v)\nabla v) = \phi'(v)(\partial_t v - \Delta v) - \underbrace{\phi''(v)}_{\ge 0}\underbrace{|\nabla v|^2}_{\ge 0} \le 0$

(iii) Let be $u \in C^3(U_T)$ with $u_t - \Delta u = 0$ in Ω_T.

$$\Rightarrow (\partial_t - \Delta)v = (\partial_t-\Delta)((\partial_t u)^2 + |\nabla u|^2) = 2(\partial_t u\,\partial_t^2 u) + 2\nabla u(\nabla\partial_t u) - 2\partial_t u\,\Delta\partial_t u - 2|\nabla\partial_t u|^2 - 2\nabla u\cdot\nabla\Delta u - 2\sum_{i=1}^n|\nabla\partial_i u|^2$$

$$= 2\partial_t u(\underbrace{\partial_t(\partial_t u - \Delta u)}_{=0}) + 2\nabla u\cdot\nabla(\underbrace{\partial_t u - \Delta u}_{=0}) - 2\underbrace{|\nabla\partial_t u|^2}_{\ge 0} - 2\sum_{i=1}^n\underbrace{|\nabla\partial_{x_i}u|^2}_{\ge 0} \le 0$$

E4 Let $U \subseteq \mathbb{R}^n$ be open and bounded with smooth boundary $\partial U \in C^1$, $T>0$.
Assume $u_1, u_2 \in C^2(U_T)\cap C^0(\overline{U_T})$ are solutions of the (nonlinear) initial boundary value problem $(\partial_t - \Delta)u_i(t,x) = f(t,x,u_i(t,x))$ for all $(t,x)\in U_T$

$u_i|_{\Gamma_T} = g_i$

where $f \in C^0(U_T\times\mathbb{R})$ and $g_i \in C^0(\Gamma_T)$ for $i\in\{1,2\}$

$\Rightarrow$ If $f(t,x,u_1(t,x)) \le f(t,x,u_2(t,x))$ for all $(t,x)\in U_T$ and $g_1 \le g_2 \Rightarrow u_1 \le u_2$.

Proof. v ~~subharmonic~~ subsolution $\Rightarrow$ E3(i)

$0 \le \frac{1}{4r^n}\iint_{E(t,x;r)} (v(s,y) - v(t,x))\frac{|x-y|^2}{(t-s)^2}\,ds\,dy$ (1)

suppose there exists a maximum of v at $(t,x)\in \mathring{U}_T \Rightarrow v(s,y) - v(t,x) \le 0$ f.a. $(s,y)\in U_T$

(1) yields, that this is only possible for $v \equiv$ const. $\Rightarrow \max_{(t,x)\in U_T} v = \max_{(t,x)\in\partial U_T} v$

$w = u_1 - u_2$ is a subsolution, since $(\partial_t - \Delta)w(t,x) = f(t,x,u_1(t,x)) - f(t,x,u_2(t,x)) \le 0$

$\Rightarrow \max_{(t,x)\in U_T} w(t,x) = \max_{(t,x)\in\Gamma_T} w(t,x) = \max g_1 - g_2 \le 0$