Ex.1: $u,v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

(a) $\mathcal{F}(u), \mathcal{F}(v) \in L^2(\mathbb{R}^n) \rightsquigarrow \int_{\mathbb{R}^n} \bar{v} u \, dx = \int_{\mathbb{R}^n} \overline{\mathcal{F}(v)}\, \mathcal{F}(u) \, dx$

For $\alpha \in \mathbb{C}$ we have

$$\underbrace{\int_{\mathbb{R}^n} |u+\alpha v|^2 \, dx}_{\text{Plancherel}} = \int_{\mathbb{R}^n} \overline{(u+\alpha v)}(u+\alpha v)\, dx = \int_{\mathbb{R}^n} \left(|u|^2 + |\alpha v|^2 + \bar{u}\alpha v + u\overline{\alpha v}\right) dx$$

$$= \int_{\mathbb{R}^n} |\mathcal{F}(u+\alpha v)|^2 \, dx = \int_{\mathbb{R}^n} |\mathcal{F}(u) + \alpha \mathcal{F}(v)|^2 \, dx$$

$$= \int_{\mathbb{R}^n} \left(|\mathcal{F}(u)|^2 + |\alpha \mathcal{F}(v)|^2 + \overline{\mathcal{F}(u)}\, \alpha \mathcal{F}(v) + \mathcal{F}(u)\, \overline{\alpha \mathcal{F}(v)} \right) dx$$

$$\overset{\text{(Plancherel)}}{=} \int_{\mathbb{R}^n} \left(|u|^2 + |\alpha v|^2 + \overline{\mathcal{F}(u)}\, \alpha \mathcal{F}(v) + \mathcal{F}(u)\, \overline{\alpha \mathcal{F}(v)} \right) dx$$

Overall for all $\alpha \in \mathbb{C}$ have

$$\int_{\mathbb{R}^n} \left(\alpha \bar{u} v + \bar{\alpha} u \bar{v}\right) dx = \int_{\mathbb{R}^n} \left(\alpha \overline{\mathcal{F}(u)}\, \mathcal{F}(v) + \bar{\alpha} \mathcal{F}(u)\, \overline{\mathcal{F}(v)}\right) dx.$$

We evaluate the equation for $\alpha = 1$ and $\alpha = i$:

- $$\int_{\mathbb{R}^n} (\bar{u}v + u\bar{v})\, dx = \int_{\mathbb{R}^n} \left(\overline{\mathcal{F}(u)}\, \mathcal{F}(v) + \mathcal{F}(u)\, \overline{\mathcal{F}(v)}\right) dx \qquad (1)$$

- $$i\int_{\mathbb{R}^n} (\bar{u}v - u\bar{v})\, dx = i\int_{\mathbb{R}^n} \left(\overline{\mathcal{F}(u)}\, \mathcal{F}(v) - \mathcal{F}(u)\, \overline{\mathcal{F}(v)}\right) dx \qquad (2)$$

Finally, $(1) + i(2)$ reads

$$\int_{\mathbb{R}^n} u\bar{v} \, dx = \int_{\mathbb{R}^n} \mathcal{F}(u)\, \overline{\mathcal{F}(v)} \, dx$$

b) Let $u \in C^2(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq 2$, s.t.
$D^\alpha u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

Case 1: $|\alpha| = 1$, i.e. $\alpha = e_i$ $(i = 1, \dots, n)$.

First assume that $u \in C_c^2(\mathbb{R}^n)$ (comp. supp.)

$$(*) \left\{ \begin{aligned} \mathcal{F}(D^\alpha u)(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} u_{x_i}(y)\, dy \\ &\overset{\text{P.I.}}{=} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} i x_i e^{-ixy} u(y)\, dy \\ &= i x_i \mathcal{F}(u)(x) = (ix)^\alpha \mathcal{F}(u)(x) \qquad (x \in \mathbb{R}^n) \end{aligned} \right.$$

Now let $u \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \underbrace{\cap L^2(\mathbb{R}^n)}$, $D^\alpha u \in L^1(\mathbb{R}^n)$.

Then there ex. a sequence $\left(\begin{smallmatrix} L^2 \subset C^2 \cap L^1 \\ \text{for } \Omega = \mathbb{R}^n \end{smallmatrix}\right)$ of fct.'s
$(u_k)_{k \in \mathbb{N}} \subset C_c^2(\mathbb{R}^n)$ s.t.

$$u_k \xrightarrow{k \to \infty} u \quad \text{in } L^1(\mathbb{R}^n)$$
$$D^\alpha u_k \to D^\alpha u \quad \text{in } L^1(\mathbb{R}^n)$$

for u_k
$(*)$ together with Leb's Thm. on dom. conv.
yields $(*)$ for the fct. u.

Case 2: $|\alpha| = 2$, i.e. $\alpha = e_i + e_j$ $(i, j = 1, \dots, n)$

$$\mathcal{F}(D^\alpha u)(x) = \mathcal{F}(\partial_{x_i} u_{x_j})(x) = i x_i \mathcal{F}(u_{x_j})(x)$$
$$= (ix_i)(ix_j) \mathcal{F}(u)(x)$$
$$= (ix)^\alpha \mathcal{F}(u)(x)$$

again holds for $u \in C_c^2(\mathbb{R}^n)$. The claim then follows from the same approximation argument as in Case 1.

c) Because $\int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} dy \, |u(\xi)\, v(y-\xi)| = \|u\|_{L^1} \|v\|_{L^1} < \infty \quad (*)$

the fol. $u * v \in L^1$ and $\mathcal{F}(u*v)$ well-def. with

$$\mathcal{F}(u*v)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} \Big(\int_{\mathbb{R}^n} u(\xi)\, v(y-\xi)\, d\xi \Big) dy$$

applicable b/c of $(*)$ $\longrightarrow$ (Fubini)

$$= \int_{\mathbb{R}^n} e^{-ix\xi}\, u(\xi)\, \mathcal{F}(v)(x)\, d\xi$$

$$= (2\pi)^{\frac{n}{2}}\, \mathcal{F}(u)(x)\, \mathcal{F}(v)(x).$$

d) Let $v \in C_c^\infty(\mathbb{R}^n)$, then via Fubini's Thm.

$$\int_{\mathbb{R}^n} \overbrace{\mathcal{F}^{-1}(\mathcal{F}(u))(x)}^{=(\mathcal{F}^{-1}\circ\mathcal{F}(u))} v(x)\, dx = \int_{\mathbb{R}^n} \mathcal{F}(u)(x)\, \underbrace{\mathcal{F}^{-1}(v)(x)}\, dx$$

$\Big(= \overline{\mathcal{F}(\overline{v})(x)}$ by a direct computation$\Big)$

$$= \int_{\mathbb{R}^n} \mathcal{F}(u)(x)\, \overline{\mathcal{F}(\overline{v})(x)}\, dx$$

$$\overset{(a)}{=} \int_{\mathbb{R}^n} u(x)\, v(x)\, dx$$

and the formula follows from the Fund. Lemma in Calc. of Var. (for inst. H.W.-session 7)

Ex. 2: ($\mathcal{F}(u)$ is the Fourier transform on $\mathbb{R}^n$ only (i.e. of $u(t,\cdot)$))

a) For $t>0$:

$$\partial_t \mathcal{F}(u)(t,x) + |x|^2 \mathcal{F}(u)(t,x)$$

$$= \partial_t \mathcal{F}(u)(t,x) - \sum_{i=1}^{n} (ix_i)^2 \mathcal{F}(u)(t,x)$$

(Ex. 1)

$$= \partial_t \mathcal{F}(u)(t,x) - \sum_{i=1}^{n} \mathcal{F}(u_{x_i x_i})(t,x)$$

t-der. exchanged with integral with standard corollary

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ixy} \{u_t(t,y) - \Delta u(t,y)\} dy$$

$$= \mathcal{F}(\underbrace{u_t - \Delta u}_{=0})(x,t) = 0$$

$t=0$:

$$\mathcal{F}(u)(0,x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} \overbrace{u(0,y)}^{=g(y)} dy = \mathcal{F}(g)(x)$$

b)

$$u(t,x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ixy} e^{-t|y|^2} \mathcal{F}(g)(y)\, dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ixy} e^{-t|y|^2} \left(\int_{\mathbb{R}^n} e^{-iyz} g(z) \frac{dz}{(2\pi)^{n/2}} \right) dy$$

$$\underset{g \in C_c^2}{\overset{\text{(Fubini)}}{=}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(z)\, \mathcal{F}^{-1}(e^{-t|\cdot|^2})(x-z)\, dz$$

$$= \frac{1}{(2\pi)^{n/2}} \left[g * \mathcal{F}^{-1}(e^{-t|\cdot|^2}) \right](x)$$

(Hence with Ex. 1: $\mathcal{F}(u)(t,x) = e^{-t|x|^2} \mathcal{F}(g)(x)$

$\xrightarrow{t=0} \mathcal{F}(u)(0,x) = \mathcal{F}(g)(x) \quad (x \in \mathbb{R}^n)$)

and for $t > 0$:

$$\partial_t \mathcal{F}(u)(t,x) = -|x|^2 e^{-t|x|^2} \mathcal{F}(g)(x)$$
$$= -|x|^2 \mathcal{F}(u)(t,x) \qquad (x \in \mathbb{R}^n)$$

c) $$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(z) \underbrace{\int_{\mathbb{R}^n} e^{i(x-z)y} e^{-t|y|^2} \, dy}_{= \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|x-z|^2}{4t}}} dz$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-z|^2}{4t}} g(z) \, dz$$

[I.e. u is the candidate for a solution of the initial value problem (1)]

Ex. 3 a) Assume u sol. of $\begin{cases} u_t - \Delta u + b\cdot\nabla u & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,\cdot) = g & \text{on } \mathbb{R}^n \end{cases}$ (*)

then $v(t,x) := u(t, x+tb)$ sol. of $\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,\cdot) = g & \text{on } \mathbb{R}^n \end{cases}$

b/c.
- $v(0,x) = u(0,x) = g(x)$
- $\partial_t v(t,x) = u_t(t, x+tb) + b\cdot\nabla u(t, x+tb)$
- $\Delta u(t, x+tb) = \Delta v(t,x)$

By the assumptions on g ... (sorry, forgot the $g \in C^\infty$ in the 1^{st} version of the sheet 8)

$$v(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy$$

Hence, b/c we also have $\left\{ \begin{matrix} v \text{ sol. of} \\ \text{H.E.} \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} u \text{ sol. of} \\ \text{H.E. with} \\ \text{convection} \end{matrix} \right\}$

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y-tb)\, g(y)\, dy \quad (= v(t, x-tb))$$

sol. of (*)

b) The arg. is as in part (a).

Assume u is a sol. of $\begin{cases} u_t - \Delta u + cu = 0 & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,\cdot) = g & \text{on } \mathbb{R}^n \end{cases}$ (**)

Then $v(t,x) := e^{ct} u(t,x)$ a sol. of the Heat-eq. with initial data g

and $u(t,x) = e^{-ct} \int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy$

solves (**) w. arguments as above.