Partielle Differentialgleichungen I

WiSe 16/17 – S. Bachmann, LMU

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    ↳ Anmeldung über Lecture Assistent.
    Bonus: +0.3 auf der Klausurnote, falls genügend
    Bedingung: 50% der Aufgabepunkte
  - Literatur: i) Partial Differential Equations, L.C. Evans.
    ii) Elliptische Differentialgleichungen zweiter Ordnung, E. Wienholtz, H. Kalf, T. Kriecherbauer.
- Inhalt:
  1) Lineare Transportgl.
  2) Laplace- und Poissongl.
  3) Wärmeleitungsgl.
  4) Wellengl.
  5) Methode der Charakteristiken.
- Webseite!

0. Einleitung

- Eine PDG ist eine Gleichung für eine Funktion

$$u : \Omega \to \mathbb{R}^N \text{ oder } \mathbb{C}$$

$\Omega \subset \mathbb{R}^n$ ein Gebiet, die als Gleichheit zweier Funktionen von x, $u(x)$ und den partiellen Ableitungen von u gegeben ist.

- PDG's sind unter anderen die grundlegenden Gleichungen der Natur- und Ingenieurwissenschaften. $\leadsto$ Naturgesetze

<u>Beispiel</u>: Stab der Länge $L > 0$

Temperaturverteilung $u : [0,L] \times \mathbb{R} \to \mathbb{R}_+$, $u(x,t)$

- Prinzipien. i) Energiebilanz: zeitliche Änderung der thermischen Energie in $[a,b]$ ist gleich der Wärmefluss durch die Ränder.

ii) Energiedichte ist proportional zur Temperatur und zur Massendichte.

iii) Fourieffekt: Der Wärmefluss ist proportional zum (räumlichen) Gradienten der Temperaturvert.

- Übersetzt in Formeln bei konstanter Dichte $\rho > 0$.

$$\frac{d}{dt}\int_a^b c\,\rho\, u(t,x)\,dx = -k\frac{\partial u}{\partial x}(a,t) + k\frac{\partial u}{\partial x}(b,t)$$

$$\Leftrightarrow \int_a^b \left(c\rho \frac{\partial u}{\partial t}(x,t) - k\frac{\partial^2 u}{\partial x^2}(x,t)\right)dx = 0$$

- Da die Gleichung für alle $0 < a < b < L$ gilt, muss der Integrand verschwinden, also

$$u_t - \alpha u_{xx} = 0 \quad , \quad \alpha = \frac{k}{\rho C}$$

und in höheren Dimensionen:

$$u_t - \alpha \Delta u = 0 \qquad \text{Wärmeleitungsgleichung.}$$

- Notationen: Multiindex: $\alpha = (\alpha_1, \dots, \alpha_n) \quad \alpha_i \in \mathbb{N}$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \quad ; \quad \alpha! = \alpha_1! \dots \alpha_n!$$

und für $x \in \mathbb{R}^n$: $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Partielle Ableitungen: $$(D^\alpha u)(x) = \frac{(\partial^{|\alpha|} u)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x)$$

und falls $|\alpha|$ klein ist, oft auch z.B $\frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j}, \dots$

wir schreiben auch $D^k u$, $k \in \mathbb{N}$ für die Menge aller $D^\alpha u$, $|\alpha| = k$.

Spezialfälle: * $k=1$: $Du = (u_{x_1}, \dots, u_{x_n})$ der Gradient.

$$u_r = \frac{x}{|x|} \cdot Du \qquad \text{die radiale Ableitung.}$$

* $k=2$: $(D^2 u)_{ij} = (u_{x_i x_j})_{ij}$ die Hessematrix.

$$\Delta u = \mathrm{Tr}(D^2 u)$$

- Def: Eine PDG k-ter Ordnung ($k \in \mathbb{N}$) ist eine Gleichung der Form:

$$F(D^k u(x), D^{k-1} u(x), \dots, u(x), x) = 0$$

wobei die Funktion

$$F: \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R} \times \Omega \to \mathbb{R}$$

gegeben ist, und $u: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$) die unbekannte Funktion ist.

Eine klassische Lösung ist eine Funktion $u \in C^k(\Omega)$ welche die PDG erfüllt. (manchmal unter zusätzlichen Bedingungen auf Teilen von $\partial\Omega$).

- Bem: Wir werden fraktionelle Ableitungen hier nicht betrachten

- Eine PDG ist wohlgestellt (inkl. Randbedingungen) falls:
  i) Eine Lösung existiert (und zwar auf ganz Ω).
  ii) Die Lösung ist eindeutig
  iii) Die Lösung hängt stetig von den Parametern ab.

- Ziele der Theorie?
  * Wohlgestelltheit (unter Bedingungen)
  * Explizite Lösungen.
  * Eigenschaften von Lösungen wie Schranken, Asymptotisches Verhalten
  * Glattheit der Lösungen.
  * Oder eben: Blow-up, Singularitäten

  !

- Bem: Es gibt auch schwache Lösungen: u ist nicht $\in C^k$, z.B. Schockwellen

- Beispiele:
  * Laplacegl.: $-\Delta u(x) = 0$, $x \in \Omega \subset \mathbb{R}^n$
  * Poisson-gl: $-\Delta u(x) = f(x)$, $x \in \Omega \subset \mathbb{R}^n$
  * Helmholtz-gl.: $-\Delta u(x) = \lambda u(x)$, $x \in \Omega \subset \mathbb{R}^n$.
  * Transportgl: $u_t(t,x) + b(x) \cdot \nabla u(t,x)$
    $(t,x) \in (0,\infty) \times \Omega$, $\Omega \subset \mathbb{R}^n$
    und b : $\Omega \to \mathbb{R}^n$.

  * Fokker-Planck-Gl: $(t,x) \in (0,\infty) \times \Omega$, $\Omega \subset \mathbb{R}$

$$\left.\begin{matrix}\sigma\\ D\end{matrix}\right\} \; [0,\infty)\times\Omega \to \begin{cases}\mathbb{R}\\ [0,\infty)\end{cases}$$

$$\frac{\partial}{\partial t}u(t,x) = -\frac{\partial}{\partial x}\Big(\sigma(t,x)u(t,x)\Big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(D(t,x)u(t,x)\Big)$$

mit $\sigma\equiv 0$ und $D(t,x) = 2\alpha$: Wärmeleitungsgl.

mit $D=0$ und $\sigma(t,x)=\sigma(x)$: Transport-gl.

- Schrödinger-gl. : $i u_t(t,x) + \Delta u(t,x) = 0$

  $(t,x)\in\mathbb{R}\times\Omega$, $u(t,x)\in\mathbb{C}$

  und Zusatzbedingung $\int_\Omega |u(t,x)|^2 dx < \infty$.

- Wellengl : $u_{tt}(t,x) - \Delta u(t,x) = 0$, $(t,x)\in(0,\infty)\times\Omega$
- Nichtlineare Poisson-gl. $-\Delta u(x) = f(u(x))$, $x\in\Omega$.
- Monge-Ampère-Gl : $\det(D^2u(x)) = f(x)$
- Burgers-Gl : $u_t(t,x) + u\cdot u_x(t,x) = \mu u_{xx}(t,x)$
- Navier-Stokes-Gl : $u : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$

  $f,p : (0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ gegeben.

$$\begin{cases}\frac{\partial}{\partial t}u + (u\cdot\nabla)u = \nu\Delta u - \nabla p + f\\ \nabla\cdot u = 0\end{cases}$$

„Millenium problem": Wohlgestellt ?

- Eine PDG ist

i) <u>linear</u>, falls: $\sum_{|\alpha|\le n} a_\alpha(x)D^\alpha u(x) = 0$

$a_\alpha : \Omega\to\mathbb{R}$ gegeben.

Inhomogene Gl: $\sum_{|\alpha|\le n} a_\alpha(x)D^\alpha u(x) = f(x)$

ii) Semilinear, falls:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + f\left(D^{k-1}u(x), \dots, u(x), x\right) = 0$$

iii) Quasi-linear, falls

$$\sum_{|\alpha|=k} F_\alpha\left(D^{k-1}u(x) \dots u(x), x\right) D^\alpha u(x) + f\left(D^{k-1}u(x), \dots, u(x), x\right) = 0.$$

iv) Vollkommen nichtlinear, falls $f(D^k u(x), \dots, u(x), x) = 0$
wobei f eine nicht-lineare Funktion von $D^k u$ ist.

- Klassifizierung der linearen PDGs 2. Ordnung.
Allgemeine Form: $x \in \Omega \subseteq \mathbb{R}^n$, $f: \Omega \to \mathbb{R}$

$$(*) \quad -\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u(x) = f(x)$$

wobei $A = (a_{ij})_{i,j=1}^n$ eine symmetrische Matrix für jeden $x \in \Omega$ ist.
Die Gleichung $(*)$ ist:

i) <u>elliptisch</u> in $x \in \Omega$, falls alle Eigenwerte von $A(x)$ das gleiche Vorzeichen haben.

ii) <u>parabolisch</u> in $x \in \Omega$, falls es genau einen Eigenwert von $A(x)$ gibt, der 0 ist.
(oder allgemeiner: $A(x)$ ist singulär)

iii) <u>hyperbolisch</u> in $x \in \Omega$, falls es genau einen Eigenwert von $A(x)$ gibt, der ein anderes Vorzeichen hat als alle anderen

Beispiele: i) $-\Delta u = 0$; ii) $u_t - \Delta u = 0$; iii) $u_{tt} - \Delta u = 0$.

Bem.: Für {quasi-lineare / semilineare} Gleichungen höherer Ordnung können diese Definition mit Hilfe von Charakteristischen Ebenen erweitert werden.

* Im Prinzip kann sich der Typ einer Gleichung vom $x \in \Omega$ abhängen. Falls z.B. $A(x) \geq C\mathbb{I}$ mit $C > 0$ für alle $x \in \Omega$ redet man von <u>gleichmässiger Elliptizität</u>.

1. The linear transport equation.

- We consider the following initial value problem.

$$\begin{cases} u_t(t,x) + b(t,x)\cdot Du(t,x) = f(t,x) & (t,x)\in(0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) & x\in\mathbb{R}^n \end{cases} \qquad \text{(TE)}$$

where b, f and g are given functions.

This will serve as a first elementary example of the method of characteristics: the idea being to reduce a general PDE to a system of ordinary differential equations.

- Case 1: $b(t,x) = b$ is a constant vector,
  $f(t,x) = 0$
  $g\in C^1(\mathbb{R}^n)$.

Lemma: Consider the equation

$$u_t(t,x) + b\cdot Du(t,x) = 0 \qquad (t,x)\in(0,\infty)\times\mathbb{R}^n \qquad (*)$$

(i) For any $v\in C^1(\mathbb{R}^n)$, the function $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ defined by $u(t,x) = v(x-bt)$ is a solution of $(*)$ $u\in C^1((0,\infty)\times\mathbb{R}^n;\mathbb{R})\cap C^0([0,\infty)\times\mathbb{R}^n;\mathbb{R})$ with $u(0,x) = v(x)$.

(ii) If $u\in C^1((0,\infty)\times\mathbb{R}^n;\mathbb{R})\cap C^0([0,\infty)\times\mathbb{R}^n;\mathbb{R})$ is a solution of $(*)$, then $\exists\, v\in C^1(\mathbb{R}^n)$ s.t. $u(t,x) = v(x-bt)$.

Proof: i) By a direct computation, $\frac{\partial}{\partial t}u(t,x) = -b\cdot Dv(x-bt)$ and the claim follows from $Du(t,x) = Dv(x-bt)$.

ii) Consider the auxiliary function

$$z_{t,x} : [-t,\infty) \to \mathbb{R}$$
$$s \mapsto u(t+s, x+bs)$$

Note that $z \in C^1((-t,\infty);\mathbb{R}) \cap C^0([-t,\infty);\mathbb{R})$ and

$$z'(s) = u_t(t+s, x+bs) + b\cdot Du(t+s, x+bs)$$
$$= 0 \qquad \forall\, s \in (-t,\infty)$$

with $z(-t) = u(0, x-bt)$. Hence z is constant, and

$$u(t,x) = z(0) = z(-t) = u(0, x-bt) =: v(x-bt)$$

where $v : \mathbb{R}^n \to \mathbb{R}$, $v(y) = u(0,y)$, $v \in C^1(\mathbb{R}^n)$. $\square$

Rem: * The key of the proof of (ii) is to transform $(*)$ into a ordinary differential equation for z.

* The lemma shows that any solution of $(*)$ is constant along the lines

$$\{(t,x) \in [0,\infty)\times\mathbb{R}^n;\ x = bt + x_0,\ x_0 \in \mathbb{R}^n\}$$

called characteristics.

Proposition: Consider (TE) with $f \equiv 0$, $b(t,x) = b \in \mathbb{R}^n$ and $g \in C^1(\mathbb{R}^n)$. Then (TE) has a unique solution $u \in C^1((0,\infty)\times\mathbb{R}^n;\mathbb{R}) \cap C^0([0,\infty)\times\mathbb{R}^n;\mathbb{R})$ given by

$$u(t,x) = g(x-bt)$$

Proof: * Existence: Part (i) of the lemma.

* Uniqueness: Consider two solutions u_1, u_2, and let $w = u_1 - u_2$. Then w is a solution of

$$\begin{cases} w_t(t,x) - b\cdot Dw(t,x) = 0 & (t,x) \in (0,\infty)\times\mathbb{R}^n \\ w(0,x) = 0 \end{cases}$$

By (ii) of the lemma, $\exists\, v \in C^1(\mathbb{R}^n)$ s.t.

$w(t,x) = v(x-bt)$ and $v(x) = w(0,x) = 0$ for all $x \in \mathbb{R}^n$. Hence $w=0$.

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- Case 2: For any $t \in [0,\infty)$, $b(t,\cdot) \in C^1(\mathbb{R}^n;\mathbb{R}^n)$.

As above, we look for curves in $[0,\infty)\times\mathbb{R}^n$ along which the solution is constant. Parametrisation of the curve:

$$\Gamma(s) = (s, \gamma(s)) \quad , \gamma : \mathbb{R} \to \mathbb{R}^n$$

and $$\frac{d}{ds} u(s,\gamma(s)) = u_s(s,\gamma(s)) + \dot\gamma(s)\cdot Du(s,\gamma(s))$$

If u is a solution, then u is constant along $\Gamma(s)$ iff

$$\dot\gamma(s) = b(s,\gamma(s))$$

(again an ODE!) with "initial" condition $\gamma(t) = x$:

$$u(t,x) = u(t,\gamma(t)) = u(0,\gamma(0)) = g(\gamma(0))$$

In other words, $\gamma(0)$ must be chosen so that the characteristic $\Gamma(s) \ni (t,x)$. We have proved:

Proposition: Let $u \in C^1((0,\infty)\times\mathbb{R}^n) \cap C^0([0,\infty)\times\mathbb{R}^n)$ be a solution of (TE) with $f=0$, $b(t,\cdot) \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$. Assume that for $(t,x) \in (0,\infty)\times\mathbb{R}^n$, the problem

$$\begin{cases} \dot\gamma(s) = b(s,\gamma(s)) & s \in (0,t) \\ \gamma(t) = x \end{cases} \qquad (C)$$

has a solution $\gamma \in C^1((0,t);\mathbb{R}^n) \cap C^0([0,t];\mathbb{R}^n)$.

Then $$u(t,x) = g(\gamma(0))$$

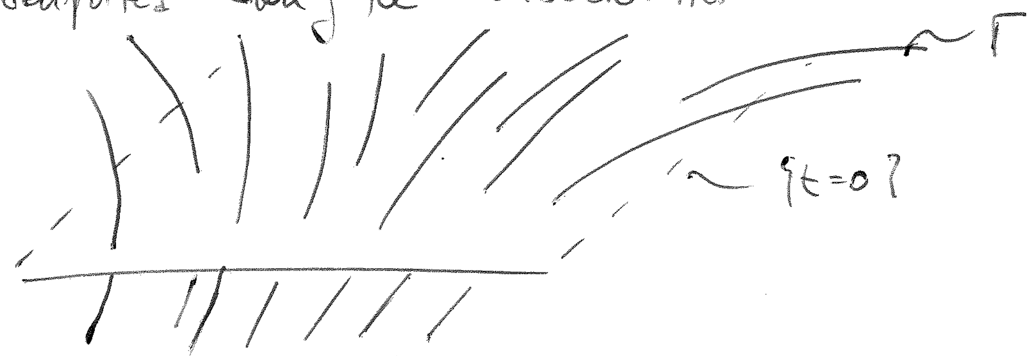
and $$u(t,x) = u(s,\gamma(s)) \quad \forall s \in [0,t].$$

- Rem: Hence: If the ODE (C) has a solution on $[0,T)$, then (TE) has at most one solution.
  - But in general: if $(t,x) \in \Gamma(s)$, there may

be no s for which $\Gamma(s) \cap \{t=0\} \neq \phi$.

Put differently: the set of characteristics Γ that cut the plane $\{(0,x), x\in\mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ do in general not cover all of $[0,\infty) \times \mathbb{R}^n$.

- As in case 1: The initial condition $g(x)$ is transported along the characteristics.



- Examples:

i) $b(t,x) = b$ (Case 1): (C) reads $\begin{cases} \dot{\gamma}(s) = b \\ \gamma(t) = x \end{cases}$

with solution $\gamma(s) = \gamma_0 + bs$ where γ_0 is determined by $x = \gamma_0 + bt$. Hence, for any $(t,x) \in (0,\infty)\times\mathbb{R}^n$, (C) has a (unique) solution $\gamma(s) = (x-bt)+bs$.

We set $u(t,x) = g(x-bt)$ and easily check that this solves the initial value problem (TE) on $[0,\infty)\times\mathbb{R}^n$.

ii) $b(t,x) = x$. Hence $\begin{cases} \dot{\gamma}(s) = \gamma(s) \\ \gamma(t) = x \end{cases}$

with solution $\gamma(s) = \gamma_0 \exp(s)$ and γ_0 is determined by $x = \gamma_0 e^t$

Hence $\gamma(s) = x e^{s-t}$ and

$u(t,x) = g(xe^{-t})$ is a solution.

iii) $b(t,x) = -x^2$ in the case $n=1$, and $g \in C_c^1(\mathbb{R})$ (compactly supported).

(c) $\begin{cases} \dot\gamma(s) = -(\gamma(s))^2 \\ \gamma(t) = x \end{cases}$

with solution $\gamma(s) = \begin{cases} \left((s-t)+\frac{1}{x}\right)^{-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

We look for the maximal domain of existence of this solution around $s=t$.

$$J(t,x) = \begin{cases} \left(t-\frac{1}{x}, \infty\right) & \text{if } x>0 \\ \left(-\infty, t-\frac{1}{x}\right) & \text{if } x<0 \\ \mathbb{R} & \text{if } x=0 \end{cases}$$

Let $t>0$. If $\exists y \in \mathbb{R}: (0,y) \in J(t,x)$, then

$t-\frac{1}{x} < 0$ and $x>0$, or $x<0$, namely

$x < \frac{1}{t}$. Hence

$$u(t,x) = g\left(\frac{x}{1-tx}\right)$$

is the unique solution in

$$G = \{(t,x) \in (0,\infty)\times\mathbb{R} : x < \tfrac{1}{t}\}.$$

Now: since g is compactly supported we have that $u(t,x) = 0$ in a neighbourhood of the critical curve $\{(t,\frac{1}{t}), t>0\}$.

Let $h \in C_c^1(\mathbb{R})$ be an arbitrary function. Then

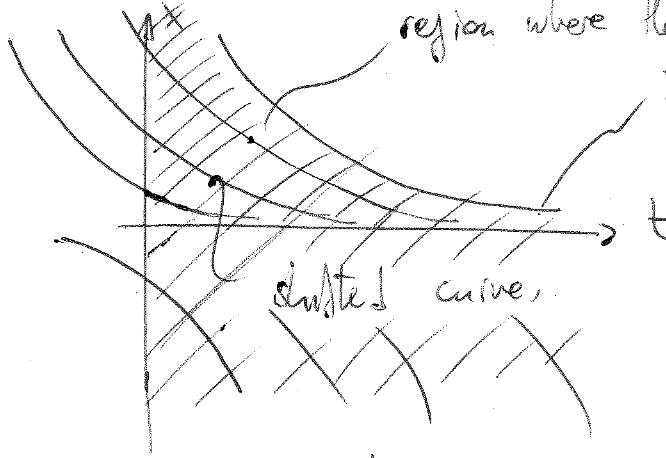
$$u(t,x) = \begin{cases} g\left(\frac{x}{1-tx}\right) & (t,x) \in (0,\infty)\times\mathbb{R} : x<\frac{1}{t} \\ 0 & x = t^{-1} \\ h\left(\frac{x}{1-tx}\right) & \text{— " —} \quad x > \frac{1}{t} \end{cases}$$

is a solution of $\begin{cases} u_t(t,x) - x^2 u_x(t,x) = 0 \\ u(0,x) = g(x) \end{cases}$

s.t. $u \in C^1((0,\infty)\times\mathbb{R};\mathbb{R}) \cap C^0([0,\infty)\times\mathbb{R};\mathbb{R})$

Check: $\lim_{t\searrow 0} u(t,x) = g(x)$ for all $x \in \mathbb{R}$

Geometric situation:



$\Gamma(s) = \left(s, \frac{1}{s + (\frac{1}{x} - t)}\right)$

• Case 3: The inhomogeneous equation: $f \in C^0([0,\infty)\times\mathbb{R}^n;\mathbb{R})$

Proposition: Under the assumption of the previous Proposition, where $f \equiv 0$ is replaced by $f \in C^0([0,\infty)\times\mathbb{R}^n;\mathbb{R})$

$$u(t,x) = g(\gamma(0)) + \int_0^t f(s,\gamma(s))\,ds$$

Proof: We compute

$$\frac{d}{ds} u(s,\gamma(s)) = u_t(s,\gamma(s)) + Du(s,\gamma(s)) \cdot \underbrace{\dot\gamma(s)}_{= b(s,\gamma(s))}$$

$$= f(s,\gamma(s))$$

because $u(t,x)$ solves (TE). Hence,

$$u(t,x) = u(t,\gamma(t)) = u(0,\gamma(0)) + \int_0^t f(s,\gamma(s))\,ds$$

by the fundamental theorem of calculus. $\square$

• Rem: In the inhomogeneous case, g is not simply transported along the characteristics, but $u(s,\gamma(s))$ is still uniquely determined by $u(0,\gamma(0)) = g(\gamma(0))$.

Example: $u_t(t,x) + x \cdot Du(t,x) = t \quad , \quad u(0,x) = g(x)$

(c) has already been solved to yield $\gamma(s) = x e^{s-t}$ so that we get

$u(t,x) = g(x e^{-t}) + \frac{1}{2} t^2$

as a candidate solution.

2. The Laplace & Poisson equations $(\Omega \neq \emptyset)$

- The equations: Let $\Omega \subseteq \mathbb{R}^n$ open and $f: \Omega \to \mathbb{R}$.

$$-\Delta u(x) = 0 \quad \text{(Laplace)}$$
$$-\Delta u(x) = f(x) \quad \text{(Poisson)} \qquad x \in \Omega.$$

One seeks solutions $u \in C^2(\Omega;\mathbb{R})$, usually also $u \in C^0(\overline{\Omega};\mathbb{R})$ with a given boundary condition (the so-called Dirichlet problem).

- Def. Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega;\mathbb{R})$. The function u is called <u>harmonic</u> if $\Delta u(x) = 0$ for all $x \in \Omega$.

<u>Rem</u>: Harmonic functions are ubiquitous in math and physics: gravitational potential, real part of complex differentiable functions, ...

We shall first study general properties of harmonic functions.

- Mean-value property:

<u>Theorem</u>. Let $\Omega \subset \mathbb{R}^n$ open, $u: \Omega \to \mathbb{R}$. T.f.a.e.:

i) u is harmonic

ii) u is continuous and for all $x \in \Omega$, $r > 0$ s.t. $\overline{B_r(x)} \subset \Omega$

$$u(x) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(x)} u(y)\, dS(y) \qquad \text{(MVP I)}$$

iii) u is continuous and for all $x \in \Omega$, $r > 0$ s.t. $\overline{B_r(x)} \subset \Omega$

$$\int_{B_r(x)} (u(x) - u(y))\, dy = 0 \qquad \text{(MVP II)}$$

<u>Note</u>: * ω_n is the surface of the unit sphere in $\mathbb{R}^n$, $\omega_n = \int_{\partial B_1(0)} 1 \cdot dS(y) = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

* Using polar coordinates:

$$\int_{B_r(x)} u(y)\,dy = \int_0^r \left(\int_{\partial B_s(x)} u(y)\,dS(y) \right) ds$$

so that (MVP I) implies

$$\int_{B_r(x)} u(y)\,dy = \int_0^r \omega_n s^{n-1}\,ds\; u(x) = \alpha_n r^n u(x)$$

where $\alpha_n = \frac{\omega_n}{n}$ is the volume of the unit ball in $\mathbb{R}^n$. Hence, if u is harmonic,

$$u(x) = \frac{1}{r^n \alpha_n} \int_{B_r(x)} u(y)\,dy \quad , \text{ which is (MVP II)}$$

* Notation: we shall often write:

$$⨍_{\partial B_r(x)} (\dots) \quad \text{for} \quad \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(x)} (\dots) \qquad \text{and}$$

$$⨍_{B_r(x)} (\dots) \quad \text{for} \quad \frac{1}{r^n \alpha_n} \int_{B_r(x)} (\dots)$$

<u>Proof</u>: (i) $\Rightarrow$ (ii): Let $\phi(r) := ⨍_{\partial B_r(x)} u(y)\,dS(y)$

Since u is harmonic, $u \in C^1(\overline{B_r(x)})$, so that by ex of Sheet 2, $\phi \in C^1((0,r))$ with

$$\phi'(r) = ⨍_{\partial B_1(0)} \nabla u(x+rz) \cdot z\,dS(z)$$

(by writing $y = x+rz$, $z \in \partial B_1(0)$)

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u being harmonic, $Du \in C^1(\overline{B_r(x)})$, so that the divergence theorem yields

$$\phi'(r) = \fint_{\partial B_r(x)} Du(y) \cdot \frac{y-x}{r} \, dS(y) = \frac{r}{n} \fint_{B_r(x)} \Delta u(y) \, dy \quad (\Diamond)$$

$$= 0$$

since $-\Delta u = 0$ for all $x \in \Omega$.

Hence, $\phi(r)$ is constant

$$\phi(r) = \lim_{s \searrow 0} \phi(s) = \lim_{s \searrow 0} \frac{1}{s^{n-1}\omega_n} \int_{\partial B_s(x)} u(y) \, dS(y) = u(x)$$

since u is continuous on $B_r(x)$.

(ii) ⇒ (iii). Integrating (MVPI) (multiplied by $\omega_n r^{n-1}$) yields:

$$\int_0^r u(x) s^{n-1} \omega_n \, ds = \int_0^r \left(\int_{\partial B_s(x)} u(y) \, dS(y) \right) ds$$

$$\alpha_n r^n u(x) \qquad\qquad \int_{B_r(x)} u(y) \, dy$$

$$\int_{B_r(x)} u(x) \, dy$$

(iii) ⇒ (ii) Since (iii) holds for all $s \in (0, r)$:

$$0 = \frac{d}{ds} \int_{B_s(x)} (u(x) - u(y)) \, dy = \int_{\partial B_s(x)} (u(x) - u(y)) \, dS(y)$$

~~and~~ by ex of Sheet 2. Carrying out the integration of the first term yields the claim.

(ii) ⇒ (i). Assume first that $u \in C^2(\Omega)$. Consider the function $\phi(r)$ defined above. Integrating $(\Diamond)$ yields

$$\phi(r) = u(x) + \frac{1}{n} \int_0^r s \fint_{B_s(x)} \Delta u(y) \, dy \quad (\Diamond\Diamond)$$

where we used that $\lim_{s \searrow 0} \phi(s) = u(x)$.

Assume that x is so that $\Delta u(x) > 0$. By continuity, $\exists r > \varepsilon > 0$ s.t. $\Delta u(y) > 0$ for all $y \in B_\varepsilon(\cancel{x})$. Then, $(\ast\ast) \Rightarrow u(x) < \phi(\varepsilon)$, namely

$$u(x) < \fint_{\partial B_\varepsilon(x)} u(y)\, dS(y)$$

which contradicts (MVPI).

* It remains to prove $(ii) \Rightarrow u \in C^\infty(\Omega;\mathbb{R})$, and by the above, it is sufficient to prove $(iii) \Rightarrow u \in C^\infty(\Omega;\mathbb{R})$.

We need the following lemma:

<u>Lemma</u>: Let $\varepsilon > 0$ and

$$\Omega_\varepsilon := \{x \in \Omega : \overline{B_\varepsilon(x)} \subset \Omega\}.$$

Let $f \in C^0(\Omega)$. Then, the function $F : \Omega_\varepsilon \to \mathbb{R}$,

$$F(x) = \int_{B_\varepsilon(x)} f(y)\, dy$$

is in $C^1(\Omega_\varepsilon)$, and

$$F_{x_i}(x) = \int_{\partial B_\varepsilon(x)} f(y) \frac{(y-x)_i}{\varepsilon}\, dS(y)$$

We prove by induction $u \in C^k(\Omega;\mathbb{R})$ $\forall k \in \mathbb{N} \cup \{0\}$, with $k=0$ holding by assumption. Now: if $u \in C^k(\Omega;\mathbb{R})$, then the map $x \longmapsto \int_{B_r(0)} u(x+z)\, dz$ is also $C^k(\Omega_r;\mathbb{R})$, with $D^\alpha \int(\ldots) = \int D^\alpha_x u(x+z)\, dz$ for any $|\alpha| = k$. By (MVPII):

$$D^\alpha_x u(x) = D^\alpha_x \fint_{B_r(x)} u(y)\, dy = D^\alpha_x \fint_{B_r(0)} u(x+z)\, dz$$

$$= \fint_{B_r(0)} D^\alpha_x u(x+z)\,dz = \fint_{B_r(x)} D^\alpha u(y)\,dy$$

and the lemma ensures that the r.h.s. is $C^1(\Omega_r;\mathbb{R})$ since $u \in C^k(\Omega;\mathbb{R})$. Hence $u \in C^{k+1}(\Omega_r;\mathbb{R})$. Since $\forall x \in \Omega,\ \exists r>0: x \in \Omega_r$, this implies that $u \in C^{k+1}(\Omega;\mathbb{R})$. $\square$

- We have proved the following proposition along the way:

Proposition: If $u:\Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is harmonic, then $u \in C^\infty(\Omega;\mathbb{R})$

In fact, any partial derivative of u is again harmonic, since
$$\Delta u_{x_i}(x) = \frac{\partial}{\partial x_i}\Delta u(x) = 0.$$

- We finally prove the lemma:

It suffices to prove differentiability in a neighbourhood of an arbitrary $x_0 \in \Omega_\varepsilon$. $\exists R>0: \overline{B_R(x_0)} \subset \Omega$ and we consider all $x \in \Omega$: $\overline{B_\varepsilon(x)} \subset B_R(x_0)$. By A4, Sheet 1, T4, Sheet 1, $\exists (f_n)_{n\in\mathbb{N}}$, $f_n \in C^1(B_R(x))$ converging uniformly on $\overline{B_\varepsilon(x)}$. It follows that

$$F_n(x) := \int_{B_\varepsilon(x)} f_n(y)\,dy \longrightarrow F(x)$$

Furthermore, $$\frac{\partial}{\partial x_i}\int_{B_\varepsilon(x)} f_n(y)\,dy = \int_{B_\varepsilon(0)} \frac{\partial}{\partial x_i} f_n(x+z)\,dz$$
$$= \int_{\partial B_\varepsilon(x)} f_n(y)\frac{(y-x)_i}{\varepsilon}\,dy$$

by Gauss-Green's theorem, since $f_n \in C^1(\overline{B_\varepsilon(x)})$. Since, furthermore, $\int_{\partial B_\varepsilon(x)} f_n(y)\frac{(y-x)_i}{\varepsilon}\,dy \longrightarrow \int_{\partial B_\varepsilon(x)} f(y)\frac{(y-x)_i}{\varepsilon}\,dy$

uniformly on $\{x \in \Omega : \overline{B_\varepsilon(x)} \subset B_R(x_0)\}$, namely

$$f_n \to f$$
$$\frac{\partial}{\partial x_i} f_n \xrightarrow{u} \int_{\partial B_\varepsilon(x)} f(y) \frac{(y-x)_i}{\varepsilon}\, dy\,,$$

the claim follows. $\square$

- We now obtain a sequence of important corollaries of MVP:

<u>Theorem</u> (Harnack I) $\Omega \subset \mathbb{R}^n$ open and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of harmonic functions in Ω which converges locally uniformly. Then

i) $(u_n)_{n\in\mathbb{N}}$ converges pointwise in Ω, $u_n \to u$ $(n\to\infty)$, and $u: \Omega \to \mathbb{R}$ is harmonic

ii) For any multiindex α, $(D^\alpha u_n)_{n\in\mathbb{N}}$ converges locally uniformly to $D^\alpha u$.

<u>Recall</u>: $u_n \to u$ locally uniformly if for any $\Omega' \subset \Omega$, Ω' open and $\overline{\Omega'} \subset \Omega$, $u_n\restriction_{\Omega'} \xrightarrow{u} u\restriction_{\Omega'}$.

<u>Proof</u>: i) Since $(u_n)_{n\in\mathbb{N}}$ converges locally uniformly in Ω, it converges pointwise to a $u: \Omega \to \mathbb{R}$, and u is continuous. Now: let $x \in \Omega$, $r > 0$: $B_r(x) \subset \Omega$. Since u_n is harmonic,

$$\int_{B_r(x)} (u_n(x) - u_n(y))\, dy = 0 \qquad \text{for all } n \in \mathbb{N}\,;$$

The uniform convergence of $(u_n)_{n\in\mathbb{N}}$ in $\overline{B_r(x)}$ implies

$$\int_{B_r(x)} (u(x) - u(y))\, dy = \lim_{n\to\infty} \int_{B_r(x)} (u_n(x) - u_n(y))\, dy = 0$$

hence u is harmonic in Ω.

ii) It also follows that $u \in C^\infty(\Omega;\mathbb{R})$. We prove inductively that $(D^\alpha u_n)_{n\in\mathbb{N}}$ converges locally uniformly for $|\alpha|=k$, and all $k\in\mathbb{N}\cup\{0\}$. $k=0$ just proved. Now:

$$D^\alpha u_j(x)=\frac{1}{\alpha_n r^n}\int_{B_r(x)} D^\alpha u_j(y)\,dy$$

and since $(D^\alpha u_j)$ is continuous,

$$\frac{\partial}{\partial x_i}(D^\alpha u_j)(x)=\frac{1}{\alpha_n r^n}\int_{\partial B_r(x)} D^\alpha u_j(y)\cdot\frac{(y-x)_i}{r}\,dS(y)$$

Since the r.h.s converges by the uniform convergence of $D^\alpha u_j$ on $B_r(x)$, so does the l.h.s, and the convergence is again locally uniform. Hence $(D^\alpha u_n)\to D^\alpha u$ $\square$

Note: The theorem also goes by the name of Weierstrass who proved its analogon in complex analysis.

In fact, one can prove more than this, namely that the set of harmonic function on a given Ω is compact:

Theorem: Let $\Omega\subset\mathbb{R}^n$ open and $(u_n)_{n\in\mathbb{N}}$ be a sequence of harmonic functions. Assume $\exists\, M>0$: $|u_n(x)|\le M$ for all $x\in\Omega$, $n\in\mathbb{N}$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges locally uniformly.

- Theorem (Liouville) If $u:\mathbb{R}^n\to\mathbb{R}$ is harmonic and bounded below or above, then u is constant.

~~Prop~~ Lemma: Let $\Omega\subset\mathbb{R}^n$ open, $u:\Omega\to\mathbb{R}$ harmonic and $u\ge 0$. Let $x,y\in\Omega$ s.t. $|x-y|=d$. Then:

$$u(x)\le\left(1+\frac{d}{r}\right)^n u(y)$$

for all $r>0$ s.t. $\overline{B_{r+d}(y)}\subset\Omega$.

Proof of the lemma: By (MVP II) and since $B_r(x) \subset B_{r+d}(y)$:

$$\alpha_n r^n u(x) = \int_{B_r(x)} u(z)\,dz \leq \int_{B_{r+d}(y)} u(z)\,dz = \alpha_n (r+d)^n u(y)$$

□

Proof of Liouville's theorem: W.l.o.g., u is bounded below: $u(x) \geq C$ $\forall x \in \mathbb{R}^n$. Let $w := u - C \geq 0$.

Since w is harmonic in $\mathbb{R}^n$, the lemma implies that $\forall x, y \in \mathbb{R}^n$, with $d = |x-y|$, $w(x) \leq (1 + d/r)^n w(y)$ for all $r > 0$. Since r is arbitrary: $w(x) \leq w(y)$. Repeating the argument with x and y exchanged yields $w(y) \leq w(x)$, hence w is constant. □.

- **Theorem** (Harnack II) Let $\Omega \subset \mathbb{R}^n$ open and connected. Then for any $\Omega_0 \subset \Omega$, open, $\overline{\Omega_0} \subset \Omega$ compact, $\exists\ C_0 = C_0(\Omega_0) > 0$ s.t. if $u : \Omega \to \mathbb{R}$ is harmonic and $u \geq 0$, then

$$u(x) \leq C_0 u(y) \quad , \quad x, y \in \Omega_0 .$$

Note: * In particular:

$$\sup_{x \in \Omega_0} u(x) \leq C_0 \inf_{x \in \Omega_0} u(y)$$

uniformly for all non-negative harmonic functions in Ω: C_0 depends on Ω but not on u!

→ effect of the mean-value property away from $\partial\Omega$.

* If Ω is connected and u is harmonic and $u \geq C$ on Ω. Let $w = u - \inf u$, $w \geq 0$ and harmonic. If $\exists\ x_0 \in \mathring{\Omega}$: $u(x_0) = \inf u$, then $w(x_0) = 0$

and by Harnack's theorem: $w(x) = w(x_0) = 0$ for all $x \in \Omega$. In other words: if u attains its minimum inside Ω, then it is constant.

$\rightsquigarrow$ Minimum Principle.

<u>Proof</u>: Let

$$d := \begin{cases} \operatorname{dist}(\Omega_0, \partial\Omega) & \text{if } \operatorname{dist}(\Omega_0, \partial\Omega) < \infty \\ 1 & \text{if } \operatorname{dist}(\Omega_0, \partial\Omega) = \infty. \end{cases}$$

Since $\overline{\Omega_0}$ is compact, there is a <u>finite</u> cover of Ω_0: $\exists\, n_0 \in \mathbb{N}$, $\{x_i : i = 1, \dots, n_0\}$, $x_i \in \overline{\Omega_0}$, s.t.

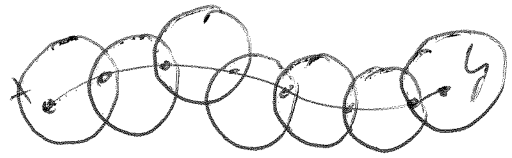
$$\overline{\Omega_0} \subset \bigcup_{i=1}^{n_0} B_{d/4}(x_i)$$

Let $\Gamma := \{B_{d/4}(x_i) : 1 \le i \le n_0\}$.

Since Ω is connected and open, if $x, y \in \Omega_0$, $\exists$ $m \le n_0$ and element $B^1, \dots, B^m \in \Gamma$ s.t.

$$x \in B^1,\ y \in B^m \text{ and } B^i \cap B^{i+1} \neq \emptyset$$

Let $z_i \in B^i \cap B^{i+1}$ with $z_0 = x$, $z_m = y$:



we have $|z_{i+1} - z_i| < \frac{d}{2}$

We also have that $\overline{B_{\frac{3d}{4}}(z_i)} \subset \Omega$. By the lemma with $r = d/4$:

$$u(z_i) \le \left(1 + \frac{d/2}{d/4}\right)^n u(z_{i+1}) = 3^n u(z_{i+1})$$

for all $i \in \{0, \dots, m-1\}$. Hence $u(x) \le 3^{mn} u(y)$ and $m \le n_0$ implies that $c_0 = 3^{n_0 n}$, where $n_0 = n_0(\Omega_0, \Omega)$. $\square$

- **Theorem**: Let $\Omega \subset \mathbb{R}^n$ open, $u: \Omega \to \mathbb{R}$ harmonic. Then the set of minimal points

$$M := \{ x_0 \in \Omega : u(x_0) \le u(x) \ \forall x \in \Omega \}$$

is open, possibly empty.

Proof: • The empty set is open.
• If $M \neq \emptyset$. Let $x_0 \in M$. $\exists\, r > 0$: $\overline{B_r(x_0)} \subset \Omega$, so that (MVP II) implies $\int_{B_r(x_0)} (u(x_0) - u(y))\, dy = 0$ and the integrand is continuous and ≤ 0. Hence $u(y) = u(x_0)$ for all $y \in B_r(x_0)$, namely $B_r(x_0) \subset M$ □

- Remark: By replacing u by $(-u)$, the claim of the theorem remains true for the set of maximal points.

- Corollary: Let $\Omega \subset \mathbb{R}^n$ open, $u: \Omega \to \mathbb{R}$ harmonic. If u has a minimal point $x_0 \in \Omega$, then u is constant on the connected component $Z \ni x_0$, $Z \subset \Omega$.

Proof. We have that Z is open and $u\restriction_Z$ is harmonic. Let M_Z be the set of minimal points of u in Z. By the theorem, M_Z is open. Since $u\restriction_Z$ is continuous, if x is an accumulation point of M_Z in Z, then $x \in M_Z$, i.e. M_Z is relatively closed in Z. Since Z is connected, M_Z is therefore either $\emptyset$ or all of Z. $x_0 \in M_Z \Rightarrow M_Z = Z$ □

- Corollary: Let $\Omega \subset \mathbb{R}^n$ open, connected, $u: \Omega \to \mathbb{R}$ harmonic and $u \in C^0(\overline{\Omega}, \mathbb{R})$. Then:

$$\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x)$$

and

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$$

Proof: Clearly, $\min_{x\in\bar\Omega} u(x) \leq \min_{x\in\partial\Omega} u(x)$

If $\exists x_0\in\Omega$: $\min_{x\in\Omega} u(x) < \min_{x\in\partial\Omega} u(x)$, then by the first corollary u is constant in Ω and by continuity in $\bar\Omega$, contradiction. $\square$

- We have already proved: u harmonic $\Rightarrow u\in C^\infty(\Omega;\mathbb{R})$. More is true: u is in fact real analytic. For this, we need estimates on the derivatives of u:

Proposition: $\Omega\subset\mathbb{R}^n$ open, $u:\Omega\to\mathbb{R}$ harmonic. Then, for any $x\in\Omega$, $r>0$: $B_r(x)\subset\Omega$ and any multiindex α $|\alpha|=k$:

$$|D^\alpha u(x)| \leq \frac{C(n,k)}{r^k}\sup_{y\in B_r(x)}|u(y)|$$

where $C(n,k) = n^k k!\, e^{k-1}$.

Proof: Recall: Any derivative of u is harmonic.

The case $k=1$ follows from Ex. 2, Sheet 3.

Let $k\geq 2$, $|\alpha|=k$, and $s=\frac{k}{k+1}r<r$

$$|D^\alpha u_{x_i}(x)| \leq \frac{C(n,k)}{s^k}\sup_{y\in B_s(x)}|u_{x_i}(y)| \quad \text{by the ind. hyp.}$$

By the case $k=1$: $|u_{x_i}(y)| \leq \frac{n}{r-s}\underbrace{\sup_{z\in B_{r-s}(y)}|u(z)|}$

$$\leq \sup_{z\in B_r(x)}|u(z)|$$

Gathering all constants,

$$\frac{C(n,k)\,n}{\left(\frac{k}{k+1}\right)^k\left(\frac{1}{k+1}\right)r^{k+1}} = \frac{n^{k+1}(k+1)!\,e^{k-1}}{r^{k+1}}\underbrace{\left(1+\frac{1}{k}\right)^k}_{\leq e} \leq \frac{C(n,k+1)}{r^{k+1}}$$ $\square$

- Technical lemma: Let $u \in C^\infty(B_s(x_0))$. If $\exists M, C>0$ s.t. $|D^\alpha u(x)| \leq C^k k!\, M$ for all $x \in B_s(x_0)$, $|\alpha|=k$, $k\in\mathbb{N}$, then u is analytic at x_0.

Proof: By Taylor's theorem: if $h \in \mathbb{R}^n$, $|h|<s$, $\exists\, 0<\tau<1$:

$$u(x_0+h) = \sum_{j=0}^{m-1}\frac{1}{j!}\left((h\cdot D)^j u\right)(x_0) + \underbrace{\frac{1}{m!}\left((h\cdot D)^m u\right)(x_0+\tau h)}_{=:R_m}$$

In order to estimate R_m, we apply

$$(a_1+\dots+a_n)^m = \sum_{\mu_1+\dots+\mu_n=m}\left(\frac{m!}{\mu_1!\cdots\mu_n!}\right)a_1^{\mu_1}\cdots a_n^{\mu_n}$$

to $(h\cdot D)^m = \left(\sum_{i=1}^n h_i\frac{\partial}{\partial x_i}\right)^m$ to get

$$|R_m| \leq \frac{1}{m!}\sum_{\mu_1+\dots+\mu_n=m}\frac{m!}{\mu_1!\cdots\mu_n!}|h_1|^{\mu_1}\cdots|h_n|^{\mu_n}\,|D^\mu u(x_0+\tau h)|$$

Since $|\mu|=m$, the derivative is bounded by hypothesis, so that

$$|R_m| \leq \frac{1}{m!}\left(\sum_{i=1}^n|h_i|\right)^m C^m m!\, M$$

i.e. $|R_m| \leq \left(C\|h\|_{\ell^1}\right)^m M$

and $|R_m| \to 0$ $(m\to\infty)$ whenever $\|h\|_{\ell^1} < 1/C$ $\square$.

- Theorem: $\Omega\subset\mathbb{R}^n$ open, $u:\Omega\to\mathbb{R}$ harmonic. Then, u is real analytic in Ω.

Proof: First of all, $u \in C^\infty(\Omega;\mathbb{R})$. Let $x_0 \in \Omega$ and $r > 0$ s.t. $\overline{B_r(x_0)} \subset \Omega$. By continuity, $\exists K$ s.t. $|u(x)| \le K \quad \forall x \in \overline{B_r(x_0)}$. By the proposition

$$|D^\alpha u(x)| \le \frac{C(n,k)}{(r-s)^k} \underbrace{\sup_{y \in B_r(x_0)} |u(y)|}_{\le K}$$

for all $x \in B_s(x_0)$ (since then $\overline{B_{r-s}(x)} \subset B_r(x_0)$)
The claim follows by applying the lemma with ($s = \frac{r}{2}$)
$C = \frac{2ne}{r}$, $M = \frac{K}{e}$ and the fact that x_0 is arbitrary in Ω. $\square$

- Remark (elliptic regularity) Any solution of the Laplace equation which is a priori required to be only $C^2(\Omega;\mathbb{R})$ is not only $C^\infty(\Omega;\mathbb{R})$ but even real analytic in Ω.
- Finally, we extend the minimum principle to superharmonic functions:

Theorem (weak minimum principle). Let $\Omega \subset \mathbb{R}^n$ open and s.t. $\overline{\Omega}$ is compact. Let $u: \Omega \to \mathbb{R}$ be s.t. $u \in C^2(\Omega;\mathbb{R}) \cap C^0(\overline{\Omega};\mathbb{R})$, and $-\Delta u \ge 0$ in Ω. Then there is at least one minimal point of u in $\partial\Omega$.

Proof: Since u is continuous on $\overline{\Omega}$, then for any $\varepsilon > 0$, the function $v_\varepsilon(x) := u(x) - \varepsilon|x|^2$ defined on $\overline{\Omega}$ has a minimal point x_ε.
We assume that $x_\varepsilon \in \Omega$. Then the Hessian $D^2 v_\varepsilon(x_\varepsilon)$ is non-negative definite, so that

$$\Delta v_\varepsilon(x_\varepsilon) = \mathrm{Tr}\left(D^2 v_\varepsilon(x_\varepsilon)\right) \geq 0.$$

On the other hand, the superharmonicity of u implies

$$\Delta v_\varepsilon(x) = \Delta u(x) - 2\varepsilon n \leq -2\varepsilon n < 0 \quad \forall x \in \Omega,$$

which is a contradiction. Hence $x_\varepsilon \in \partial\Omega$, and

$$u(x) = v_\varepsilon(x) + \varepsilon|x|^2 \geq v_\varepsilon(x) \geq v_\varepsilon(x_\varepsilon) = u(x_\varepsilon) - \varepsilon|x_\varepsilon|^2$$
$$\geq \min_{y\in\partial\Omega} u(y) - \varepsilon|x_\varepsilon|^2 \quad \text{for all } x \in \bar\Omega.$$

Since this holds $\forall \varepsilon > 0$: $u(x) \geq \min_{y\in\partial\Omega} u(y) \; \forall x \in \bar\Omega$. $\square$

- We now turn to solving the Dirichlet Problem for Laplace's equation, namely: Let $\Omega \subset\subset \mathbb{R}^n$, $g: \partial\Omega \to \mathbb{R}$ continuous and given. Solve:

$\uparrow$ $\bar\Omega \in \mathbb{R}^n$, $\bar\Omega$ compact, Ω open.

$$\begin{cases} -\Delta u(x) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \qquad (DP)$$

where $u \in C^2(\Omega;\mathbb{R}) \cap C^0(\bar\Omega;\mathbb{R})$.

We'll look at: existence, uniqueness, explicit solutions, stability.

In other words: We look for a continuous extension of g to $\bar\Omega$, which is harmonic in Ω. The answer is positive and unique under a very mild assumption on $\partial\Omega$.

- We start with uniqueness:

Theorem: $\Omega \subset\subset \mathbb{R}^n$ and let $g \in C^0(\partial\Omega;\mathbb{R})$. Then there exists at most one function $u: \Omega \to \mathbb{R}$, $u \in C^2(\Omega;\mathbb{R}) \cap C^0(\bar\Omega;\mathbb{R})$ solving (DP).

Proof: Let u, v be two solutions of (DP) and let $w = u - v$.
Then $w \in C^0(\bar{\Omega})$ and w is harmonic in Ω.
By the minimum principle
$$u(x) - v(x) \geq \min_{y \in \bar{\Omega}} w(y) = \min_{y \in \partial\Omega} w(y) = 0 \quad \forall x \in \bar{\Omega}$$
By the same argument applied to $\tilde{w} = v - u$, $v(x) - u(x) \geq 0$
Hence $u(x) = v(x)$ for all $x \in \bar{\Omega}$. $\square$.

- We continue with stability with respect to the boundary condition: (continuity in the L^∞-norm).

Theorem: $\Omega \subset\subset \mathbb{R}^n$, let $u, v \in C^2(\Omega; \mathbb{R}) \cap C^0(\bar{\Omega}; \mathbb{R})$ be harmonic functions, such that
$$\sup_{y \in \partial\Omega} |u(y) - v(y)| \leq \varepsilon .$$
Then:
$$\sup_{y \in \bar{\Omega}} |u(y) - v(y)| \leq \varepsilon$$

Proof: By the minimum principle applied to $u - v$:
$$u(x) - v(x) \geq \min_{y \in \bar{\Omega}} (u(y) - v(y)) = \min_{y \in \partial\Omega} (u(y) - v(y)) \geq -\varepsilon \quad \forall x \in \bar{\Omega}$$
Similarly with $v - u$: $u(x) - v(x) \leq \varepsilon$
Hence $|u(x) - v(x)| \leq \varepsilon \quad \forall x \in \bar{\Omega}$. $\square$.

- Remarks: * One can prove slightly more: monotonicity, namely $g_1(x) \geq g_2(x) \ \forall x \in \partial\Omega \Rightarrow u_1(x) \geq u_2(x) \ \forall x \in \bar{\Omega}$.
  * The weak minimum principle extends to solutions of more general uniformly elliptic equations, so that the above conclusions hold similarly.

- We first solve (DP) for the ball.

Theorem: Let

$$u(x) := \begin{cases} \dfrac{1}{r\omega_n} \displaystyle\int_{\partial B_r(0)} \dfrac{|y|^2 - |x|^2}{|y-x|^n} g(y)\, dS(y) & |x| < r \\ g(x) & |x| = r \end{cases}.$$

$u : \overline{B_r(0)} \to \mathbb{R}$ is a solution of (DP).

Notes: 1. By "a solution of (DP)" we mean $u \in C^2(B_r(0);\mathbb{R}) \cap C^0(\overline{B_r(0)};\mathbb{R})$, u is harmonic in $B_r(0)$ and $u\restriction_{\partial B_r(0)} = g$.

2. The kernel $K_r : B_r(0) \times \partial B_r(0) \to \mathbb{R}$,

$$K_r(x,y) = \frac{1}{r\omega_n} \frac{r^2 - |x|^2}{|y-x|^n}$$

is called the Poisson kernel.

Proof: * Let $\Omega \subset B_r(0)$: $\overline{\Omega} \subset B_r(0)$, Ω open. We first prove that u is harmonic on Ω. For any fixed $y \in \partial B_r(0)$, the map $\Omega \ni x \mapsto \frac{|y|^2 - |x|^2}{|y-x|^n} g(y)$ is $C^\infty(\Omega;\mathbb{R})$.
It follows that all derivatives of the integrand are bounded continuous functions, so that $u(x) \in C^\infty(\Omega;\mathbb{R})$.
Furthermore:

$$\Delta u(x) = \Delta_x \frac{1}{r\omega_n} \int_{\partial B_r(0)} (\dots) = \frac{1}{r\omega_n} \int_{\partial B_r(0)} \Delta_x \frac{|y|^2 - |x|^2}{|y-x|^n} g(y)\, dS(y)$$

$$= 0$$

since $K_r(\cdot, y)$ is harmonic on Ω for any fixed $y \in \partial B_r(0)$

Hence u is harmonic on Ω for any Ω, i.e. u is harmonic in $B_r(0)$.

* It remains to prove continuity at $\partial B_r(0)$. First, let

$$I(x) := \frac{1}{r\omega_n} \int_{\partial B_r(0)} \frac{|y|^2 - |x|^2}{|y-x|^n} \, dS(y) \quad \text{for any } |x| < r.$$

By the above, I is harmonic on $B_r(0)$. By the minimum principle applied to $B_R(0)$, $R < r$, $\exists\, x_{min}, x_{max} \in \partial B_R(0)$ s.t.

$$I(x_{min}) \leq I(x) \leq I(x_{max})$$

for all $x \in \overline{B_R(0)}$. Now: for any orthogonal transformation O (rotation!), we have $I(Ox) = I(x)$, in particular for a O s.t. $x_{max} = O x_{min}$. Hence $I(x_{min}) = I(x_{max})$

$\Rightarrow$ $x \mapsto I(x)$ is constant on $B_r(0)$, and

$$I(x) = I(0) = \frac{1}{r\omega_n} \int_{\partial B_r(0)} \underbrace{|y|^{2-n}}_{r^{2-n}} \, dS(y) = \frac{1}{r^{n-1}\omega_n} \cdot r^{n-1}\omega_n = 1.$$

Let now $y_0 \in \partial B_r(0)$ and pick $(x_j)_{j\in\mathbb{N}}$: $x_j \in B_r(0)$, $x_j \to y_0$ $(j \to \infty)$. By the above:

$$u(x_j) - g(y_0) = \frac{1}{r\omega_n} \int_{\partial B_r(0)} \frac{|y|^2 - |x_j|^2}{|y - x_j|^2} \big(g(y) - g(y_0)\big) \, dS(y)$$

Let $\varepsilon > 0$. $\exists\, \delta > 0$: $y \in B_{2\delta}(y_0) \cap \partial B_r(0)$ implies $|g(y) - g(y_0)| < \varepsilon/2$ by the continuity of g. Splitting

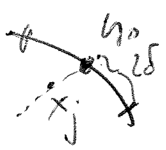
$$\int_{\partial B_r(0)} (\ldots) = \int_{B_{2\delta}(y_0) \cap \partial B_r(0)} (\ldots) + \int_{\partial B_r(0) \setminus (B_{2\delta}(y_0) \cap \partial B_r(0))} (\ldots) = I_1 + I_2$$

we first have

$$|I_1| \le \frac{\varepsilon}{2} \int_{B_{2\delta}(y_0)\cap\partial B_r(0)} (\dots) \le \frac{\varepsilon}{2} r\omega_n \int_{\partial B_r(0)} K_r(x,y)\,dS(y) = \frac{r\omega_n \varepsilon}{2}.$$

For I_2: We note that $\exists\, M > 0$ s.t. $|g(y) - g(y_0)| < M$ for all y in the compact domain of integration by $g \in C^0(\partial B_r(0))$.

Furthermore: if j is s.t. $|x_j - y_0| < \delta$, then



$$|y - x| \ge |y - y_0| - |y_0 - x_j| > \delta, \text{ so that}$$

$$\frac{|y|^2 - |x|^2}{|y - x_j|^n} = \frac{(r + |x_j|)(|y_0| - |x_j|)}{|y - x_j|^n} < \frac{2r\,|y_0 - x_j|}{\delta^n}$$

Altogether: $|I_2| \le \cancel{\frac{1}{r\omega_n}} \cdot r^{n-1}\omega_n \cdot \frac{2r|y_0 - x_j|}{\delta^n} M$ (j large enough)

$$\Rightarrow |u(x_j) - g(y_0)| \le \frac{\varepsilon}{2} + \frac{2Mr^{n-1}}{\delta^n}|y_0 - x_j| < \varepsilon$$

for j large enough, so that $|y_0 - x_j| < \min\left\{\delta, \frac{\varepsilon\delta^n}{4Mr^{n-1}}\right\}$. $\square$

- The representation can of course be translated to any ball:

$$K_{B_r(x_0)}(x,y) = \frac{1}{r\omega_n} \frac{|y|^2 - |x - x_0|^2}{|y - x|^n}$$

With this, one obtains another self-consistent equation for harmonic functions (compare with (MVPI)):

- <u>Theorem</u>: Let $\Omega \subset \mathbb{R}^n$ be open and $u: \Omega \to \mathbb{R}$ harmonic. For any $x_0 \in \Omega$, $r > 0$: $\overline{B_r(x_0)} \subset \Omega$:

$$u(x) = \int_{\partial B_r(x_0)} K_{B_r(x_0)}(x,y)\, u(y)\, dS(y)$$

for all $x \in B_r(x_0)$

- Remarks: * Here harmonicity is assumed
  * Contrary to (MVPI), x is any point in $B_r(x_0)$ (not necessarily x_0), see Cauchy's integral formula in complex analysis.

- Proof: By the previous thm, the r.h.s defines a harmonic function on $B_r(x_0)$, which is continuous on $\overline{B_r(x_0)}$ with

$$\int_{\partial B_r(x_0)} K_{B_r(x_0)}(\cdot, y)\, u(y)\, dS(y) \Big|_{\partial B_r(x_0)} = u\Big|_{\partial B_r(x_0)}$$

By the uniqueness of the solution of (DP):

$$\int(\ldots) = u(x) \qquad \text{for all } x \in \overline{B_r(x_0)} \qquad \square$$

- We now turn to the existence of a solution of (DP) for an arbitrary $\Omega \subset\subset \mathbb{R}^n$ $\rightsquigarrow$ "Perron's method".

If $u \in C^0(\overline{\Omega})$, we define for any $B := B_r(x_0) \subset\subset \Omega$ a linear map on $C^0(\overline{\Omega})$ by

$$\left(H_B(u)\right)(x) := \begin{cases} \displaystyle\int_{\partial B} K_B(x,y)\, u(y)\, dS(y) & \text{if } x \in B \\ u(x) & \text{if } x \in \overline{\Omega} \setminus B \end{cases}$$

By the above: for any $u \in C^0(\overline{\Omega})$, $H_B(u)$ is harmonic on B,

$$H_B(u)\Big|_{\partial B} = u\Big|_{\partial B} \text{ and } H_B(u) \in C^0(\overline{\Omega}).$$

- We slightly generalize the concept of a superharmonic function:

- Def: $\Omega \subset \mathbb{R}^n$ open; A function $u: \Omega \to \mathbb{R}$ is called superharmonic in Ω, if
  i) $u \in C^0(\Omega)$
  ii) $H_B(u) \leq u$ for any $B = B_r(x_0) \subset\subset \Omega$.

Remarks: * If u were harmonic, then i) would be strengthened to $C^2(\Omega)$ and (ii) would hold with $=$.
* Furthermore: Let $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a harmonic function s.t.

$$v\restriction_{\partial B} = u\restriction_{\partial B}$$

where u is a superharmonic function. Then:

$$u(x) \geq (H_B(u))(x) = (H_B(v))(x) = v(x) \quad \forall x \in \overline{B}$$

i.e. $u \geq v$ on $\overline{B}$, hence the name.
* Not surprisingly: superharmonic functions satisfy a minimum principle:

- Theorem: (i) Let $\Omega \subset \mathbb{R}^n$ be open and $u: \Omega \to \mathbb{R}$ be a superharmonic function. Then, the set of minimal points of u is open.
  (ii) Let $\Omega \subset\subset \mathbb{R}^n$, open and $u: \Omega \to \mathbb{R}$ be a superharm. function s.t. $u \in C^0(\bar{\Omega})$. Then there is at least one minimal point of u on $\partial\Omega$.

Proof: (i) Let $x_0 \in \Omega$ be a minimal point of u. Let $R > 0$ be s.t. $B_R(x_0) \subset\subset \Omega$. Let $B := B_r(x_0)$ for a fixed $0 < r < R$. We define

$w : \Omega \to \mathbb{R}$ by $w := H_B(u)$. Then

$$\left.\begin{array}{l} w \in C^0(\Omega) \\ w \le u \\ w\restriction_{\partial B} = u\restriction_{\partial B} \end{array}\right\} \Rightarrow \begin{array}{l} w \text{ has a minimal point} \\ \text{in } B \text{ (since } x_0 \in B \text{ is} \\ \text{a minimal point of } u \text{)} \end{array}$$

Since $w\restriction_B$ is harmonic, it must be constant in B and by continuity in $\overline{B}$, so that $u\restriction_{\partial B}$ is constant.

Hence: $\forall x \in \partial B$: $u(x) = w(x) = w(x_0) \le u(x_0) \le u(x)$

$\Rightarrow u(x) = u(x_0) \quad \forall x \in \partial B$

and all points $x \in \partial B$ are minimal points of u.

Since this hold for all $B_r(x_0)$ with $0 < r < R$, and u is continuous: $B_R(x_0)$ is a set of minimal points.

(ii) $\overline{\Omega}$ compact and $u \in C^0(\overline{\Omega}) \Rightarrow \exists$ minimal point $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$, then u is constant on the connected component $Z \subset \Omega$, $Z \ni x_0$. By continuity, the minimum is also attained in $\partial Z \subset \partial\Omega$.

Since Ω is bounded, $\partial Z \ne \emptyset$. $\square$

- Now: H_B is a positive map:

Lemma: Let $u, v \in C^0(\overline{\Omega})$. If $u \le v$, then $H_B(u) \le H_B(v)$ for all balls $B \subset \Omega$.

Proof: • If $x \in \overline{\Omega} \setminus B$, then $(H_B(v) - H_B(u))(x)$

$= (H_B(v-u))(x) = (v-u)(x) \ge 0$.

• On ∂B: for $x \in \partial B$:

$(H_B(v) - H_B(u))(x) = (v-u)(x) \ge 0$,

in particular: $\min\limits_{x \in \partial B} \left((H_B(v))(x) - (H_B(u))(x)\right) \ge 0$.

* $H_B(v-u)$ is harmonic on B. By the minimum principle, if $x \in B$:

$$\left(H_B(v-u)\right)(x) \geq \min_{y \in B} \left(H_B(v-u)\right)(y)$$

$$= \min_{y \in \partial B} \left(H_B(v-u)\right)(y) \geq 0 \text{ by above.}$$

□.

• Proposition: $\Omega \subset \mathbb{R}^n$ open.

(i) If u is superharmonic in Ω, then $H_B(u)$ is superharmonic in Ω for any ball $B \subset\subset \Omega$.

(ii) Let $S \neq \phi$ be a set of superharmonic functions in Ω. If $u : \Omega \to \mathbb{R}$, $u(x) = \inf_{v \in S} v(x)$ is continuous, then u is superharmonic.

(iii) The min of finitely many superharmonic functions is superharmonic.

Proof. (i) By definition: for any ball $B \subset\subset \Omega$: $H_B(u) \leq u$.

Let $K \subset\subset \Omega$ be another ball. By the lemma:

$$H_K H_B(u) \leq H_K(u) \qquad (*)$$

We claim: $H_K H_B(u) \leq H_B(u)$. Indeed:

* if $x \in \bar{\Omega} \setminus B$: $H_B(u)(x) = u(x) \geq H_K(u)(x)$

hence by $(*)$ $H_B(u)(x) \geq H_K H_B(u)(x)$.

* if $x \in \bar{\Omega} \setminus K$: $H_K H_B(u)(x) = H_B(u)(x)$

* if $x \in B \cap K$, then

$$v := H_K H_B(u) - H_B u$$

is harmonic on $B \cap K$ (as the difference of two harmonic functions), and by the two points above.

$$v\big|_{\partial(B\cap K)} \geq 0.$$

By the minimum principle, if $x \in B \cap K$

$$v(x) \geq \min_{y\in B\cap K} v(y) = \min_{y\in\partial(B\cap K)} v(y) \geq 0.$$

which finishes the proof of the claim.

(ii) Let $v \in S$ and $B \subset\subset \Omega$ a ball. Then:

$$u(x) = \inf\{w(x) : w \in S\} \leq v(x) \quad \forall x \in \bar{B}$$

Hence $\;\; H_B u \leq H_B v$ by the lemma

$\qquad\qquad\quad \leq v$ by superharmonicity.

Since this holds for any $v \in S$: $H_B u \leq \inf\{v : v \in S\} = u$

and u is superharmonic □

(iii) Follows from (ii) and the fact that the min of finitely many superharmonic functions – being continuous – is continuous.

- In order to prove existence of the solution of (DP) for a $g \in C^0(\partial\Omega;\mathbb{R})$, $\Omega \subset\subset \mathbb{R}^n$, we consider

$$S(g) := \{ v \in C^0(\bar{\Omega}) : \begin{array}{l} v \text{ is superharmonic}, \\ v(x) \leq \max\{g(y) : y \in \partial\Omega\} \\ \qquad\text{whenever } x \in \Omega \\ v(x) \geq g(x) \text{ for } x \in \partial\Omega \end{array} \}$$

Note: the constant function $u(x) = \max g$ is in $S(g)$ so that $S(g) \neq \emptyset$ for any g.

Due to such: $v \in S(g) \to v(x) \geq \min_{x \in \partial\Omega} g \quad \forall x \in \bar{\Omega}$.

- Lemma: If $h \in C^2(\Omega; \mathbb{R}) \cap C^0(\bar{\Omega}; \mathbb{R})$ is a solution of (DP), then:
  (i) $h \in S(g)$
  (ii) $h \leq v \quad \forall v \in S(g)$

Proof: (i) * h is harmonic, hence superharmonic

* By the maximum principle:

$$h(x) \leq \max_{y \in \partial\Omega} h(y) = \max_{y \in \partial\Omega} g(y) \quad \forall x \in \bar{\Omega}$$

* If $x \in \partial\Omega$, $h(x) = g(x)$ indeed.

(ii) If $v \in S(g)$, then $v - h$ is superharmonic and by the weak minimum principle

$$v(x) - h(x) \geq \min_{y \in \partial\Omega} (v(y) - \underbrace{h(y)}_{= g(y)}) \geq 0. \quad \forall x \in \bar{\Omega}$$

□

In other words: A solution of (DP) is an infimum of $S(g)$.
Key now is to prove the converse.

Theorem (Perron): $\Omega \subset\subset \mathbb{R}^n$, open. Let $g: \partial\Omega \to \mathbb{R}$ be a bounded function. Then the function $u: \Omega \to \mathbb{R}$ defined by

$$u(x) := \inf \{ v(x) : v \in S(g) \} \qquad x \in \Omega$$

is such that

i) $\inf g \leq u \leq \sup g$

ii) u is harmonic in Ω.

Note: Since g is not assumed continuous here, the condition $v(x) \leq \max g$ in the def of $S(g)$ must be replaced by $\leq \sup g$.

• Proof: (i). Since $S(g) \neq \phi$ and $S(g)$ is bounded below (i.e. $v \in S(g) \Rightarrow v \geq \inf g$), the infimum u exists.

Now $u(x) \leq v(x)$ for any $v \in S(g)$, and by definition $v(x) \leq \sup g$.

Furthermore, as noted above: $v(x) \geq \inf g$ for any $v \in S(g)$, so that $u \geq \inf g$.

(ii) We prove: $\forall$ ball $B \subset\subset \Omega$, $H_B u = u$ which implies that the infimum u is harmonic on Ω.

a) $H_B u \leq u$: By the proposition (iii), it suffices to prove that u is continuous. Let $v \in S(g)$. Then by the proposition (i), $v_B := H_B v$ is superharmonic, and $v_B \leq v \leq \sup g$ (*).

Moreover, if $x \in \partial\Omega$: $v_B(x) = v(x) \geq g(x)$. Hence $v \in S(g) \Rightarrow v_B \in S(g)$. By (i)

$$\inf g \leq u \leq v_B \leq \sup g \qquad (**).$$

which shows that $v_B\restriction_B$ is uniformly bounded for all $v \in S(g)$. Since $v_B\restriction_B$ is harmonic, this implies (see Ex on Sheet) that $\{v_B\restriction_B : v \in S(g)\}$ is uniformly equicontinuous, on $B' \subset\subset B$, namely: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|v_B(x) - v_B(y)| < \varepsilon/2$ if $x, y \in B'$, $|x-y| < \delta$ (for all $v \in S(g)$)

Furthermore, by the definition of inf, for any $y \in B'$,

$\exists\, v \in S(g)$ s.t.

$$v(y) < u(y) + \frac{\varepsilon}{2}$$

Altogether: if $x \in B'$ and $|x-y| < \delta$:

$$u(x) - u(y) < u(x) - v(y) + \frac{\varepsilon}{2}$$
$$\underset{(\text{by } (**))}{\leq} v_B(x) - v_B(y) + \frac{\varepsilon}{2} \underset{(\text{by } (*))}{<} \varepsilon \quad \text{by uniform equicontinuity.}$$

Repeating the argument with $x \leftrightarrow y$, we obtain:

$\forall \varepsilon > 0, \exists \delta > 0$: $|x-y| < \delta$, $x, y \in B'$ implies
$$|u(x) - u(y)| < \varepsilon$$

Hence u is continuous in B' and hence in Ω.
$\Rightarrow u$ is superharmonic: $H_B(u) \leq u$.

(b) $u \leq H_B u$:

Let $x \in \Omega$, $\varepsilon > 0$. There exists $v \in S(g)$: $v(x) \leq u(x) + \frac{\varepsilon}{3}$.
Furthermore, since $u, v \in C^0(\Omega)$, $\exists\, \delta > 0$ s.t. if $y \in B_\delta(x)$, then $|u(x) - u(y)| < \varepsilon/3$ and $|v(x) - v(y)| < \varepsilon/3$. Hence:

$$v(y) = v(y) - v(x) + v(x) - u(x) + u(x) - u(y) + u(y)$$
$$< 3 \cdot \frac{\varepsilon}{3} + u(y) \qquad \forall y \in B_\delta(x).$$

Now: any ball $B \subset\subset \Omega$ can be covered with finitely many such $B_\delta(x)$, say $B_1, \dots, B_n$, with corresponding functions $v_1, \dots, v_n$, $v_i \in S(g)$. Let

$$w := \min_i v_i$$

Then: for any $y \in B$,

$$w(y) \leq u(y) + \varepsilon \qquad (***)$$

and w is superharmonic as the minimum of finitely many superharmonic functions. As above, $w_B \in \mathcal{S}(y)$ also. Hence, $\forall \varepsilon > 0$:

$$u(y) \leq w_B(y) \leq H_B(u+\varepsilon)(y) = (H_B(u))(y) + \varepsilon$$

H_B is a positive map and $(***)$

which proves that $u \leq H_B(u)$ for any ball $B \subset\subset \Omega$. $\square$

- final question: Having u as in the theorem, when does it extend to a continuous function on $\bar{\Omega}$?

Let $h \in C^2(\Omega, \mathbb{R}) \cap C^0(\bar{\Omega}, \mathbb{R})$ be a solution of (DP) with

$$g_{x_0} : \partial\Omega \to \mathbb{R}$$
$$x \mapsto g_{x_0}(x) = |x - x_0|.$$

Then
- h is harmonic, in particular superharmonic
- $\min_{x \in \bar{\Omega}} h(x) = \min_{x \in \partial\Omega} h(x) = \min_{x \in \partial\Omega} g_{x_0}(x) = 0$

so that $h \geq 0$.

Since, furthermore, $h \neq$ constant (it is not constant on $\partial\Omega$), $h(x) > 0 \quad \forall x \in \bar{\Omega} \setminus \{x_0\}$, with $h(x_0) = 0$.

Hence: If (DP) has a solution for any $g \in C^0(\partial\Omega; \mathbb{R})$, then there is, for any $x_0 \in \partial\Omega$, a function $b \in C^0(\bar{\Omega})$ with the following properties:

i) b is superharmonic in Ω

ii) $b(x) > 0 \quad \forall x \in \bar{\Omega} \setminus \{x_0\}$

iii) $b(x_0) = 0$.

Def: • A function $b \in C^0(\bar{\Omega}; \mathbb{R})$ satisfying properties (i), (ii), (iii) is called a barrier at x_0

* A point $x_0 \in \partial\Omega$ is called regular (w.r.t. $-\Delta$) if there exists a barrier at x_0.

• Theorem: Let $\Omega \subset\subset \mathbb{R}^n$, $\Omega \neq \emptyset$ and $g \in C^0(\partial\Omega; \mathbb{R})$.

Let
$$\begin{cases} -\Delta u(x) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \qquad (DP)$$

i) If (DP) has a solution $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\bar{\Omega}; \mathbb{R})$ for any $g \in C^0(\partial\Omega)$, then every point $x \in \partial\Omega$ is regular.

ii) If every point $x \in \partial\Omega$ is regular, then there exist a unique solution $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\bar{\Omega}; \mathbb{R})$ of (DP) for any $g \in C^0(\partial\Omega; \mathbb{R})$.

Proof: i) has just been proved.

ii) Let $g \in C^0(\partial\Omega; \mathbb{R})$ be fixed and let
$$u := \inf\{v : v \in S(g)\}$$
be the harmonic function constructed above. It remains to prove: If x_0 is regular, then
$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = g(x_0).$$

By continuity: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$$x \in \partial\Omega, \ |x - x_0| < \delta \implies |g(x) - g(x_0)| < \frac{\varepsilon}{2}$$

Let b be a barrier at x_0, and $c>0$. For $x \in \bar{\Omega}$, let $g_\pm(x) := g(x_0) \pm \left(cb(x) + \frac{\varepsilon}{2}\right)$.

Then for $x \in \partial\Omega$, $|x-x_0| < \delta$:

$$(g_+ - g)(x) = \frac{\varepsilon}{2} - \big(g(x) - g(x_0)\big) + cb(x) \geq 0.$$
$$(g_- - g)(x) = -\frac{\varepsilon}{2} - \big(g(x) - g(x_0)\big) - cb(x) \leq 0.$$
$\forall c > 0.$

Now: if $x \in \partial\Omega \setminus B_\delta(x_0)$, then $\min b(x) =: b_0 > 0$.

Hence, choosing $c = \frac{2}{b_0} \sup|g|$, we get

$$(g_+ - g)(x) \geq 0 \text{ and } (g_- - g)(x) \leq 0 \quad x \in \partial\Omega \setminus B_\delta(x_0)$$

so that the inequalities hold for all $x \in \partial\Omega$. We note that

* g_+ is superharmonic in Ω, $g_+ \geq g$ on $\partial\Omega$, hence $\min\{g_+, \sup g\} \in S(g)$ so that $u \leq \min\{g_+, \sup g\} \leq g_+$ in Ω.
* $\forall v \in S(g)$: $v + (-g_-)$ is superharmonic, and

$$v - g_- \underset{(S_g \text{ def})}{\geq} g - g_- \geq 0 \quad \text{on } \partial\Omega.$$

By the minimum principle: $v \geq g_-$ on $\bar{\Omega}$.
And since v is arbitrary in $S(g)$, this implies $u \geq g_-$ on $\bar{\Omega}$. Altogether:

$$g_- \leq u \leq g_+ \quad \text{on } \bar{\Omega}, \text{ namely}$$

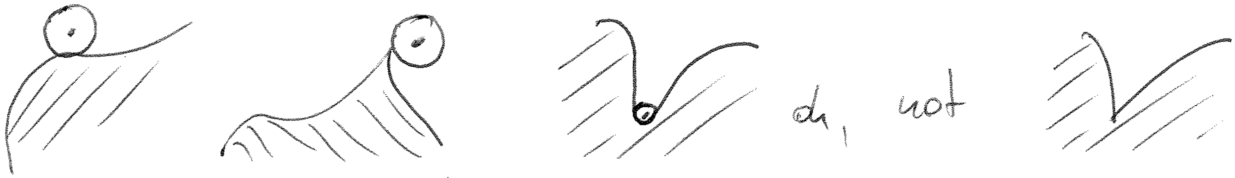
$$-\left(cb(x) + \frac{\varepsilon}{2}\right) \leq u(x) - g(x_0) \leq \left(cb(x) + \frac{\varepsilon}{2}\right)$$

which concludes the proof since $\exists\, \tilde{\delta} > 0 : |x - x_0| < \tilde{\delta} \Rightarrow cb(x) < \varepsilon/2$ $\square$

- The existence of a barrier at each $x \in \partial\Omega$ is therefore the necessary and sufficient condition for the existence of a solution of the Dirichlet problem for all $g \in C^0(\partial\Omega)$ in Ω.

  It yields a number of natural sufficient geometric/analytic conditions. E.g.: the external ball condition:

<u>Proposition</u>. $\Omega \subset\subset \mathbb{R}^n$, $x_0 \in \partial\Omega$: $\exists$ a ball $B \subset \mathbb{R}^n$ such that: $\bar{\Omega} \cap \bar{B} = \{x_0\}$. Then x_0 is a regular point.



<u>Proof</u>: We construct a barrier at x_0: Let $r > 0$ and $a \in \mathbb{R}^n \setminus \{x_0\}$ be such that $B = B_r(a)$. Let $x \in \mathbb{R}^n \setminus \{a\}$, and

$$b(x) := \begin{cases} r^{2-n} - |x-a|^{2-n} & (n \geq 3) \\ \ln|x-a| - \ln r & (n = 2) \end{cases}$$

Then by direct calculation, b is harmonic in $\mathbb{R}^n \setminus \{a\}$, $b \in C^0(\bar{\Omega})$ since $a \notin \bar{\Omega}$, and $b(x_0) = 0$, and if $x \in \bar{\Omega} \setminus \{x_0\}$, then $|x-a| > |x_0 - a| - r$ so that $b(x) > 0$ for all $x \in \bar{\Omega} \setminus \{x_0\}$. Hence b is a barrier. $\square$

- <u>Remark</u>: There are other sufficient conditions, e.g.:
  - $\partial\Omega$ is C^2.
  - Ω is convex
  - Exterior cone condition,

There are also counterexamples:

Lebesgue. Let

$$D = \{(x,y,z) \in \mathbb{R} : 0 < x < \infty,\ y^2 + z^2 < e^{-1/x}\}$$

and let $\Omega = B_1(0) \setminus \overline{D}$. Then the function

$$u(x,y,z) = \int_0^1 \frac{t}{\sqrt{(t-x)^2 + y^2 + z^2}}\, dt$$

defined on $\mathbb{R}^3 \setminus ([0,1] \times \{0\} \times \{0\})$ is such that:

i) $u\restriction_\Omega$ is harmonic and $u \in C^0(\overline{\Omega} \setminus \{0\}; \mathbb{R})$

ii) there is no continuous extension of $u\restriction_\Omega$ to $\overline{\Omega}$

iii) $u\restriction_{\partial\Omega \setminus \{0\}}$ has a continuous extension to $\partial\Omega$.

iv) $u\restriction_\Omega$ is bounded.

$\Rightarrow$ (see EX Sheet 6) there is no $\tilde{u} \in C^2(\Omega;\mathbb{R}) \cap C^0(\overline{\Omega};\mathbb{R})$ harmonic and $\tilde{u}\restriction_{\partial\Omega} = u\restriction_{\partial\Omega}$

- The inhomogeneous equation. <u>Poisson's equation</u>

$$\begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases} \qquad \text{(PP)}$$

Key: use linearity and one specific (particular) solution of (PP).

<u>Method 1</u>: Fundamental solution / Newton's potential:

i) find a particular solution $V \in C^2(\Omega;\mathbb{R}) \cap C^0(\overline{\Omega};\mathbb{R})$ of $\quad -\Delta V(x) = f(x) \quad x \in \Omega$

with $V\restriction_{\partial\Omega}$ irrelevant

ii) Use the solution of (DP) for Laplace's equation

$H \in C^2(\Omega;\mathbb{R}) \cap C^0(\bar{\Omega};\mathbb{R})$:

$$\begin{cases} -\Delta H(x) = 0 & x \in \Omega \\ H(x) = g(x) - V(x) & x \in \partial\Omega \end{cases}$$

Then: $u := V + H$ is a solution of (PP)

<u>Method 2</u>: Green's function:

i) Find a particular solution $W \in C^2(\Omega;\mathbb{R}) \cap C^0(\bar{\Omega};\mathbb{R})$:

$$\begin{cases} -\Delta W(x) = f(x) & x \in \Omega \\ W(x) = 0 & x \in \partial\Omega \end{cases}$$

ii) Solve the (DP) $\begin{cases} -\Delta \tilde{H}(x) = 0 & x \in \Omega \\ \tilde{H}(x) = g(x) & x \in \partial\Omega \end{cases}$

Again: $u := W + \tilde{H}$ is a solution of (PP).

- First remark: Linearity implies immediately uniqueness of the solution of (PP): if u_1, u_2 are two solutions, then $v := u_1 - u_2$ solves

$$\begin{cases} -\Delta v(x) = -\Delta u_1(x) - \Delta u_2(x) = 0 & x \in \Omega \\ v(x) = 0 & x \in \partial\Omega \end{cases}$$

and by the min and max principles, $v(x) = 0 \ \forall x \in \bar{\Omega}$.

- <u>Definition</u>: The function $\Phi : \{(x,y) \in \mathbb{R}^{2n} : x \neq y\} \to \mathbb{R}$ defined by

$$\Phi(x,y) := \begin{cases} \frac{1}{(n-2)\omega_n} |x-y|^{2-n} & (n \geq 3) \\ -\frac{1}{2\pi} \ln |x-y| & (n = 2) \end{cases}$$

is called the fundamental solution of $(-\Delta)$.

- Remarks: * Φ is sometimes extended by $\Phi(x,x)=\infty$.
  * Symmetry: $\Phi(x,y)=\Phi(y,x)$
  * Asymptotics: $\lim\limits_{\substack{x\to y\\ x\neq y}} \Phi(x,y)=\infty$.

* By a direct calculation:

$$\Phi_{x_i}(x,y) = -\frac{1}{\omega_N}\frac{x_i-y_i}{|x-y|^n} = -\Phi_{y_i}(x,y)$$

$$\Phi_{x_i x_k}(x,y) = -\frac{1}{\omega_N|x-y|^n}\left\{\delta_{ik} - n\frac{(x_i-y_i)(x_k-y_k)}{|x-y|^2}\right\}$$

And in particular:

i) $|D\Phi(x,y)| \leq \frac{1}{\omega_N}|x-y|^{1-n}$

ii) If $x\neq y$: $-\Delta\Phi(x,y)=0$ (i.e. $\Phi(\cdot,y)$ is harmonic on $\mathbb{R}^n\setminus\{y\}$).

as well as $|\Phi_{x_i x_k}(x,y)| \leq \frac{n+1}{\omega_N|x-y|^n}$

* Let $\Omega\subset\mathbb{R}^n$ be open, $x\in\Omega$ and $\varepsilon>0$: $\overline{B_\varepsilon(x)}\subset\Omega$.
for any $f\in C^0(\Omega)$:

$$\lim_{\varepsilon\downarrow 0}\int_{\partial B_\varepsilon(x)} f(y)\frac{x-y}{|x-y|}\cdot D_y\Phi(x,y)\,dS(y)$$

$$= \lim_{\varepsilon\downarrow 0}\frac{1}{\omega_n}\int_{\partial B_\varepsilon(x)} f(y)\underbrace{\frac{|x-y|^2}{|x-y|^{n+1}}}_{=\varepsilon^{-(n-1)}}dS(y) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{n-1}\omega_n}\int_{\partial B_\varepsilon(x)} f(y)\,dS(y)$$

$$= f(x) \qquad (\lozenge)$$

* $\Phi(x,\cdot)$ is locally integrable. In the case $n \geq 3$ let $\Omega \subset\subset \mathbb{R}^n$ with $D := \operatorname{diam}(\Omega)$ and $x \in \overline{\Omega}$:

$$\int_\Omega |\Phi(x,y)|\,dy \leq \frac{1}{(n-2)\omega_n} \int_{B_D(x)} |x-y|^{2-n}\,dy$$

$$= \frac{\alpha_n}{(n-2)\omega_n} \int_0^D r^{2-n+n-1}\,dr = \frac{1}{2n(n-2)} D^2$$

* Def: Let $\Omega \subset\subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a measurable, bounded function. The Newtonpotential (or Coulomb potential) of f is the function $V_f : \mathbb{R}^N \to \mathbb{R}$ defined by

$$V(x) := \int_\Omega \Phi(x,y) f(y)\,dy$$

* Theorem: Let $f \in C_c^2(\mathbb{R}^n)$ with $\operatorname{supp} f \subset\subset \Omega \subset\subset \mathbb{R}^n$. Then:
  i) $V \in C^2(\mathbb{R}^n;\mathbb{R})$
  ii) $-\Delta V(x) = f(x)$ for all $x \in \mathbb{R}^n$.

namely: V is a strong solution to Poisson's equation, if $f \in C_c^2$!

Proof: (i) By a change of variables: $(z = x-y)$

$$V(x) = \int_{\mathbb{R}^n} \Phi(x,y) f(y)\,dy = \int_{\mathbb{R}^n} \underbrace{\Phi(z,0)}_{\text{denote } \Phi(z)\text{, singularity at } z=0.} f(x-z)\,dz$$

Hence:

$$\varepsilon^{-1}\big(V(x+\varepsilon e_i) - V(x)\big) = \int_{\mathbb{R}^n} \Phi(z)\, \underbrace{\varepsilon^{-1}\big(f(x+\varepsilon e_i - z) - f(x-z)\big)}_{\xrightarrow{\varepsilon \downarrow 0}\ f_{x_i}(x-z) \text{ uniformly}}\,dz$$

Hence: $V_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(z)\, f_{x_i}(x-z)\, dz$

and similarly $V_{x_i x_k}(x) = \int_{\mathbb{R}^n} \Phi(z)\, f_{x_i x_k}(x-z)\, dz$.

Since the right hand side is continuous in x: $V \in C^2(\mathbb{R}^n;\mathbb{R})$.

(ii) Let $\varepsilon > 0$ and decompose for any $x \in \mathbb{R}^n$:

$$\Delta V(x) = \Bigg(\int_{B_\varepsilon(0)} + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Bigg) \Big(\Phi(z) \underbrace{\Delta_x f(x-z)}_{= \Delta_z f(x-z)} \Big)\, dz = I_\varepsilon^{int} + I_\varepsilon^{out}$$

* $|I_\varepsilon^{int}| \leq \sup \{ |\Delta f(y)| : y \in B_\varepsilon(x) \} \underbrace{\int_{B_\varepsilon(0)} |\Phi(z)|\, dz}_{\leq C\varepsilon^2 \ (\text{if } n \geq 3)}$

* By Gauss-Green: ($\mathbb{R}^n \setminus B_\varepsilon(0)$ does not contain $z=0$)

$$I_\varepsilon^{out} = - \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} D_z\Phi(z)\, D_z f(x-z)\, dz + \int_{\partial B_\varepsilon(0)} \Phi(z) \frac{\partial f}{\partial_z \nu_{in}}(x-z)\, dS(z)$$

↑ inward normal derivative

The second term vanishes again as $\varepsilon^{2-n+n-1} = \varepsilon$ as $\varepsilon \downarrow 0$.
As for the first one, another integration by parts yields

$$-\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} (\dots) = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta_z \Phi(z)\, f(x-z)\, dz - \int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial \nu_{in}}(z)\, f(x-z)\, dS(z)$$

with $\frac{\partial \Phi}{\partial \nu_{in}} = -\frac{1}{\omega_n} \frac{z}{|z|^n} \cdot \frac{-z}{z} = \frac{1}{\omega_n} \frac{1}{|z|^{n-1}}$, and

the fact that $\Delta_z \Phi(z) = 0$ on $\mathbb{R}^n \setminus B_\varepsilon(0)$,

we get $\lim_{\varepsilon \downarrow 0} \left(- \int (\dots) \right) = - f(x)$, see also (◇) p. 46.

Altogether: $\Delta V(x) = - f(x) \quad \forall x \in \mathbb{R}^n$ □

- Q: Can this be extended to f's that are less smooth?

<u>Def</u>: Let $\Omega \subseteq \mathbb{R}^n$ be open. The vector space of locally <u>Hölder continuous</u> functions is given by:

$$C^H(\Omega) := \{ f : \Omega \to \mathbb{R} : \forall K \subset\subset \Omega, \exists c > 0, \text{ and } \alpha \in (0,1] \text{ such that } |u(x) - u(y)| < c|x-y|^\alpha \text{ for all } x, y \in K \}.$$

Also:

$$C^H_b(\Omega) := \{ f \in C^H(\Omega) : f \text{ is bounded} \}.$$

Note: * c and α may depend on K.
* If $\alpha = 1$ for all K, this recovers Lipschitz continuity.

- <u>Theorem</u>: Let $\Omega \subset\subset \mathbb{R}^n$ and $f \in C^H_b(\Omega)$. Then:
  (i) $V \in C^2(\Omega; \mathbb{R})$
  (ii) $-\Delta V(x) = f(x)$ for all $x \in \Omega$

- <u>Remarks</u>: * In fact one proves that for $x \in \Omega$

$$V_{x_i x_k}(x) = \int_\Omega (f(y) - f(x)) \, \Phi_{x_i x_k}(x,y) \, dy + f(x) \frac{\partial^2}{\partial x_i \partial x_k} \int_\Omega \Phi(x,y) \, dy$$

yielding $+\Delta V(x) = 0 + f(x) \underbrace{\Delta \int_\Omega \Phi(x,y) \, dy}_{= -1 \text{ by the theorem above}}$.

* For Laplace's equation, $f \equiv 0$, the Coulomb potential V is the trivial solution $V = 0$.
* Formally (or strictly if) everything is interpreted "in the sense of

distributions") $\quad -\Delta\Phi(x) = \delta(x)$, meaning

$$-\Delta \int \Phi(z) f(x-z)\,dz \;"="\; \int (-\Delta\Phi)(z) f(x-z)\,dz$$
$$= \int \delta(z) f(x-z)\,dz = f(x)$$

* Hölder continuity of f in Ω is necessary. $f\in C^0(\Omega)$ does not suffice to imply $V\in C^2(\Omega)$ [Petrini]
* Since $V\in C^0(\bar\Omega)$, the results of Perron's method yield:

If $\Omega \subset\subset \mathbb{R}^n$, $f\in C_b^H(\Omega)$ and $g\in C^0(\partial\Omega)$, then (PP) has exactly one solution $u\in C^2(\Omega;\mathbb{R})\cap C^0(\bar\Omega;\mathbb{R})$ if and only if every $x\in\partial\Omega$ is regular.

- We move to Method 2: the Green's function. One seeks a function $G(x,y)$ s.t.

$$w(x) := \int_\Omega G(x,y) f(y)\,dy \quad \text{solves}$$

$$\begin{cases} -\Delta w(x) = f(x) & x\in\Omega \\ w(x) = 0 & x\in\partial\Omega \end{cases}$$

Natural choice: * $y\mapsto G(x,y) = 0$ whenever $x\in\partial\Omega$ which satisfies the B.C.

* $x\mapsto G(x,y) - \Phi(x,y)$ to be a harmonic function for any $y\in\Omega$ which satisfies the equation.

<u>Def</u>: Let $\Omega\subset\subset\mathbb{R}^n$ open. A function $G:\bar\Omega\times\Omega\to\mathbb{R}$ is a <u>Green's function</u> for Ω if for any $y\in\Omega$:

i) $x\mapsto G(x,y) - \Phi(x,y)$ is continuous on $\bar\Omega$ and harmonic on Ω.

ii) $x \mapsto G(x,y) = 0$ on $\partial\Omega$.

- Theorem: $\Omega \subset\subset \mathbb{R}^n$, open, $\Omega \neq \emptyset$. If every $x \in \partial\Omega$ is regular, then there exists a unique Green's function.

Proof: Let $y \in \Omega$ fixed. The function $\partial\Omega \ni x \mapsto \Phi(x,y)$ being continuous, there exists a unique $\phi^y \in C^2(\Omega) \cap C^0(\bar{\Omega})$ s.t.

$$\begin{cases} -\Delta \phi^y(x) = 0 & x \in \Omega \\ \phi^y(x) = \Phi(x,y) & x \in \partial\Omega. \end{cases}$$

Let $G(x,y) := \Phi(x,y) - \phi^y(x)$. Then G is the Green's fct.

i) $G(x,y) - \Phi(x,y) = -\phi^y(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

ii) For any $y \in \Omega$ and $x \in \partial\Omega$: $G(x,y) = 0$.

Uniqueness follows from the maximum principle for $G_1 - G_2 = (G_1 - \Phi) - (G_2 - \Phi)$ □

- Remarks: * Here, we use the definition of $\Phi(t,t) = 0$, which implies that $G(\cdot,y) - \Phi(\cdot,y)$ is well-defined on all of $\bar{\Omega}$.

  * Since $G(\cdot,y) - \Phi(\cdot,y)$ is continuous on $\bar{\Omega}$, the Green's function has the same singular behaviour as Φ in the neighbourhood of the diagonal.

  * In the following, we consider only $n \geq 3$. The case $n=2$ would be similar.

- Lemma: Let $\Omega \subset\subset \mathbb{R}^n$ be connected, and G be the Green's
($n \geq 3$) function for Ω. Then if $y \in \Omega$:

$$0 < G(x,y) \leq \Phi(x,y) \quad \text{for } x \in \Omega \setminus \{y\}$$

Proof: * Since $x \mapsto \Phi(x,y)$ is harmonic on $\Omega \setminus \{y\}$, so is $x \mapsto G(x,y)$. We consider $(x_n)_{n\in\mathbb{N}}$ converging to $x_0 \in \partial(\Omega \setminus \{y_0\})$, $x_n \in \Omega \setminus \{y_0\}$.

* Case 1: $x_0 = y$: Recall that $G - \Phi = -\phi^y$ is continuous in a neighbourhood of y, hence bounded. Since $\Phi(x_n, y) \to \infty$, this implies that

$$G(x_n, y) \to \infty, \text{ and in particular}$$

$$\liminf_{n\to\infty} G(x_n, y) \geq 0.$$

* Case 2: $x_0 \in \partial\Omega$. Again by construction:

$$G - \Phi = -\phi^y \longrightarrow -\Phi(\cdot, y) \quad \text{by continuity of } \phi^y.$$

Hence $\liminf_{n\to\infty} G(x_n, y) = 0$.

Altogether: $\liminf_{n\to\infty} G(x_n, y) \geq 0$
(i.e. $x_n \to \partial(\Omega \setminus \{y\})$)

and by the minimum principle: $G(x,y) \geq 0$
If $\exists\, x \in \Omega \setminus \{y\}$ s.t. $G(x,y) = 0$, then $G \equiv 0$, which contradicts the fact that $G - \Phi$ is continuous.
$\Rightarrow G(x,y) > 0 \quad \forall x \in \Omega \setminus \{y\}$.

* Since ϕ^y is harmonic for any $y \in \Omega$:

$$\min_{x\in\bar\Omega} \phi^y(x) = \min_{x\in\partial\Omega} \phi^y = \min_{x\in\partial\Omega} \Phi(x,y) \geq 0.$$

namely $\phi^y(x) = -G(x,y) + \Phi(x,y) \geq 0$ □

- Lemma: Let $\Omega \subset\subset \mathbb{R}^n$ and G be the Green's function. Then the function $G - \Phi$ is jointly continuous on $\bar\Omega \times \Omega$.

Proof. See Ex. Sheet. □

- Theorem: Let $\Omega \subset\subset \mathbb{R}^n$, G the Green's function. If $f \in C^H_b(\Omega)$, then the function

$$w(x) := \int_\Omega G(x,y) f(y) dy \qquad (x \in \bar\Omega)$$

is s.t. $w \in C^2(\Omega) \cap C^0(\bar\Omega)$ and solves

$$\begin{cases} -\Delta w(x) = f(x) & x\in\Omega \\ w(x) = 0 & x \in \partial\Omega. \end{cases}$$

Remark: $w \in C^0(\bar\Omega)$ follows from the weaker condition f being measurable and bounded.

- Proof. Continuity follows from the estimates of the first lemma. With this and the second condition for G: $w(x) = 0$, $x \in \partial\Omega$. It remains to prove that w solves the equation. For $x \in \bar\Omega$, let

$$w(x) = V(x) - h(x)$$

where V is the Newton potential, and

$$h(x) = \int_\Omega \phi^y(x) f(y) dy.$$

Since $-\Delta V(x) = f(x)$ for all $x\in\Omega$ and $V\in C^2(\Omega)$, it suffices to prove that h is harmonic in Ω.

Let $x_0\in\Omega$, $r>0$: $\overline{B_r(x_0)}\subset\Omega$. Then map

$$\overline{B_r(x_0)}\times\Omega \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \left(\phi^y(x_0) - \phi^y(x)\right) f(y)$$

is jointly continuous by the lemma, hence measurable. Moreover,

$$|\phi^y(x)| = |\Phi(x,y) - G(x,y)| \leq 2|\Phi(x,y)|$$

so that $|(\phi^y(x_0) - \phi^y(x))||f(y)|$ is absolutely integrable.

By Fubini:

$$\int_{B_r(x_0)} \left(h(x_0) - h(x)\right) dx = \int_\Omega \Big(\underbrace{\int_{B_r(x_0)} \left(\phi^y(x_0) - \phi^y(x)\right) dx}_{=0 \text{ since } \phi^y \text{ is harmonic, by (MVP II)}} \Big) f(y)\, dy$$

$$= 0$$

so that h is harmonic by (MVP II). $\square$

- Summarizing: T.f.a.e for $\Omega \subset\subset \mathbb{R}^n$:
  - i) The Green's function exists for Ω.
  - ii) Every $x\in\partial\Omega$ is regular
  - iii) The (PP) has a solution for any $f\in C^H_b(\Omega)$ and any $g\in C^0(\partial\Omega)$.

The only implication that is not proved yet is (i) $\to$ (ii).

Let $x_0\in\partial\Omega$, and let

$$b(x) := \frac{1}{2n}|x-x_0|^2 + \int_\Omega G(x,y)\,dy \quad ,\ x\in\overline{\Omega}$$

Then b is a barrier at x_0.

- Proposition: Let $\Omega \subset\subset \mathbb{R}^n$ and $x, y \in \Omega$ s.t. $x \neq y$. Then
$$G(x,y) = G(y,x).$$

Proof: Let $v_x, v_y : \overline{\Omega} \to \mathbb{R}$ be defined by
$$v_x(z) := G(z,x)$$
$$v_y(z) := G(z,y)$$
v_x, resp. v_y, is harmonic on $\Omega \setminus \{x\}$, resp. $\Omega \setminus \{y\}$, and $v_x(z) = v_y(z) = 0$ for all $z \in \partial\Omega$.

Let $\varepsilon > 0$ and let
$$\Omega_\varepsilon := \Omega \setminus \{ \overline{B_\varepsilon(x)} \cup \overline{B_\varepsilon(y)} \}$$
By Gauss-Green:
$$0 = \int_{\Omega_\varepsilon} \left(v_x \Delta v_y - v_y \Delta v_x \right) dz$$
$$= \int_{\partial B_\varepsilon(x)} \left(v_x \frac{\partial v_y}{\partial \nu} - v_y \frac{\partial v_x}{\partial \nu} \right) dS(z) + \int_{\partial B_\varepsilon(y)} \left(v_x \frac{\partial v_y}{\partial \nu} - v_y \frac{\partial v_x}{\partial \nu} \right) dS(z)$$
where ν is the inward unit normal to $\partial B_\varepsilon(\cdot)$.

- First integral: we recall that $v_x(z) = \Phi(z-x) - \phi^x(z)$ and that v_y is continuous on $\overline{B_\varepsilon(x)}$ and in fact $C^2(B_\varepsilon(x))$.

Hence,
- $\displaystyle\int_{\partial B_\varepsilon(x)} v_y(z) \frac{\partial \phi^x(z)}{\partial \nu} dS(z) \xrightarrow{\varepsilon \searrow 0} 0$ since $\phi^x \in C^2$
- $\displaystyle\int_{\partial B_\varepsilon(x)} v_y(z) \frac{\partial \Phi}{\partial \nu}(z,x) dS(z) \xrightarrow{\varepsilon \searrow 0} -v_y(x)$ by the remark on p. 46.

$$\left| \int_{\partial B_\varepsilon(x)} v_x(z) \frac{\partial v_y(z)}{\partial \nu} \, dS(z) \right|$$

$$\leq \sup_{z \in \partial B_\varepsilon(x)} \left| \frac{\partial v_y(z)}{\partial \nu} \right| \int_{\partial B_\varepsilon(x)} |G(z,x)| \, dS(z)$$

since furthermore $|G(z,x)| \leq \Phi(z,x)$, the integral is bounded by $C\varepsilon$ as $\varepsilon \downarrow 0$.

Altogether: $\displaystyle\int_{\partial B_\varepsilon(x)} \left(v_x \frac{\partial v_y}{\partial \nu} - v_y \frac{\partial v_x}{\partial \nu} \right) dS \xrightarrow{\varepsilon \downarrow 0} G(x,y)$

A similar calculation for the second integral yields

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(y)} (\ldots) = -v_x(y) = -G(y,x),$$

so that $\quad 0 = G(x,y) - G(y,x) \quad$ for $x, y \in \Omega$, $x \neq y$ $\quad \square$.

- It follows that for any $x \in \Omega$, the function $y \mapsto G(x,y)$ is harmonic on $\Omega \setminus \{x\}$ and has a continuous extension to $\bar{\Omega} \setminus \{x\}$ with $\quad G(x,y) = 0 \quad$ for all $y \in \partial\Omega$.
- The key interest of the Green's function is an explicit representation of the solution of (PP).

<u>Theorem</u> Let $\Omega \subset\subset \mathbb{R}^n$, $f \in C_b^H(\Omega)$ and $g \in C^0(\partial\Omega)$.

If $u \in C^2(\bar{\Omega})$ is a solution of

$$\begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega, \end{cases}$$

then:

$$u(x) = \begin{cases} \int_\Omega G(y,x) f(y)\,dy - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(y,x)\, g(y)\, dS(y) & x \in \Omega \\ g(x) & x \in \partial\Omega \end{cases}$$

for all $x \in \overline{\Omega}$.

Proof: Let $x \in \Omega$, $\varepsilon > 0$: $\overline{B_\varepsilon(x)} \subset \Omega$. For $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon(x)}$,

$$\int_{\Omega_\varepsilon} \left(u(y) \Delta_y \Phi(y,x) - \Phi(y,x) \Delta_y u(y) \right) dy$$
$$= \int_{\partial\Omega_\varepsilon} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y,x) - \Phi(y,x) \frac{\partial u}{\partial \nu}(y) \right) dS(y)$$

Again: $\lim_{\varepsilon \searrow 0} \int_{\partial B_\varepsilon(x)} u(y) \frac{\partial \Phi}{\partial \nu}(y,x)\, dS(y) = u(x)$

$\lim_{\varepsilon \searrow 0} \int_{\partial B_\varepsilon(x)} \Phi(y,x) \frac{\partial u}{\partial \nu}(y)\, dS(y) = 0$, so that

$$u(x) = -\int_\Omega \Phi(y,x) \Delta u(y)\, dy$$
$$- \int_{\partial\Omega} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y,x) - \Phi(y,x) \frac{\partial u}{\partial \nu}(y) \right) dS(y) \quad (*)$$

where we have used that $\Delta_y \Phi(y,x) = 0$ on Ω_ε.

Now for $x \in \Omega$, the function $y \mapsto \phi^x(y) = \Phi(y,x) - G(y,x)$ satisfies

$$\int_\Omega \phi^x(y) \Delta u(y)\, dy = \int_{\partial\Omega} \left(\underbrace{\phi^x(y)}_{= \Phi(y,x) \text{ for any } y \in \partial\Omega} \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \phi^x}{\partial \nu}(y) \right) dS(y)$$

Using this in $(*)$, we get:

$$u(x) = \int_\Omega \left(-\Phi(y,x) + \phi^x(y) \right) \Delta u(y)\, dy$$
$$- \int_{\partial\Omega} \left(\frac{\partial \Phi}{\partial \nu}(y,x) - \frac{\partial \phi^x}{\partial \nu}(y) \right) u(y)\, dy$$

which concludes the proof since $-\Delta u(y) = f(y)$ $\forall y \in \Omega$, $u(y) = g(y)$ $\forall y \in \partial\Omega$, and $G(y,x) = \Phi(y,x) - \phi^x(y)$ $\square$

- Remarks: * For simple geometries G can be computed explicitly and the theorem yields an explicit form for the solution of (PP) for any f, g.
  * Here again, the Green's function solves formally

$$\begin{cases} -\Delta_y G = \delta_x & \text{in } \Omega \\ G = 0 & \text{in } \partial\Omega. \end{cases}$$

- The Green's function for the sphere $B_R(0)$:

Consider the reflection $x \mapsto \tilde{x} := \left(\frac{R}{|x|}\right)^2 x$, $x \in B_R(0) \setminus \{0\}$.

For any fixed $y \in \mathbb{R}^n$, the <u>Kelvin transform</u> of Newton's potential

$$\Phi(x,y) \mapsto \left(\frac{|x|}{R}\right)^{2-n} \Phi(\tilde{x}, y) =: \phi^y(x)$$

defines a harmonic function in $B_R(0) \setminus \{0\}$, such that $\phi^y \in C^0(\overline{B_R(0) \setminus \{0\}})$ with $\phi^y(x) = \Phi(x,y)$ for all $x \in \partial B_R(0)$.

It follows that

$$G(x,y) := \Phi(x,y) - \left(\frac{|x|}{R}\right)^{2-n} \Phi(\tilde{x}, y)$$

is the Green's function for $\Omega = B_R(0)$ if ϕ^y has a harmonic extension to $B_R(0)$, of $y \neq 0$. Indeed:

$$\left(\frac{|x|}{R}\right)^{2-n} \Phi(\tilde{x}, y) = \frac{1}{(n-2)\omega_n} \left| x \frac{R}{|x|} - y \frac{|x|}{R} \right|^{2-n}$$

and the function $\left(R^2 - 2x\cdot y + \frac{1}{R^2}|x|^2|y|^2 \right)^{\frac{1}{2}}$ is a harmonic extension of $\left| x\frac{R}{|x|} - y\frac{|x|}{R} \right|$ to $B_R(0)$.

- Remark: For $y \in \underset{\partial B_R(0)}{\partial\Omega}$, a direct calculation yields

$$\frac{\partial G}{\partial \nu}(y, x) = \frac{1}{R\omega_n} \frac{R^2 - |x|^2}{|x-y|^n}$$

so that if $f = 0$, one recovers the Poisson representation for the solution of (DP).

Elements of calculus of variations

- We illustrate here ideas and methods to solve the Poisson equation that play a major role in modern PDE's and extend to very general classes of non-linear equations.

Key: One considers a functional $E : \mathcal{M} \to \mathbb{R}$ that is defined over a set $\mathcal{M}$ of functions, and such that

$u \in \mathcal{M}$ is a minimizer of E

$\Longrightarrow$ u is a solution of the PDE.

This also naturally yield weaker notions of a solution of a PDE.

- Concretely for Laplace's equation on $\Omega \subset\subset \mathbb{R}^n$ with $\partial\Omega$ in C^1

Let $\mathcal{M} := \{ v \in C^2(\bar{\Omega}) : v\restriction_{\partial\Omega} = g \}$, and

$$E : \mathcal{M} \to \mathbb{R}$$
$$v \mapsto E(v) := \frac{1}{2} \int_\Omega |Dv(x)|^2 dx - \int_\Omega v(x) f(x) dx$$

the energy functional!

<u>note</u>: For E to be well-defined, $v \in C^1(\bar{\Omega})$ would suffice – and in fact even weaker limitations would be enough.

We first use the functional to prove again uniqueness of the solution of (PP). Let $u_1, u_2 \in \mathcal{M}$ be two solutions of (PP). Then $w = u_2 - u_1$ ~~$\in \mathcal{M}$~~ is such that $-\Delta w(x) = 0$ for all $x \in \Omega$ and $w\restriction_{\partial\Omega} = 0$. Then:

$$0 = -\int_\Omega w \Delta w = \int_\Omega |Dw|^2 - \int_{\partial\Omega} w \frac{\partial w}{\partial \nu}$$

and since $w\restriction_{\partial\Omega} = 0$, this implies $\int_\Omega |\nabla w(x)|^2 dx = 0$.
With $|Dw(x)|^2 \geq 0$ and continuous, we obtain $Dw(x) = 0$ $\forall x \in \Omega$ and with the B.C. $w(x) = 0$ $\forall x \in \bar{\Omega}$.

<u>Note</u>:
- In the above calculation, $u_i \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is not sufficient.
- Below, we prove that in this limited class of functions, the set of minimizers is equal to the set of solutions of (PP).

- **Theorem.** Let $\Omega \subset\subset \mathbb{R}^n$ be such that $\partial\Omega$ is C^1, $f : \Omega \to \mathbb{R}$ bounded and ~~measurable~~ continuous and $g \in C^0(\partial\Omega)$.

  i) If $u \in C^2(\overline{\Omega})$ is a solution of (PP), then
  $$E(u) = \min_{v \in \mathcal{M}} E(v)$$
  ii) If $u \in \mathcal{M}$ is such that $E(u) \le E(v)$ for all $v \in \mathcal{M}$, then u is a solution of (PP).

Proof. (i). Let $w \in \mathcal{M}$. Then
$$0 = \int_\Omega \left(-\Delta u(x) - f(x)\right)\left(u(x) - w(x)\right) dx$$
$$= \int_\Omega \Big(Du(x) \cdot D(u-w)(x) - f(x)(u(x) - w(x)\Big) dx$$
$$+ \int_{\partial\Omega} \left(-\Delta u(x) - f(x)\right) \underbrace{\left(u(x) - w(x)\right)}_{=0 \text{ since } u, w \in \mathcal{M}} dx$$

Hence: $\displaystyle\int_\Omega \left(|Du(x)|^2 - f(x)u(x)\right) dx = \int_\Omega \left(Du(x) \cdot Dw(x) - f(x)w(x)\right) dx$

By Cauchy-Schwarz and the arithmetic-geometric mean inequality:
$$|Du(x) \cdot Dw(x)| \le |Du(x)|\,|Dw(x)| \le \frac{1}{2}\left(|Du(x)|^2 + |Dw(x)|^2\right)$$
yielding $\quad E(u) \le E(w) \quad (\forall w \in \mathcal{M})$.

This concludes the proof since $u \in \mathcal{M}$.

ii) Let $\xi \in C_c^\infty(\Omega)$, $\varepsilon \in \mathbb{R}$. Then $u \in \mathcal{M}$ implies $u + \varepsilon\xi \in \mathcal{M}$. Consider the function $e : \mathbb{R} \to \mathbb{R}$ defined by
$$e(\varepsilon) := E(u + \varepsilon\xi)$$

Since
$$e(\varepsilon) = E(u) + \varepsilon \int_\Omega \big(Du(x)\cdot D\xi(x) - f(x)\xi(x)\big)dx + \frac{\varepsilon^2}{2}\int_\Omega |D\xi(x)|^2 dx ,$$
we have
$$e'(0) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\big(e(\varepsilon) - e(0)\big) = \int_\Omega \big(Du\cdot D\xi - f\xi\big)\,dx$$
$$= \int_\Omega \big(-\Delta u(x)\xi(x) - f(x)\xi(x)\big)dx$$
where we used that $\xi\restriction_{\partial\Omega} = 0$. Since this holds for any $\xi \in C_c^\infty(\Omega)$ and $(-\Delta u - f)$ is continuous, this implies $-\Delta u(x) - f(x) = 0 \quad \forall x\in\Omega$ by the fundamental lemma of the calculus of variations. □

- Remarks: * It is often useful to search for minimizers in a larger set $\mathcal{M}$, typically some L^p spaces. Here: $u \in L^2$ s.t. $Du \in L^2$
  * The problem of existence of a solution can also be handled by variational methods via compactness arguments.
  * Very useful methods for non-linear equations:

  Consider $L \in C^2(\Omega\times\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$, $\Omega \subset\subset \mathbb{R}^n$ and
  $$E(u) := \int_\Omega L(x, u(x), Du(x))\,dx$$
  defined on $\mathcal{M} = \{u \in C^1(\Omega)\cap C^0(\bar\Omega) : u\restriction_{\partial\Omega} = g\}$.

  If E has a minimizer $u \in \mathcal{M}\cap C^2(\Omega)$, then

u is a solution of the Euler-Lagrange equation

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial u_i}\right) + \frac{\partial L}{\partial u} = 0 \quad \text{in } \Omega. \qquad (*)$$

Denoting $L = L(x, u, \xi)$, the reverse holds under additional assumptions on L, typically convexity of $(u, \xi) \mapsto L(x, u, \xi)$, namely: If u is a solution of $(*)$, then it is a minimizer of $E(u)$.

3. The heat equation

- Linear parabolic equation of 2^{nd} order : $\Omega \subseteq \mathbb{R}^n$, open.

$$u_t(t,x) - \Delta u(t,x) = 0 \qquad (t,x) \in \mathbb{R}_+ \times \Omega \qquad \text{(HE)}$$

possibly inhomogeneous : $= f(t,x)$

- Note: stationary solutions, namely solutions s.t. $u_t(t,x) = 0$, solve Laplace's equation.

Not surprisingly: Most fundamental properties of harmonic functions, when suitably adapted, have an equivalent for solutions of the heat equation : Mean-value property, Maximum principle, smoothness, existence and uniqueness.

- Interpretation: It is also a transport equation; as already seen, it arises from $\begin{pmatrix}\text{energy}\\\text{heat}\end{pmatrix}$ transport / conservation

Also: transport of probability : Starting a Brownian motion at $x \in \Omega$ at $t = 0$, and denoting $p_x(t,y)$ the probability density that the B.M. is at $y \in \Omega$ at time t, then p_x solves the heat equation (called "diffusion" here).

- A first useful property of solutions of (HE): scale invariance.

<u>Lemma</u>. Let $\Omega = \mathbb{R}^n$, and $u \in C^2((0,\infty) \times \mathbb{R}^n)$ be a solution of (HE). Then $v \in C^2((0,\infty) \times \mathbb{R}^n)$ defined by

$$v(t,x) := u(\lambda^2 t, \lambda x) \qquad (\lambda > 0)$$

is another solution of (HE).

<u>Proof</u>. By computation:
$$v_t(t,x) = \lambda^2 u_t(\lambda^2 t, \lambda x)$$
$$v_{x_i x_k}(t,x) = \lambda^2 u_{x_i x_k}(\lambda^2 t, \lambda x)$$
so that
$$v_t(t,x) - \Delta v(t,x) = \lambda^2 (u_t - \Delta u)(\lambda^2 t, \lambda x) = 0. \qquad \square$$

- The first natural problem to study is the initial value problem, namely solving (HE) with a given $u(0,x)$. Just as the Poisson kernel for the spherically symmetric ball is spherically symmetric, we look for a scaling invariant, and rotation invariant solution here:
$$u(t,x) = \lambda^\alpha u(\lambda^2 t, \lambda |x|)$$
for an $\alpha > 0$ to be determined, and all $\lambda > 0$. Choosing $\lambda^2 = t^{-1}$:
$$u(t,x) = t^{-\alpha/2} u(1, |x|/\sqrt{t}) =: t^{-\alpha/2} v(z)$$
With this:
- $u_{x_i}(t,x) = t^{-\alpha/2} v'(z) \frac{x_i}{|x|} \frac{1}{\sqrt{t}}$
- $u_{x_i x_k}(t,x) = t^{-\alpha/2} \left[v''(z) \frac{x_i x_k}{|x|^2} \frac{1}{t} + v'(z) \frac{1}{\sqrt{t}|x|} \left(\delta_{ik} - \frac{x_i x_k}{|x|^2} \right) \right]$
- $-u_t(t,x) = t^{-\alpha/2} \left[\frac{\alpha}{2} \frac{1}{t} v(z) + \frac{1}{2} \frac{|x|}{t^{3/2}} v'(z) \right]$

If u is a solution of (HE), then:
$$0 = t^{-\alpha/2} \left[v''(z) \frac{1}{t} + v'(z) \frac{1}{\sqrt{t}} \left(\frac{n-1}{|x|} \right) + \frac{|x|}{2t^{3/2}} v'(z) + \frac{\alpha}{2t} v(z) \right]$$
which yields an equation involving only z $(t > 0!)$.
$$v''(z) + \frac{n-1}{z} v'(z) + \frac{z}{2} v'(z) + \frac{\alpha}{2} v(z) = 0$$
Setting $\alpha = n$, this is equivalent for $z \neq 0$ to

$$\left(z^{n-1}v'(z)\right)' + \frac{1}{2}\left(z^n v(z)\right)' = 0$$

so that $\quad z^{n-1}v'(z) + \frac{1}{2}z^n v(z) = C$

and assuming that $v(z), v'(z) \to 0 \quad (z \to \infty)$ faster than any inverse power, $C = 0$. v is then obtained as

$$v(z) = D\exp\left(-\frac{z^2}{4}\right),$$

namely:

$$u(t,x) = D t^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right)$$

• Def: The function $\Phi : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ given by

$$\Phi(t,x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right) & (t,x)\in(0,\infty)\times\mathbb{R}^n \\ 0 & (t,x)\in\{0\}\times\mathbb{R}^n \end{cases}$$

is called the <u>fundamental solution</u> (or <u>heat kernel</u>) of the heat equation.

The normalization is chosen so that

$$\int_{\mathbb{R}^n}\Phi(t,x)\,dx = 1 \qquad (t>0).$$

indeed:

$$\int_{\mathbb{R}^n}\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}dx = \underbrace{\int_{\mathbb{R}^n}\pi^{-n/2}e^{-|y|^2}dy}_{y_i = x_i/2\sqrt{t}} = 1.$$

Now: the heat kernel can be used to represent the solution of the initial value problem for (HE)

<u>Theorem</u>: Let $g\in C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, and define $u : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ by

$$u(t,x) := \begin{cases} \int_{\mathbb{R}^n} \Phi(t,x-y)\, g(y)\, dy & (t,x) \in (0,\infty)\times\mathbb{R}^n \\ g(x) & (t,x) \in \{0\}\times\mathbb{R}^n \end{cases}$$

Then i) $u \in C^\infty((0,\infty)\times\mathbb{R}^n) \cap C^0([0,\infty)\times\mathbb{R}^n)$

ii) $u_t(t,x) - \Delta u(t,x) = 0$ for all $(t,x) \in (0,\infty)\times\mathbb{R}^n$.

- Proof: Clearly, $\Phi \in C^\infty((0,\infty)\times\mathbb{R}^n)$. Furthermore, $D^\alpha \Phi$ is bounded on $[\delta,\infty)\times\mathbb{R}^n$ for any $\delta > 0$.

Hence, for any $t > 0$ and $0 < \varepsilon < t$:

$$\varepsilon^{-1}\left[u(t+\varepsilon,x) - u(t,x)\right] = \int_{\mathbb{R}^n} \varepsilon^{-1}\left[\Phi(t+\varepsilon,x-y) - \Phi(t,x-y)\right] g(y)\, dy$$
$$\longrightarrow \int_{\mathbb{R}^n} \partial_t \Phi(t,x-y)\, g(y)\, dy$$

and similarly for the spatial derivatives. Hence

$$x \mapsto \int_{\mathbb{R}^n} \Phi(t,x-y)\, g(y)\, dy \in C^\infty((0,\infty)\times\mathbb{R}^n)$$

It also follows that if $t > 0$

$$(\partial_t - \Delta_x) \int_{\mathbb{R}^n} \Phi(t,x-y) g(y)\, dy = \int_{\mathbb{R}^n} (\partial_t - \Delta_x)(\cdots)$$
$$= 0 \quad \text{for any } x \in \mathbb{R}^n.$$

It remains to prove continuity. Let $\varepsilon > 0$. $\exists \delta > 0$:

$$|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \varepsilon/2$$

With the Lemma: For $t > 0$ and any $x \in \mathbb{R}^n$

$$u(t,x) - g(x_0) = \int_{\mathbb{R}^n} \Phi(t,x-y)\left(g(y) - g(x_0)\right) dy$$
$$= \left(\int_{B_\delta(x_0)} + \int_{\mathbb{R}^n \setminus B_\delta(x_0)}\right) \Phi(t,x-y)\left(g(y) - g(x_0)\right) dy$$

For the integral over $B_\delta(x_0)$:

$$\left| \int_{B_\delta(x_0)} (\dots) \right| \leq \frac{\varepsilon}{2} \int_{B_\delta(x_0)} \Phi(t, x-y)\, dy \leq \frac{\varepsilon}{2}.$$

If, furthermore $|x-x_0| < \delta/2$, then $|y-x_0| \geq \delta$ implies

$$|y-x| \geq |y-x_0| - |x-x_0| > \frac{1}{2}|y-x_0|.$$

Hence:

$$\left| \int_{\mathbb{R}^n \setminus B_\delta(x_0)} (\dots) \right| \leq 2 \sup\{|g(x)| : x \in \mathbb{R}^n\} \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \Phi(t, x-y)\, dy$$

and since $z \mapsto \Phi(t,z)$ is a decreasing function of $|z|$,

$$\leq \text{———"———} \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \Phi(t, \tfrac{1}{2}(x_0-y))\, dy$$

We conclude by

$$\int_{\mathbb{R}^n \setminus B_\delta(x_0)} \Phi(t, \tfrac{1}{2}(x_0-y))\, dy = \int_{\mathbb{R}^n \setminus B_\delta(0)} (4\pi t)^{-n/2} e^{-\frac{|z|^2}{16t}}\, dz$$

$$= \left(\frac{4}{\pi}\right)^{n/2} \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{4\sqrt{t}}}(0)} e^{-|w|^2}\, dw \leq C e^{-\frac{\delta^2}{16t}}$$

Altogether, if $|x-x_0| < \frac{\delta}{2}$, then for $t>0$ and small enough,

$$|u(t,x) - g(x_0)| < \varepsilon$$

namely $\lim_{\substack{(t,x)\to(0,x_0) \\ t>0}} u(t,x) = g(x_0)$.

$\square$

- <u>Remark</u>: Again, in the sense of distributions:

$$\begin{cases} \partial_t \Phi - \Delta \Phi = 0 & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ \Phi = \delta_0 & (t,x) \in \{0\} \times \mathbb{R}^n \end{cases}$$

Here: $(t,x) \mapsto \Phi(t, x-y)$ is smooth in $x \in \mathbb{R}^n$ for any fixed $t>0$ and $y \in \mathbb{R}^n$, but its singular behaviour as $(t,x) \to (0,y)$ allow for an integral representation of the solution of the initial value problem.

- Proposition: Let $g \in C_c^0(\mathbb{R}^n)$ with $\operatorname{supp} g = K$, and $g \geq 0$. Then for any $t>0$ and all $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy > 0 .$$

Proof: Let $x \in \mathbb{R}^n$ and $d = \operatorname{diam}(K) + \operatorname{dist}(x, K)$, where

$$\operatorname{diam}(K) = \sup \{ |y_1 - y_2| : y_1, y_2 \in K \}$$
$$\operatorname{dist}(x, K) = \inf \{ |x-y| : y \in K \}.$$

Then for any $y \in K$: $|x-y| \leq d < d+1$, so that

$$\Phi(t, x-y) > (4\pi t)^{-n/2} \exp\left(-\tfrac{(d+1)^2}{4t}\right)$$

Since $g \geq 0$:

$$\int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y)\, dy > \frac{e^{-\frac{(d+1)^2}{4t}}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} g(y)\, dy > 0$$

□

In other words: even if the initial function $g(x) = u(0,x)$ is compactly supported, the solution $u(t,x)$ of (HE) has a support in all of $\mathbb{R}^n$ for any $t>0$. There is infinite <u>propagation</u> speed.

At the same time: if $\operatorname{dist}(x, K) \gg \sqrt{t}$, then $u(t,x)$ is exponentially small

$\leadsto$ diffusive propagation

- We now introduce <u>Duhamel's Principle</u> to solve the inhomogeneous equation having a solution of the homogeneous initial value problem.

In a somewhat general – but formal – way:

Consider

$$(P)\quad \begin{cases} u_t(t,x) + Lu(t,x) = f(t,x) & (t,x)\in(0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) & (t,x)\in\{0\}\times\mathbb{R}^n \end{cases}$$

where L is a linear differential operator in the x variable.

Assume that the solution $u(t,x)$ can be constructed in the case $f\equiv 0$, and define a linear operator

$$S(t)g := u(t,\cdot) \qquad t\geq 0.$$

with $S(0)$ being the identity. In other words:

$$(\partial_t + L)(S(t)g) = 0.$$

Then the <u>Claim</u> is: Denoting $f(t,x) = f_t(x)$:

$$u(t,x) = (S(t)g)(x) + \int_0^t (S(t-s)f_s)(x)\,ds$$

is a solution of (P). Indeed

i) Initial condition: the integral vanishes at $t=0$, and $(S(0)g)(x) = g(x)$ ✓

ii) Equation: Applying $(\partial_t + L)$ to the right hand side:

- $(\partial_t+L)(S(t)g)(x) = 0 \quad \forall (t,x)\in(0,\infty)\times\mathbb{R}^n.$
- $(\partial_t+L)\int_0^t (S(t-s)f_s)(x)\,ds$

$$= \underbrace{(S(0)f_t)(x)}_{= f_t(x) = f(t,x)} + \int_0^t \underbrace{(\partial_t+L)(S(t-s)f_s)}_{=0}(x)\,dx$$

i.e. $(\partial_t + L)(\text{right hand side}) = f(t,x)$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$ indeed.

In practice, one needs to carefully carry out the formal manipulations above:

- In the case of (HE), namely $Lu(t,x) = -\Delta u(t,x)$ and by the theorem

$$(S(t)g)(x) = \int_{\mathbb{R}^n} \Phi(t, x-y) g(y)\, dy \, ,$$

we have the following theorem:

<u>Theorem</u>: Consider (P) with $g \in C_b^0(\mathbb{R}^n)$ and $f \in C^{1,2}([0,\infty) \times \mathbb{R}^n)$, with $f(t,\cdot)$ compactly supported for any $t \in [0,\infty)$. Then,

$$u(t,x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \Phi(t, x-y) g(y)\, dy + \int_0^t \left(\int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y)\, dy \right) ds & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ g(x) & (t,x) \in \{0\} \times \mathbb{R}^n \end{cases}$$

is a solution of (P), namely:

i) $u \in C^{1,2}((0,\infty) \times \mathbb{R}^n) \cap C^0([0,\infty) \times \mathbb{R}^n)$

ii) $u_t(t,x) - \Delta u(t,x) = f(t,x) \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n$

iii) $\displaystyle\lim_{\substack{(t,x) \to (0,x_0) \\ x_0 \in \mathbb{R}^n,\, t>0}} u(t,x) = g(x_0)$

- <u>Proof</u>: We need only prove (i) & (ii) since (i) $\Rightarrow$ (iii)

- With the previous theorem, it is already known that the first term satisfies (i) and (ii) with $f \equiv 0$. It remains to prove that the second term satisfies (i), (ii) and (iii) with $\lim u(t,x) = 0$.

(i) & (iii): for $t>0$, let $r = t-s$, $z = x-y$ & $v(t,x) = \int_0^t (\dots)\,ds$:

$$v(t,x) = \int_0^t \left(\int_{\mathbb{R}^n} \Phi(r,z) f(t-r, x-z)\,dz \right) dr$$

Since $\Phi(r,z)$ is smooth in a neighborhood of $r = t > 0$ for any $z \in \mathbb{R}^n$, and $f \in C^2([0,\infty)\times\mathbb{R}^n)$,

$$v_t(t,x) = \int_{\mathbb{R}^n} \Phi(t,z) f(0,x-z)\,dz + \int_0^t \left(\int_{\mathbb{R}^n} \Phi(r,z) \frac{\partial}{\partial t} f(t-r, x-z)\,dz \right) dr$$

and v_t is continuous.

Similarly, $v \in C^2((0,\infty)\times\mathbb{R}^n)$.

Furthermore,

$$\sup\{|v(t,x)| : x\in\mathbb{R}^n\} \leq t \sup\left\{ \int_{\mathbb{R}^n} \Phi(r,z)\,|f(t-r,x-z)|\,dz : r\in[0,t],\ x\in\mathbb{R}^n \right\}$$

$$\leq t \sup\{ |f(r,z)| : r\in[0,t],\ z\in\mathbb{R}^n\}$$

which prove that $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0,\ x_0\in\mathbb{R}^n}} v(t,x) = 0$

(ii) Now: if $(t,x) \in (0,\infty)\times\mathbb{R}^n$.

$$v_t(t,x) - \Delta v(t,x) = \int_0^t \int_{\mathbb{R}^n} \Phi(r,z)(\partial_t - \Delta_x) f(t-r,x-z)\,dz\,dr + \int_{\mathbb{R}^n} \Phi(t,z) f(0,x-z)\,dz \qquad (*)$$

Splitting the r-integral as
$$\int_0^t (\dots)\,dr = \left(\int_0^\varepsilon + \int_\varepsilon^t\right)(\dots)\,dr =: I^1_\varepsilon + I^2_\varepsilon ,$$
for some $0 < \varepsilon < t$, we get the estimates:
$$|I^1_\varepsilon| \leq \varepsilon \sup\left\{ |(\partial_t - \Delta_x) f(t,x)| : (t,x) \in (0,\infty)\times\mathbb{R}^n \right\}$$
$$\cdot \sup\Big(\underbrace{\int_{\mathbb{R}^n} \Phi(s,z)\,dz}_{=1} ,\ s\in(0,\varepsilon)\Big) \xrightarrow[\varepsilon\downarrow 0]{} 0 .$$
Furthermore, since $(r,z) \mapsto \Phi(r,z)(\partial_t - \Delta_x) f(t-r, x-z)$ is C^2 and $(\partial_t - \Delta_x) f(t-r, x-z) = (-\partial_r - \Delta_z) f(t-r,x-z)$, we have by Gauss's theorem:
$$* \quad \int_{\mathbb{R}^n} \Phi(r,z)\Delta_z f(t-r,x-z)\,dz = \int_{\mathbb{R}^n} \Delta_z \Phi(r,z) f(t-r,x-z)\,dz$$
since $f(s,\cdot)$ is compactly supported in $\mathbb{R}^n$.

By Fubini and partial integration:
$$* \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(r,z)\,\partial_r f(t-r,x-z)\,dz\,dr$$
$$= -\int_{\mathbb{R}^n} \int_\varepsilon^t \partial_r \Phi(r,z) f(t-r,x-z)\,dr\,dz$$
$$+ \int_{\mathbb{R}^n} \Phi(r,z) f(t-r,x-z) \Big|_{r=\varepsilon}^{r=t}$$
for $r > 0$: $\quad -\partial_r \Phi(r,z) = -\Delta_z \Phi(r,z)$

while $\lim\limits_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \Phi(\varepsilon,z) f(t-\varepsilon, x-z)\,dz = f(t,x)$

by the previous theorem.

and the boundary term $r=t$ cancels the last term of (*, p. 72)

Altogether: $v_t(t,x) - \Delta v(t,x) = f(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}^n$
which concludes the proof by the initial remark. $\square$

- We now study the asymptotics of the solution $u(t,x)$ of (P) as $t \to \infty$).

Theorem Let $g \in C^0_b(\mathbb{R}^n)$ and $\bar{f} \in C^2_c(\mathbb{R}^n)$. Let $u(t,x)$ be the solution of (P) with $f(t,x) = \bar{f}(x)$ for all $t \in [0,\infty)$.
Let also $v \in C^2(\mathbb{R}^n)$ be the solution of the Poisson equation $-\Delta v(x) = \bar{f}(x)$ such that $|v(x)| \to 0$ as $|x| \to \infty$.
If $n \geq 3$, then

$$\lim_{t \to \infty} u(t,x) = v(x) \quad \text{for all } x \in \mathbb{R}^n .$$

- Proof: We first observe that v is Newton's potential,

$$v(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} |x-y|^{2-n} \bar{f}(y)\, dy .$$

Now, by dominated convergence:

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} \Phi(t,x-y) g(y)\, dy = \int_{\mathbb{R}^n} \lim \Phi(t,x-y) g(y)\, dy = 0$$

Applying again Fubini, the second term of $u(t,x)$ reads

$$\int_0^t \Big(\qquad \Big) ds = \int_{\mathbb{R}^n} \left(\int_0^t \Phi(r,z)\, dr \right) \bar{f}(z)\, dz$$

letting $\rho := \frac{|z|^2}{4r}$,

$$\int_0^t \Phi(r,z)\, dr = \frac{1}{(4\pi)^{n/2}} \int_{z^2/4t}^{\infty} \frac{(4\rho)^{n/2}}{|z|^n} e^{-\rho} \left(\frac{-|z|^2}{4\rho^2} \right) d\rho$$

so that

$$\lim_{t\to\infty} \int_0^t \Phi(r,t)\,dt = \frac{1}{4\pi^{n/2}} |z|^{2-n} \underbrace{\int_0^\infty \rho^{\frac{n}{2}-2} e^{-\rho}\,d\rho}_{=\Gamma\left(\frac{n}{2}-1\right)}$$

$$= \frac{1}{(n-2)\omega_n} |z|^{2-n}$$

which is the fundamental solution of the Laplace's equation □

- Remarks: - This proves return to equilibrium for $t\to\infty$, the solution of (HE) converges to its stationary (i.e. time independent) solution, and completely loses track of the initial condition g.
  - The result fails for $n=1,2$, in fact $\lim_{t\to\infty} u(t,x) = \infty$

    This is reminiscent to the difference between recurrence and transience of Brownian Motion.

- One can prove other asymptotic results. E.g.

  In the case $f=0$, the solution $u(t,x)$ converges to zero, and in fact

$$u(t,x) \sim \left(\int g\right) \frac{1}{\sqrt{t}}\, e^{-\frac{|x|^2}{4t}} \qquad (n=1)$$

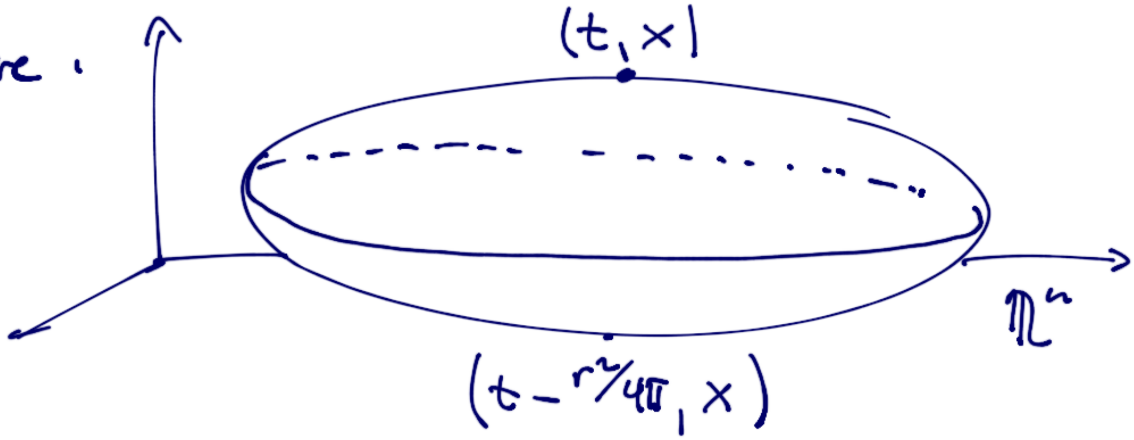
and the same result holds even if the equation is modified by nonlinear terms such as $u(t,x)^p$ for $p>3$.

- We now turn to the mean-value property for solutions of heat equation. Recall:
  for Laplace's equation, the mean is over $\partial B_r(x)$, namely the <u>level sets</u> of the fundamental solution.

Similarly here:
"heat ball". for $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$:

$$E(t,x;r) := \{(s,y) \in \mathbb{R}^{n+1} : s \le t,\ \Phi(t-s, x-y) \ge r^{-n}\}$$

In a picture:



* $s \ll t$: $t-s \gg 1$, hence Φ is very small for all $|x-y|$ $\leadsto$ no points in $E(t,x;r)$
* $t-s = r^2/4\pi$: $(4\pi(t-s))^{-n/2} = r^{n}$ so that $y=x$ belongs to $E(t,x;r)$
* $t-s < r^2/4\pi$: the set of y's in $E(t,x;r)$ is a ball around x
* $t=s$: the only point belonging to $E(t,x;r)$ is $y=x$

The boundary of $E(t,x;r)$, given by $\Phi(t-s,x-y) = r^{-n}$ is best described using:

$$\psi(s,y;r) := \log\left(r^n \Phi(t-s, x-y)\right)$$
$$= -\frac{n}{2}\log\left(4\pi(t-s)\right) - \frac{|x-y|^2}{4(t-s)} + n\log r .$$

Then: $\partial E(t,x;r) = \{(s,y)\in \mathbb{R}^{n+1} : \psi(s,y;r) = 0\}$

- <u>Lemma</u>: Let $u \in C^0(E(t,x;R))$ and for $0<r<R$:

$$\varphi(u;r) = \frac{1}{r^n}\int_{E(t,x;r)} u(s,y)\,\frac{|x-y|^2}{4(t-s)^2}\,ds\,dy .$$

Then $\varphi(1;r) = 1$.

<u>Proof</u>: We change integration variables to $rz := x-y$ and $r^2\tau := t-s$:

$$\varphi(1;r) = \int_{\Phi(\tau,z)\geq 1} r^{-n}\frac{r^2z^2}{4r^4\tau^2}\, r^{n+2}\,d\tau\,dz = \int_{\Phi(\tau,z)\geq 1} {z^2}/{4\tau^2}\,d\tau\,dz$$

$$\overset{w = z/\sqrt{\tau}}{=} \int_{\Phi(1,w)\geq \tau^{n/2}} \frac{w^2}{4}\,\tau^{n/2-1}\,d\tau\,dw \overset{\sigma=\tau^{n/2}}{=} \int_{\Phi(1,w)\geq\sigma} \frac{w^2}{2n}\,d\sigma\,dw$$

$$= \int_{\mathbb{R}^n}\left(\int_0^{\Phi(1,w)} d\sigma\right)\frac{w^2}{2n}\,dw = 1$$

by a Gaussian integration $\square$

- We shall solve the heat equation in a spatial domain $\Omega \subset\subset \mathbb{R}^n$.

- **Def**: Let $\Omega \subset\subset \mathbb{R}^n$ be open and let $T>0$. The *parabolic cylinder* Ω_T is defined by
$$\Omega_T := \Omega \times (0,T].$$

Rem: The boundary of Ω_T:
$$\partial\Omega_T = (\Omega \times \{t=0\}) \cup (\partial\Omega \times (0,T]) \cup (\Omega \times \{t=T\})$$
and we shall also use
$$\partial'\Omega_T := (\Omega \times \{t=0\}) \cup (\partial\Omega \times (0,T])$$

- **Theorem**: Let $\Omega \subset\subset \mathbb{R}^n$ be open, $T>0$, and let $u \in C^2(\Omega_T)$ be a solution of
$$\partial_t u(t,x) - \Delta u(t,x) = 0 \qquad (t,x) \in \mathring{\Omega}$$
Then for any $(t,x,r) \in (0,\infty)\times\mathbb{R}^n, (0,\infty)$ s.t. $\overline{E(t,x;r)} \subset \Omega_T$:
$$u(t,x) = \frac{1}{r^n}\int_{E(t,x;r)} u(s,y)\frac{|x-y|^2}{4(t-s)^2}\,ds\,dy \qquad \text{(HMVP)}.$$

- **Proof**: With the notation above, the r.h.s reads $\varphi(u;r)$, and
$$\varphi(u;r) = \int_{\Phi(\tau,z)\geq 1} u(t-r^2\tau, x-rz)\,|z|^2/4\tau^2\,d\tau\,dz.$$
Since $u \in C^2(\Omega_T)$, we have that
$$\frac{d}{dr}\varphi(u;r) = -\int_{\Phi(\tau,z)\geq 1}(\partial_1 u\cdot 2r\tau + D_2 u\cdot rz)\,|z|^2/4\tau^2\,d\tau\,dz$$
$$= -\frac{1}{r^{n+1}}\int_{E(t,x;r)}\left(\partial_s u(s,y)\,2(t-s) + Du(s,y)\cdot(x-y)\right)\frac{|x-y|^2}{4(t-s)^2}\,ds\,dy$$

This can be expressed in terms of the function ψ. We compute:

$$\partial_s \psi(s,y;r) = \frac{n}{2(t-s)} - \frac{|x-y|^2}{4(t-s)^2}$$

$$D\psi(s,y;r) = -(y-x)/2(t-s)$$

and hence:

$$\frac{|x-y|^2}{4(t-s)^2} = -\frac{(y-x)}{2(t-s)} \cdot D\psi(s,y;r)$$

$$= -\partial_s \psi(s,y;r) + \frac{n}{2(t-s)}$$

Altogether:

$$\frac{d}{dr}\varphi(u;r) = \frac{1}{r^{n+1}} \int_{\psi \geq 0} \Big(\partial_s u \cdot (y-x) \cdot D\psi - Du \cdot (y-x) \partial_s \psi - n\, Du \cdot D\psi \Big)\, ds\, dy$$

Integrating by parts and recalling that $\psi = 0$ at the boundary:

$$\frac{d}{dr}\varphi(u;r) = \frac{1}{r^{n+1}} \int_{\psi \geq 0} \Big(\cancel{(D\partial_s u)}(y-x) + n\partial_s u - \cancel{(D\partial_s u)}(y-x) - n\Delta u \Big) \psi\, ds\, dy$$

$$= \frac{n}{r^{n+1}} \int_{\psi \geq 0} \big(\partial_s u(s,y) - \Delta u(s,y) \big)\, ds\, dy = 0$$

since u is a solution of (HE). Hence,

$$\varphi(u;r) = \lim_{s \searrow 0} \varphi(u;s) = u(t,x) \quad \text{by the lemma}$$

$\square$

- Remarks: • As in the case of Laplace's equation, the (HMVP) does not depend on the explicit form of the solution.

* $u(t,x)$ depends on $u(s,y)$ for some $s \leq t$ (in the past of t) only.
* Having a mean value property, we can now derive a min or max principle.

• <u>Theorem</u> : Let $\Omega \subset\subset \mathbb{R}^n$ be open and connected. If $u \in C^0(\overline{\Omega_T})$ solves (HMVP) for any $\overline{E(t,x;r)} \subset \Omega_T$, then:

i) $$\min_{(t,x)\in\overline{\Omega_T}} u(t,x) = \min_{(t,x)\in\partial'\Omega_T} u(t,x)$$

ii) If $(t,x) \in \Omega_T$ is a minimal point, then u is constant in $\overline{\Omega_t}$.

<u>Rem</u> : * in (ii), note Ω_t, not Ω_T : the (HMVP) yields u = constant only in the past.

* By considering $-u$ instead of u, one obtains "max".

• <u>Proof</u> : (ii) The (HMVP) and the lemma $\psi(1;r) = 1$ yield the second form of the mean value property.

$$\int_{E(t,x;r)} \big(u(t,x) - u(s,y)\big) \frac{|x-y|^2}{(t-s)^2}\, ds\, dy = 0 \qquad \text{(HMVP II)}$$

for any $\overline{E(t,x;r)} \subset \Omega_T$

Let $m := \min_{(s,y)\in\overline{\Omega_T}} u(s,y)$ and $(t,x) \in \Omega_T$: $u(t,x) = m$.

Let $r>0$ be s.t. $\overline{E(t,x;r)} \subset \Omega_T$. With $(m-u(s,y))\frac{|x-y|^2}{(t-s)^2} \leq 0$ and (HMVP II), we have that
$$m - u(s,y) = 0 \quad \forall (s,y) \in E(t,x;r)$$
namely $u = m$ in $E(t,x;r)$.

Let now $(s,y) \in \Omega_T$ with $s<t$, and let
$$\gamma : [0,1] \longrightarrow \Omega_T$$
$$\lambda \mapsto \gamma(\lambda) = (1-\lambda)(t,x) + \lambda(s,y)$$
Let further
$$f : [0,1] \longrightarrow \mathbb{R}$$
$$\lambda \mapsto f(\lambda) := u(\gamma(\lambda)) \quad .$$
Then $f^{-1}(\{m\}) \neq \emptyset$ since $f(0) = m$. By the above, $f^{-1}(\{m\})$ is relatively open in $[0,1]$. Since u is continuous and $\{m\}$ is closed, $f^{-1}(\{m\})$ is relatively closed. Hence, $f(\lambda) = m$ for all $\lambda \in [0,1]$. Since Ω is connected, any $(s,x) \in \Omega_t$ can be connected with (t,x) by a polynomial path in Ω_t, so that $u \equiv m$ in Ω_t and by continuity in $\overline{\Omega_t}$.

i) Immediate from (ii). □

- Again this implies uniqueness of the mixed initial and boundary value problem for (HE).

<u>Corollary</u>: Let $g \in C^0(\partial' \Omega_T)$ and $f \in C^0(\Omega_T)$. Then there is at most one solution $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ of
$$\begin{cases} u_t(t,x) - \Delta u(t,x) = f(t,x) & (t,x) \in \Omega_T \\ u(t,x) = g(t,x) & (t,x) \in \partial' \Omega_T \end{cases}$$

- Proof: Let u_1, u_2 be two solutions and $v := u_2 - u_1$. Then $v_t - \Delta v = 0$ in Ω_T and $v(t,x) = 0$ for all $(t,x) \in \partial' \Omega_T$.
  By the theorem:
  $$v(t,x) \geq \min_{(s,y) \in \overline{\Omega}_T} v(s,y) = \min_{(s,y) \in \partial' \Omega_T} v(s,y) = 0 \quad \forall (t,x) \in \overline{\Omega}_T$$
  Repeating the argument with $\tilde{v} = u_1 - u_2$ yields the claim. $\square$
- Note: * Uniqueness does <u>not</u> hold without further assumptions for the pure initial value problem.
  * Stability for the mixed problem follows again: Let u_1, u_2 be solutions of $\begin{cases} \partial_t u_i - \Delta u_i = 0 & \text{in } \mathring{\Omega}_T \\ u_i = g_i & \text{in } \partial' \Omega_T \end{cases}$
  s.t. $\sup_{(t,x) \in \partial' \Omega_T} |g_1(t,x) - g_2(t,x)| \leq \varepsilon$.

  Then, $w = \pm(u_1 - u_2)$ solves $\partial_t w - \Delta w = 0$ with $w \restriction_{\partial' \Omega_T} = \pm (g_1 - g_2)$

  By the maximum principle, this implies $\sup_{(t,x) \in \Omega_T} |w(t,x)| \leq \varepsilon$.
- Proposition: Let $g \in C_b^0(\mathbb{R}^n)$ and let $u \in C^2((0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ be a solution of
  $$\begin{cases} u_t(t,x) - \Delta u(t,x) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = g(x) & x \in \mathbb{R}^n \end{cases}$$
  such that there are $A, a > 0$ with

$$u(t,x) \le A e^{+a|x|^2} \qquad (t,x) \in [0,T] \times \mathbb{R}^n .$$

Then:

$$\sup\{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}^n\} = \sup\{g(x) : x \in \mathbb{R}^n\}$$

• Proof: We assume that T is so that $4aT < 1$. In the general case, $\exists k \in \mathbb{N}$ and $\tau < 1/4a$ s.t. $[0,T] = \bigcup_{i=1}^{k} [(i-1)\tau, i\tau]$, and

$$\sup\{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}^n\}$$
$$= \max_{i \in \{1,\dots,k\}} \sup\{u(t,x) : (t,x) \in [(i-1)\tau, i\tau] \times \mathbb{R}^n\}$$

$$\le \max_{i \in \{1,\dots,k\}} \sup\{u((i-1)\tau, x) : x \in \mathbb{R}^n\} \le \sup\{u(0,x) : x \in \mathbb{R}^n\}$$

So:

* Let $\varepsilon > 0$: $4a(T+\varepsilon) < 1$. Let $y \in \mathbb{R}^n$, $\mu > 0$ and define

$$v_\mu^\varepsilon(t,x) := u(t,x) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$$

for $x \in \mathbb{R}^n$, $t \in [0,T]$. By assumption,

$$(\partial_t - \Delta) v_\mu^\varepsilon(t,x) = 0 \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

* Let now $\Omega = B_r(y)$ for some $r > 0$. By the maximum principle,

$$\max_{(t,x) \in \Omega_T} v_\mu^\varepsilon(t,x) = \max_{(t,x) \in \partial' \Omega_T} v_\mu^\varepsilon(t,x)$$

Now: at $t = 0$: $v_\mu^\varepsilon(0,x) \le u(0,x) = g(x) \quad \forall x \in \mathbb{R}^n$;

on $(0,T] \times \partial\Omega$,

$$v_\mu^\varepsilon(t,x) \le A \exp(a|x|^2) - \frac{\mu}{(T+\varepsilon)^{n/2}} \exp\left(\frac{r^2}{4(T+\varepsilon)}\right)$$

Let $\gamma > 0$ s.t. $4(T+\varepsilon) = 1/(a+\gamma)$; then:

$$v_\mu^\varepsilon(t,x) \leq -\frac{\mu}{(T+\varepsilon)^{n/2}} e^{(a+\gamma)r^2}\Big(1-\frac{A(T+\varepsilon)^{n/2}}{\mu}\underbrace{e^{a(|y|+r)^2-(a+\gamma)r^2}}_{\to 0 \quad (r\to\infty)}\Big)$$

≤ 0 for r large enough.

Altogether: $v_\mu^\varepsilon(t,x) \leq \sup\{g(x) : x\in\mathbb{R}^n\}$ for all $(t,x)\in\overline{\Omega_T}$

and in particular for $x=y$:

$$u(t,y) - \mu/(T+\varepsilon-t)^{n/2} \leq \sup\{g(x) : x\in\mathbb{R}^n\},$$

which concludes the proof since μ is arbitrary (as well as $y\in\mathbb{R}^n$). $\square$

- <u>Theorem</u>: Let $g\in C_b^0(\mathbb{R}^n)$ and $f\in C^0([0,T]\times\mathbb{R}^n)$. Then there exists at most one solution $u\in C^2((0,T]\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$ of

$$\begin{cases} u_t(t,x)-\Delta u(t,x) = f(t,x) & (t,x)\in(0,T]\times\mathbb{R}^n \\ u(0,x) = g(x) & x\in\mathbb{R}^n \end{cases}$$

satisfying the condition

$$|u(t,x)| \leq A\exp(a|x|^2) \qquad (t,x)\in[0,T]\times\mathbb{R}^n$$

for some constants $A,a>0$.

<u>Proof</u>: Immediate corollary of the previous Proposition applied to $\pm(u_1-u_2)$ $\square$

- Remark: Without additional condition, there may be (infinitely) many solutions: see Tychonoff's example corresponding to $g\equiv 0$.

- We now turn to <u>smoothing</u> properties of the flow associated to the heat equation. Recall:

if $g\in C_b^0(\mathbb{R}^n)$, then the solution $u(t,x)=\int\Phi(t,x-y)g(y)\,dy$ is $C^\infty((0,\infty)\times\mathbb{R}^n)$, and $\lim_{t\searrow 0} u(t,x) = g(x)$

In other words $\Phi(t,x-\cdot)$ is a <u>mollifier</u> (see Ex 4, Sheet 1)

- Theorem: Let $1 \le p < \infty$ and $g \in L^p(\mathbb{R}^n)$ $\left(\text{i.e. } \int |g(x)|^p dx < \infty\right)$.
  Let $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be defined by
  $$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) g(y) dy .$$
  Then $u \in C^\infty\left((0,\infty) \times \mathbb{R}^n\right)$, $u_t - \Delta u = 0$, and
  $$\lim_{t \searrow 0} \| u(t,\cdot) - g \|_p = 0 .$$

- Remark: This is a stronger smoothing statement, but the convergence as $t \searrow 0$ is not pointwise anymore.

- Proof: * We summarize the properties of the heat kernel:
  i) $\Phi_t(t,x) - \Delta \Phi(t,x) = 0 \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n$
  ii) $\Phi \in C^\infty\left((0,\infty) \times \mathbb{R}^n\right)$
  iii) For any $t > 0$, $\Phi(t,\cdot) \in L^q(\mathbb{R}^n)$ for all $1 \le q \le \infty$. Similarly, $D^\alpha \Phi(t,\cdot) \in L^q(\mathbb{R}^n)$, all by the exponential decay of the heat kernel in space.

  * Let now q be s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder's inequality:
  $$\left| \int D^\alpha \Phi(t, x-y) g(y) dy \right| \le \| D^\alpha \Phi(t, x - \cdot) \|_q \| g \|_p < \infty$$
  for any multiindex $\alpha = (\alpha_t, \alpha_{x_1}, \dots, \alpha_{x_n})$. Hence, $u \in C^\infty\left((0,\infty) \times \mathbb{R}^n\right)$ and $|D^\alpha u(t,x)| \le \| D^\alpha \Phi(t, x - \cdot) \|_q \| g \|_p$
  * It immediately follows that u solves (HE).
  * By the general results on mollifiers $u(t,\cdot) = \Phi(t,\cdot) * g$ converges to g in L^p-norm. (see Ex Sheet) $\square$

- Remarks: * Smoothing in the future implies irreversibility. In particular: if $u(t,x)$ is not $C^\infty((0,\infty)\times\mathbb{R}^n)$, then it cannot arise as a solution of the heat equation.
  * The initial value problem has another natural setup: Instead of considering $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, one defines $\tilde{u}:[0,\infty)\to$ function space, e.g. the Schwartz functions S.

This emphasises the "evolution equation" aspect of the heat equation. The concept of a strong solution corresponds to $\tilde{u}\in C^1((0,\infty);S)$, namely:

i) the pointwise derivative $\partial_t\tilde{u}(t,x)$ exists for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$.

ii) for any $t\in(0,\infty)$, $x\mapsto\partial_t\tilde{u}(t,x)$ is in S.

iii) $t\mapsto\partial_t\tilde{u}(t,\cdot)$ is $C^0((0,\infty);S)$.

One can then show (see Ex 2, Sheet 8): If $g\in S$, then there is a unique solution $\tilde{u}\in C^0([0,\infty);S)\times C^1((0,\infty);S)$. Moreover, $\tilde{u}\in C^\infty((0,\infty);S)$ and $\tilde{u}(t,x)=\int\Phi(t,x-y)g(y)\,dy$.

- Finally, we note without proof two further, general results:

Theorem: Let $\Omega\subset\subset\mathbb{R}^n$ be open and $T>0$. If $u\in C^2(\mathring{\Omega}_T)$ solves $u_t(t,x)-\Delta u(t,x)=0$ for all $(t,x)\in\mathring{\Omega}_T$, then for any $0<t\leq T$, the function $x\mapsto u(t,x)$ is analytic in Ω.

Note that analyticity does not hold for $t\mapsto u(t,x)$!

- Theorem: Let $g \in C_b^0(\mathbb{R}^n)$ and $f \in C_b^0((0,\infty)\times\mathbb{R}^n)$. If for any $t\in(0,\infty)$, $x \mapsto f(t,x)$ is locally Hölder continuous, then there exists $u \in C^2((0,\infty)\times\mathbb{R}^n) \cap C^0([0,\infty)\times\mathbb{R}^n)$ solving
$$\begin{cases} u_t(t,x) - \Delta u(t,x) = f(t,x) & (t,x)\in(0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) & x\in\mathbb{R}^n \end{cases}$$
In that case, u is given by the representation formula discussed above.

- We conclude this chapter with some variational results.

Theorem: Let $\Omega \subset\subset \mathbb{R}^n$ be open and $T>0$. Let $f\in C^0(\Omega_T)$ and $g \in C^0(\Omega)$, and let $u \in C^2(\overline{\Omega_T})$ be a solution of
$$\begin{cases} u_t(t,x) - \Delta u(t,x) = f(t,x) & (t,x)\in \mathring{\Omega}_T \\ u(t,x) = 0 & (t,x)\in(0,T]\times\partial\Omega \\ u(0,x) = g(x) & x\in\Omega. \end{cases}$$
Then:
$$\|u\|_{L^\infty((0,T);L^2(\Omega))} + \|Du\|_{L^2((0,T);L^2(\Omega))} \leq C(T)\left(\|f\|_{L^2((0,T);L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right)$$

- Remark: The smoothing effect of the heat flow can again be seen in this estimate: If $f=0$, then not only can u, but also Du be bounded in terms of the initial data g.

- Proof: We first note that
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t,x)|^2 dx + \int_\Omega |Du(t,x)|^2 dx = \int_\Omega \big(u_t(t,x) - \Delta u(t,x)\big)u(t,x)\,dx$$

$$= \int_\Omega f(t,x)\, u(t,x)\, dx$$

by integration by parts and the fact that $u(t,x) = 0$ whenever $x \in \partial\Omega$.

Integrating this from 0 to T:

$$* \quad \int_0^T \frac{d}{dt} \int_\Omega u^2\, dx\, dt = \int_\Omega |u(t,x)|^2\, dx - \int_\Omega g(x)^2\, dx$$

$$* \quad \int_0^T \int_\Omega fu\, dx\, dt \le \left(\int_0^T \int_\Omega f^2\, dx\, dt \right)^{1/2} \left(\int_0^T \int_\Omega u^2\, dx\, dt \right)^{1/2}$$

$$\le 1/4\varepsilon \int_0^T \int_\Omega f^2\, dx\, dt + \varepsilon \int_0^T \int_\Omega u^2\, dx\, dt$$

by the Cauchy–Schwarz and geometric-arithmetic mean inequality.

Altogether:

$$\left(\frac{1}{2} - \varepsilon T\right) \sup_{t\in[0,T]} \int_\Omega |u(t,x)|^2\, dx + \int_0^T \int_\Omega |Du(t,x)|^2\, dx\, dt$$

$$\le 1/4\varepsilon \int_0^T \int_\Omega f(t,x)^2\, dx\, dt + \frac{1}{2} \int_\Omega g(x)^2\, dx$$

Choosing $\varepsilon = 1/4T$ yields

$$\frac{1}{4} \sup_{[0,T]} \int_\Omega u^2 + \int_0^T \int_\Omega |Du|^2 \le T \int_0^T \int_\Omega f^2 + \frac{1}{2} \int_\Omega g^2$$

which yields the claim, see next page $\square$

- <u>Note</u>: The proof holds further under the weaker initial condition

$$\lim_{t \to 0} \| u(t,\cdot) - g \|_{L^2(\Omega)} = 0.$$

- As was the case for Laplace's equation, uniqueness can easily be proved:

The inequality $\frac{1}{4}\sup\int_\Omega u^2 + \int_0^T\int_\Omega |\nabla u|^2 \leq T\int_0^T\int_\Omega f^2 + \frac{1}{2}\int_\Omega g^2$ can be written as

$$\frac{1}{4}\|u\|^2_{L^2((0,T);L^2(\Omega))} + \|\nabla u\|^2_{L^2((0,T);L^2(\Omega))}$$
$$\leq T\,\|f\|^2_{L^2((0,T);L^2(\Omega))} + \frac{1}{2}\|g\|^2_{L^2(\Omega)}$$

The right hand side is bounded above by

$$\left(\max\left(\sqrt{T}, \frac{1}{\sqrt{2}}\right)\left(\|f\|_{..} + \|g\|_{..}\right)\right)^2$$

so that

$$\left(\frac{1}{4}\|u\|^2_{..} + \|\nabla u\|^2_{..}\right)^{1/2} \leq \max\left(\sqrt{T}, \frac{1}{\sqrt{2}}\right)\left(\|f\|_{..} + \|g\|_{..}\right)$$

Since $x \mapsto x^{1/2}$ is concave, the left hand side is bounded below by

$$\sqrt{5/4}\left(\frac{1}{5}\|u\|_{..} + \frac{4}{5}\|\nabla u\|_{..}\right) \geq \frac{1}{\sqrt{20}}\left(\|u\|_{..} + \|\nabla u\|_{..}\right)$$

Altogether:

$$\|u\|_{L^2((0,T);L^2(\Omega))} + \|\nabla u\|_{L^2((0,T);L^2(\Omega))}$$
$$\leq \max\left(\sqrt{20T}; \sqrt{10}\right)\left(\|f\|_{L^2((0,T);L^2(\Omega))} + \|g\|_{L^2(\Omega)}\right)$$

which is the claim.

Let $u_{1,2} \in C^2(\Omega_T)$ be two solutions of the heat equation on Ω_T and let

$$e : [0,T] \longrightarrow [0,\infty)$$
$$t \mapsto e(t) = \frac{1}{2}\int_\Omega |v(t,x)|^2 dx$$

where $v = u_2 - u_1$. Since $v\restriction_{\partial'\Omega_T} = 0$,

$$\frac{d}{dt} e(t) = \int_\Omega v(t,x) v_t(t,x)\, dx = \int_\Omega v(t,x)\Delta v(t,x)\, dx$$
$$= -\int_\Omega |Dv(t,x)|^2 dx \leq 0.$$

Since, moreover $e(0) = 0$, this implies $e(t) \leq 0$ for all $t \in [0,T]$, and hence $e \equiv 0$.

- A more interesting fact is the uniqueness backwards in time:

<u>Theorem</u>: Let $\Omega \subset\subset \mathbb{R}^n$ open and $T > 0$. Let $g \in C_b^0([0,T]\times\partial\Omega)$, and let $u_1, u_2 \in C^2(\overline{\Omega_T})$ be solutions of

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = 0 & (t,x) \in \mathring{\Omega}_T \\ u(t,x) = g(t,x) & (t,x) \in [0,T]\times\partial\Omega \end{cases}$$

If $u_1(T,x) = u_2(T,x)$ for all $x \in \Omega$, then $u_1(t,x) = u_2(t,x)$ for all $(t,x) \in \overline{\Omega}_T$.

<u>Proof</u>: We consider again the functional $e(t)$ defined above.

$$\frac{d}{dt} e(t) = -\int_\Omega |Dv(t,x)|^2 dx$$

$$\frac{d^2}{dt^2} e(t) = -2\int_\Omega Dv(t,x)\cdot Dv_t(t,x)\, dx$$
$$= 2\int_\Omega \Delta v(t,x)\, v_t(t,x)\, dx = 2\int_\Omega |\Delta v(t,x)|^2 dx$$

where we used that $v_t - \Delta v = 0$ and $v(t,x) = 0$ if $x \in \partial\Omega$.

Now: $\left(\frac{d}{dt}e(t)\right)^2 = \left(\int_\Omega v(t,x)\,\Delta v(t,x)\,dx\right)^2$

$$\leq \int_\Omega |v(t,x)|^2 dx \int_\Omega (\Delta v(t,x))^2 dx$$
$$= e(t)\left(\frac{d^2}{dt^2}e(t)\right) \qquad (*)$$

We have: $e \in C^0([0,T])$ with $e \geq 0$ and $e(T)=0$.
Assume that $\exists\, [t_1,t_2] \subset [0,T]$ s.t.

$$e(t) > 0 \quad \text{for} \quad t \in [t_1,t_2), \quad e(t_2) = 0.$$

Then the function $f: [t_1,t_2) \to \mathbb{R}$
$t \mapsto \log e(t)$ is well-defined, and since

$$\frac{d^2}{dt^2}f(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{(\dot{e}(t))^2}{(e(t))^2} \geq 0$$

by $(*)$, f is convex on $[t_1,t_2)$. For $t \in [t_1,t_2)$ and $\lambda \in (0,1)$: $\quad f((1-\lambda)t_1 + \lambda t) \leq (1-\lambda)f(t_1) + \lambda f(t)$
namely $\quad e((1-\lambda)t_1 + \lambda t) \leq e(t_1)^{(1-\lambda)} e(t)^\lambda$

since exp is a monotone function. Letting $t \nearrow t_2$:

$$0 \leq e((1-\lambda)t_1 + \lambda t_2) \leq e(t_1)^{(1-\lambda)}\, e(t_2)^\lambda = 0$$

which is a contradiction. Hence $e(t) = 0$ for all $t \in [0,T]$ □

- Remark: The backwards equation is however not well-posed. The smoothing "forwards in time" yields instability "backwards in time": small fluctuations can grow.

4. The wave equation.

- We finally turn to the hyperbolic equation

$$u_{tt}(t,x) - c^2 \Delta u(t,x) = 0 \qquad (t,x) \in (0,\infty) \times \Omega$$

where $c > 0$ is the propagation velocity of the wave $u(t,x)$.
By rescaling $t := s/c$ and defining $v(s,x) := u(t,x)$ for which
$v_s(s,x) = c^{-1} u_t(t,x)$
we have
$$v_{ss}(s,x) - \Delta v(s,x) = c^{-2}\left(u_{tt}(t,x) - c^2 \Delta u(t,x)\right) = 0$$
whenever u is a solution of the wave equation.
We shall henceforth always set $c = 1$.

- Since the equation is of second order in t, the natural initial value conditions are:

$$\begin{cases} u(0,x) = g(x) & x \in \Omega \\ u_t(0,x) = h(x) & x \in \Omega \end{cases}$$

- We first solve the one-dimensional case: $\Omega = \mathbb{R}$.

$$(WE^1): \begin{cases} u_{tt}(t,x) - u_{xx}(t,x) = 0 & (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = g(x) \ ; \ u_t(0,x) = h(x) & x \in \mathbb{R} \end{cases}$$

Introducing $\begin{cases} t' = x + t \\ x' = x - t \end{cases}$

and $w(t',x') := u(t,x)$, we have, assuming $u \in C^2((0,\infty)\times\mathbb{R})$:

$$u_t = w_{t'} - w_{x'} \qquad ; \qquad u_x = w_{t'} + w_{x'}$$

& $u_{tt} = w_{t't'} - 2w_{t'x'} + w_{x'x'}$

& $u_{xx} = w_{t't'} + 2w_{t'x'} + w_{x'x'}$

so that
$$u_{tt} - u_{xx} = -4w_{t'x'} .$$

Hence, u is a classical solution of the wave equation on $\mathbb{R}\times\mathbb{R}$ if and only if w is a solution of $w_{t'x'} = 0$.
The general solution thereof is given by $\xi(t') + \zeta(x')$ for arbitrary functions $\xi, \zeta \in C^2(\mathbb{R})$. Hence:
$$u(t,x) := \xi(x+t) + \zeta(x-t) \qquad (t,x) \in \mathbb{R}\times\mathbb{R} \qquad (*)$$

is a solution of the 1-D wave equation. For the initial value problem (WE¹), ξ, ζ can be determined by elementary algebra as functions of g, h:
$$\begin{cases} g(x) = \xi(x) + \zeta(x) \\ h(x) = \xi'(x) - \zeta'(x) \end{cases} \qquad (I*).$$

- $(*)$ describes two profiles ξ and ζ propagating at velocity 1 (i.e. c) to the left and the right along x. This is closely related to the 1-D transport equation $u_t \pm u_x = 0$ in $\mathbb{R}\times\mathbb{R}$:
We note that $u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u$
Assume that $w \in C^2(\mathbb{R}^2)$ is a solution of
$$w_t(t,x) - w_x(t,x) = 0 \qquad (t,x) \in \mathbb{R}\times\mathbb{R},$$
and assume that $u \in C^2(\mathbb{R}^2)$ solves
$$u_t(t,x) + u_x(t,x) = w(t,x) \qquad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

Then u solves the wave equation.
Each of the transport equations requires one initial condition:
If $g \in C^1(\mathbb{R})$, we set

$$w(0,x) = h(x) + g'(x) \qquad (x \in \mathbb{R})$$
$$u(0,x) = g(x)$$

Then, the unique solution is given by

$$w(t,x) = w(0, x+t)$$

and

$$u(t,x) = u(0, x-t) + \int_0^t w(s, x-(t-s))\, ds \, .$$

Hence:

$$u(t,x) = g(x-t) + \int_0^t w(0, x-(t-s)+s)\, ds$$
$$= g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} w(0,y)\, dy$$

which finally yields d'Alembert's Formula:

$$u(t,x) = \frac{1}{2}\Big(g(x+t) - g(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy \, . \qquad (\diamond)$$

- **Theorem.** Let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and $u : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ be defined by $(\diamond)$. Then:

(i) $u \in C^2((0,\infty)\times\mathbb{R})$

(ii) $u_{tt}(t,x) - u_{xx}(t,x) = 0 \qquad (t,x) \in (0,\infty)\times\mathbb{R}$.

(iii) $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u(t,x) = g(x_0)$; $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u_t(t,x) = h(x_0)$

Moreover, $(\diamond)$ is the unique function satisfying (i, ii, iii).

- Proof: (i, ii, iii) are immediate.
  Uniqueness follows from the fact that $u_1 - u_2$ solves (WE[1]) with $g = h = 0$ and $(*)(I*)$. $\square$
- Remark: For later purposes, we note that for a fixed (t,x), the first term of $(\diamond)$ is the average of g over the sphere of radius t, centered at x, while the second term of $(\diamond)$ is the average of $t \cdot h$ over the ball of radius t, centered at x.
- The inhomogeneous problem can again be solved by a version of Duhamel's principle: We consider

$$(\mathrm{IWE}^1)\quad \begin{cases} (\partial_{tt}^2 + L)\, u(t,x) = f(t,x) & (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = g(x), \quad u_t(0,x) = h(x) & x \in \mathbb{R} \end{cases}$$

and denote by $S(t)$ the family of solution operators for the homogeneous case $f \equiv 0$, with $g \equiv 0$:

$$\begin{cases} (\partial_{tt}^2 + L)\,(S(t)h)(x) = 0 \\ (S(0)h)(x) = 0 \quad ; \quad \lim_{t \searrow 0} (S(t)h)'(x) = h(x) \end{cases}$$

Claim:

$$u(t,x) = (S(t)g)'(x) + (S(t)h)(x) + \int_0^t (S(t-s) f_s)(x)\, ds \qquad (D)$$

is a solution of (IWE[1]).

Remark: Clearly, the claim is not restricted to the one-dim. situation.

We check the claim at the formal level:

* Initial conditions: i) $\lim_{t\searrow 0} \left(S(t)g\right)'(x) = g(x)$

$$\left(S(0)h\right)(x) = 0$$

$$\int_0^t (\dots)\,dx \,\Big|_{t=0} = 0$$

ii) $\partial_t \left(S(t)g\right)'(x) = \partial_{tt}^2 \left(S(t)g\right)(x) = -\left(L\,S(t)g\right)(x)$

and the formal limit $t \searrow 0$ yields $LS(0)g = 0$.

$\lim_{t\searrow 0} \left(S(t)h\right)'(x) = h(x)$ by assumption.

Finally: $\partial_t \int_0^t \left(S(t-s)f_s\right)(x)\,ds$

$$= \left(S(0)f_t\right)(x) + \int_0^t \left(S(t-s)f_s\right)'(x)\,ds \longrightarrow 0 \quad (t\searrow 0).$$

Together: $\lim_{t\searrow 0} u(t,x) = g(x)$

$\lim_{t\searrow 0} u_t(t,x) = h(x)$ indeed

* Equation: i) $\left(\partial_{tt}^2 + L\right)\left(S(t)g\right)'(x) = \partial_t\left(\partial_{tt}^2 - L\right)\left(S(t)g\right)(x) = 0.$

ii) $\left(\partial_{tt}^2 + L\right)\left(S(t)h\right)(x) = 0$ by definition of $S(t)$.

iii) $\partial_{tt}^2 \int_0^t (\dots)\,ds = \partial_t \left(S(0)f_t\right)(x) + \left(S(0)f_t\right)'(x)$

$$+ \int_0^t \left(S(t-s)f_s\right)''(x)\,ds$$

where the last term equals $-\int_0^t \left(L\,S(t-s)f_s\right)(x)\,ds$

Hence, $\left(\partial_{tt}^2 + L\right)\int_0^t \left(S(t-s)f_s\right)(x)\,ds = \left(S(0)f_t\right)'(x) = f(t,x)$

which concludes the derivation of the claim.

In the present case: $(Lu)(x) = -\Delta u(x)$ and by the theorem:

$$(S(t)h)(x) = \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy .$$

This yields the following proposition:

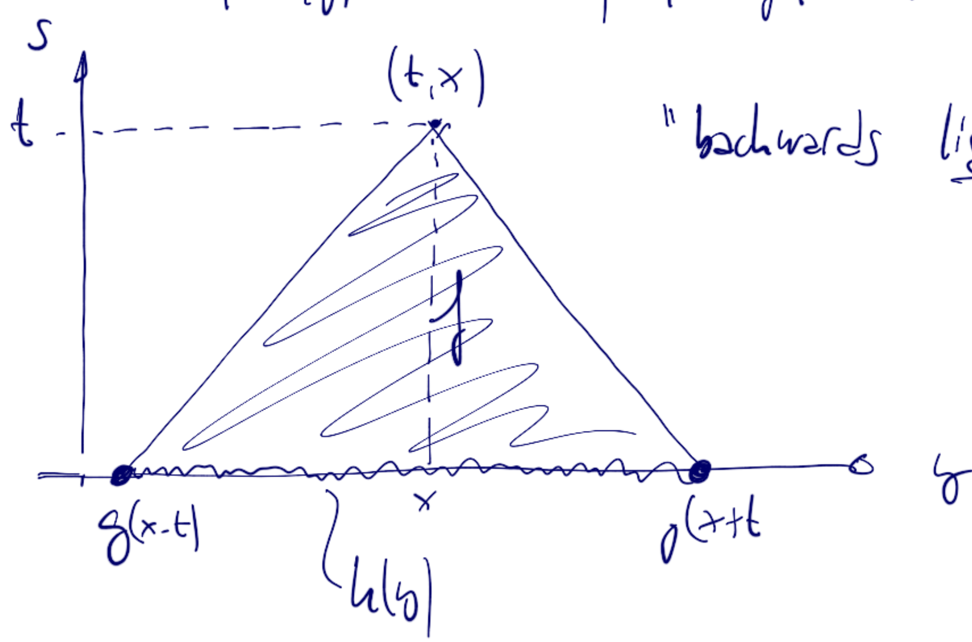
- <u>Proposition</u>: Consider (IWE 1) with $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and $f \in C^2([0,\infty)\times\mathbb{R})$. Then $u : [0,\infty)\times\mathbb{R}$

$$u(t,x) = \frac{1}{2}\left[g(t+x) + g(t-x)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$
$$- \frac{1}{2}\int_0^t \left(\int_{x-s}^{x+s} f(t-s,y)\,dy\right)ds$$

is a solution of (IWE 1).

- <u>Proof</u>: Elementary exercise. $\square$
- <u>Remark</u>: The integral over f extends over the domain

$$C^-_{t,x} := \{(s,y) : 0 \le s \le t,\ x-t+s \le y \le x+t-s\}$$
$$= \{(s,y) : 0 \le s \le t,\ |x-y| \le |t-s|\}$$



"backwards <u>light cone</u>"

- Major differences with the heat equation:
  - * Finite propagation speed
  - * Reversibility: the equation and d'Alembert's formula is invariant under the inversion $t \to -t$
  - * No smoothing
  - * No minimum principle

- <u>Proposition</u>: Let $u \in C^3(\mathbb{R} \times \Omega)$ be a solution of

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & (t,x) \in \mathbb{R} \times \Omega \\ u(0,x) = 0 \ ; \ u_t(0,x) = g(x) & x \in \Omega \end{cases}$$

where $g \in C^2(\Omega)$. Then $v = u_t$ is a solution of the same equation with initial condition

$$v(0,x) = g(x) \ ; \ v_t(0,x) = 0$$

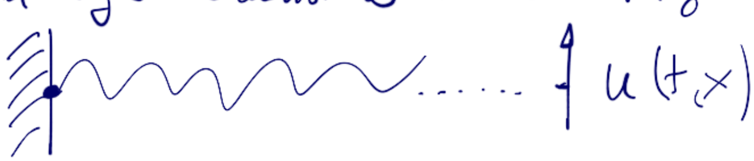
Proof: Since $u \in C^3$, we can exchange the order of differentiation, and

$$v_{tt} - \Delta v = \partial_t (u_{tt} - \Delta u) = 0.$$

Furthermore: * $v(0,x) = u_t(0,x) = g(x)$;

* $v_t(t,x) = u_{tt}(t,x) = \Delta u(t,x)$ for all $(t,x) \in \mathbb{R} \times \Omega$ and since $u(0,\cdot)$ is the constant function 0, $\Delta u(0,x) = 0 \ \forall x \in \Omega$. Hence $v_t(0,x) = 0$. $\square$

- We now turn to a 1D situation which will be useful to study solutions of the wave equation in higher dimensions: the string bound at one of its end:



Here $u(t,x)$ is the height of the string at $(t,x) \in (0,\infty) \times [0,\infty)$

Equation:

$$(*1) \quad \begin{cases} u_{tt}(t,x) - u_{xx}(t,x) = 0 & (t,x) \in (0,\infty) \times (0,\infty) \\ u(0,x) = g(x) ; \ u_t(0,x) = h(x) & x = [0,\infty) \\ u(t,0) = 0 & t \in [0,\infty) \end{cases}$$

(with $g(0) = h(0) = 0$), which is a mixed initial/boundary value problem.

Since $u(t,0) = 0$, any solution can be extended to $x \in \mathbb{R}$ by a reflection: Let $\tilde{u} : (0,\infty) \times \mathbb{R} \to \mathbb{R}$:

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{if } x \geq 0 \\ -u(t,-x) & \text{if } x < 0 \end{cases}$$

and define by a similar reflection $\tilde{g}, \tilde{h}$. Now $\tilde{u}$ solves (WE 1) and it is given by d'Alembert's formula. Hence

$$u(t,x) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy \qquad (x > 0)$$

$$= \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & \text{if } x > t \geq 0 \\ \frac{1}{2}\left[g(x+t) - g(-x+t)\right] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy & \text{if } t > x > 0 \end{cases}$$

where we used in the second case:

$$\int_{x-t}^{x+t} \tilde{h}(y)\,dy = \int_{x-t}^{0} -h(-y)\,dy + \int_0^{x+t} h(y)\,dy \qquad \text{indeed.}$$

We obtain a particularly simple picture of the wave propagation in the case $h = 0$. The initial profile g propagates to the left and to the right; when the left-moving wave reaches $x = 0$, it gets reflected with the opposite sign.

- Remark: Just as in the case of the transport equation, d'Alembert's

<u>Remark</u>: $u(t,x)$ is continuous at $x=t$. Now:

$$2u_x(t,x) = \begin{cases} g'(x+t) + g'(x-t) + h(x+t) + h(x-t) \\ g'(x+t) + g'(t-x) + h(x+t) - h(t-x) \end{cases}$$

which is continuous since $h(0)=0$. Further:

$$2u_{xx}(t,x) = \begin{cases} g''(x+t) + g''(x-t) + h'(x+t) + h'(x-t) \\ g''(x+t) - g''(t-x) + h'(x+t) + h'(t-x) \end{cases}$$

Hence, $u \in C^2((0,\infty)\times(0,\infty))$ only if $g'(0) = 0$.

formula is perfectly well-defined for a much larger class of functions, e.g. g piecewise continuous, and $h\in C^0(\mathbb{R})$ $\leadsto$ "weak solutions".

- We now turn to Kirchhoff's formula in $D=3$:

<u>Theorem</u>: Let $g\in C^3(\mathbb{R}^3)$, $h\in C^2(\mathbb{R}^3)$ and let $u:(0,\infty)\times\mathbb{R}^3\to\mathbb{R}$ be defined by

$$u(t,x)=\fint_{\partial B_t(x)}\big(g(y)+Dg(y)\cdot(y-x)+th(y)\big)dS(y)\qquad (t,x)\in(0,\infty)\times\mathbb{R}^3$$

Then:

i) $u\in C^2((0,\infty)\times\mathbb{R}^3)$

ii) $u_{tt}(t,x)-\Delta u(t,x)=0\qquad (t,x)\in(0,\infty)\times\mathbb{R}^3$

iii) $\displaystyle\lim_{(t,x)\to(0,x_0)}u(t,x)=g(x_0)$; $\displaystyle\lim_{(t,x)\to(0,x_0)}u_t(t,x)=h(x_0)$

- Although this could be proved easily from the formula, it is instructive to <u>derive</u> it by the method of spherical averaging.
- In the sequel, we assume $u\in C^2((0,\infty)\times\mathbb{R}^n)$ is a solution of (WE^n), where $n\geq 2$ first.

For $t>0$, $x\in\mathbb{R}^n$, $r>0$, let

$$U(t,x;r):=\fint_{\partial B_r(x)}u(t,x)\,dS(x)$$

$$G(t,x;r):=\fint_{\partial B_r(x)}g(y)\,dy\;;\quad H(t,x;r):=\fint_{\partial B_r(x)}h(y)\,dy$$

• Lemma: Let $\xi \in C^2(\mathbb{R}^n)$, $n \geq 2$, and let

$$M_\xi(x;r) := \fint_{\partial B_r(x)} \xi(y)\,dS(y) \quad , \quad (x;r) \in \mathbb{R}^n \times (0,\infty).$$

Then. $$\left(\frac{1}{r^{n-1}} \partial_r r^{n-1} \partial_r\right) M_\xi(x;r) = \Delta_x M_\xi(x;r)$$

Proof: Recall the proof of the mean-value property for Laplace's eq.:

$$\partial_r M_\xi(x;r) = \partial_r \fint_{\partial B_1(0)} \xi(x+rz)\,dS(z) = \int_{\partial B_r(x)} D\xi(y)\cdot \frac{y-x}{r}\,dS(y)$$

$$= \frac{r}{n} \fint_{B_r(x)} \Delta\xi(y)\,dy \quad , \text{ so that}$$

$$r^{n-1}\partial_r M_\xi(x;r) = \frac{1}{n\alpha_n} \int_{B_r(x)} \Delta\xi(y)\,dy .$$

Finally, $$\partial_r r^{n-1}\partial_r M_\xi(x;r) = \frac{1}{n\alpha_n} \int_{\partial B_r(x)} \Delta\xi(y)\,dS(y)$$

Since $n\alpha_n = \omega_n$:

$$\frac{1}{r^{n-1}} \partial_r r^{n-1}\partial_r M_\xi(x;r) = \fint_{\partial B_r(x)} \Delta\xi(y)\,dS(y)$$

$$= \int_{\partial B_r(0)} \Delta_x \xi(x+z)\,dS(z) = \Delta_x \int_{\partial B_r(x)} \xi(y)\,dS(y)$$

which is the claim. $\square$

• Rewriting $\frac{1}{r^{n-1}}\partial_r r^{n-1}\partial_r = \partial_r^2 + \frac{n-1}{r}\partial_r$, we obtain the

Euler-Poisson-Darboux equation:

- **Lemma 2:** Let $n \geq 2$ and let $u \in C^2((0,\infty)\times\mathbb{R}^n) \cap C^1_t([0,\infty)\times\mathbb{R}^n)$ be a solution of

$$(WE^n)\quad \begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & (t,x)\in(0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x)\,;\ u_t(0,x) = h(x) & x\in\mathbb{R}^n \end{cases}.$$

Then for any fixed $x\in\mathbb{R}^n$, $U \in C^2((0,\infty)\times[0,\infty)) \cap C^1_t([0,\infty)\times[0,\infty))$ and U is a solution of

$$\begin{cases} U_{tt}(t,x;r) - U_{rr}(t,x;r) - \frac{n-1}{r}U_r = 0 & (t,r)\in(0,\infty)\times(0,\infty) \\ U(0,x;r) = G(x;r)\,;\ U_t(0,x;r) = H(x;r) & r\in(0,\infty) \end{cases}$$

Proof: * The lemma applied to $\xi = u$ and the following remark implies that

$$U_{tt} - U_{rr} - \frac{n-1}{r}U_r = (\partial^2_{tt} - \Delta_x)U .$$

An explicit computation yields then

$$(\partial^2_{tt} - \Delta_x)U(t,x;r) = \fint_{\partial B_r(x)} u_{tt}(t,y)\,dS(y) - \fint_{\partial B_r(x)} \Delta_y u(t,y)\,dS(y) = 0$$

since u solves the wave equation.

* We now look at the limits: for $(t,x)\in(0,\infty)\times\mathbb{R}^n$,

$$\lim_{r\searrow 0} U_r(t,x;r) = \lim_{r\searrow 0} \frac{r}{n}\fint_{B_r(x)} \Delta u(t,y)\,dy = 0$$

$$\lim_{r\searrow 0} U_{rr}(t,x;r) = \lim_{r\searrow 0}\left(\fint_{\partial B_r(x)} \Delta u(t,y)\,dS(y) - \frac{n-1}{r}U_r(t,x;r)\right)$$

$$= \Delta u(t,x) - \frac{n-1}{n}\Delta u(t,x) = \frac{1}{n}\Delta u(t,x)$$

So that $U \in C^2((0,\infty)\times[0,\infty))$.

* Finally, for any $(x;r) \in \mathbb{R}^n \times (0,\infty)$:

$$\lim_{t\downarrow 0} U(t,x;r) = \fint_{\partial B_r(x)} \lim u(t,y)\, dS(y) = G(x;r)$$

$$\lim_{t\downarrow 0} U_t(t,x;r) = \fint_{\partial B_r(x)} \lim u_t(t,y)\, dS(y) = H(x;r)$$

since $u \in C^1_t([0,\infty)\times\mathbb{R}^n)$. $\square$

- Now: in the case $n=3$, this can be used to derive Kirchhoff's formula. Let u be as above and let

$$\tilde{U} := rU \;;\; \tilde{G} := rG \;;\; \tilde{H} := rH$$

Then: for $n=3$, $\tilde{U}$ is a solution of (M), namely

$$\begin{cases} \tilde{U}_{tt}(t,x;r) - \tilde{U}_{rr}(t,x;r) & (t;r) \in (0,\infty)\times(0,\infty) \\ \tilde{U}(0,x;r) = \tilde{G}(x;r);\ \tilde{U}_t(0,x;r) = \tilde{H}(x;r) & r \in (0,\infty) \\ \tilde{U}(t,x;0) = 0 & t \in (0,\infty) \end{cases}$$

Indeed,

* $\tilde{U}_{tt} = rU_{tt} = rU_{rr} + 2U_r$ with the E-P-D equation, $\underline{n=3}$.
  Hence, $\tilde{U}_{tt} = (rU_r + U)_r = \tilde{U}_{rr}$

* $\tilde{U}(0,x;r) = \tilde{G}(x;r)$ and $\tilde{U}_t(0,x;r) = \tilde{H}(x;r)$ immediate

- Since $U \in C^2((0,\infty)\times[0,\infty))$, we have $\lim\limits_{r \searrow 0} U(t,r) < \infty$, so that $\tilde{U}(t,r) = r\,U(t,r) \longrightarrow 0$ $(r \searrow 0)$ for any fixed $t > 0$.
- It remains to check that $\tilde{G}_{rr}(0) = 0$. We have:

$$G_r(x;r) = \frac{r}{3} \fint_{B_r(x)} \Delta g(y)\,dy \longrightarrow 0 \quad \text{as } r \searrow 0.$$

$$G_{rr}(x;r) = -\frac{2}{3\alpha_3 r^3} \int_{B_r(x)} \Delta g(y)\,dy + \frac{1}{3\alpha_3 r^2} \int_{\partial B_r(x)} \Delta g(y)\,dy$$

$$\longrightarrow \text{const} \cdot \Delta g(x) \quad \text{as } r \searrow 0.$$

Finally: $\tilde{G}_r = r\,G_r + G$

$$\tilde{G}_{rr} = 2G_r + r\,G_{rr} \longrightarrow 0 \quad \text{as } r \searrow 0 \quad .$$

Altogether:

$$\tilde{U}(t,x;r) = \frac{1}{2}\left(\tilde{G}(x;r+t) - \tilde{G}(x;t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(x;y)\,dy$$

is a C^2 solution by fixed $x \in \mathbb{R}^3$, for $t \geq r \geq 0$. We now obtain $u(t,x)$ again as a limit: for $(t,x) \in (0,\infty) \times \mathbb{R}^3$

$$u(t,x) = \lim_{r \searrow 0} U(t,x;r) = \lim_{r \searrow 0} \frac{\tilde{U}(t,x,r)}{r} = \tilde{G}_t(x;t) + \tilde{H}(x;t)$$

with $\tilde{H}(x;t) = t \fint_{\partial B_t(x)} h(y)\,dy$

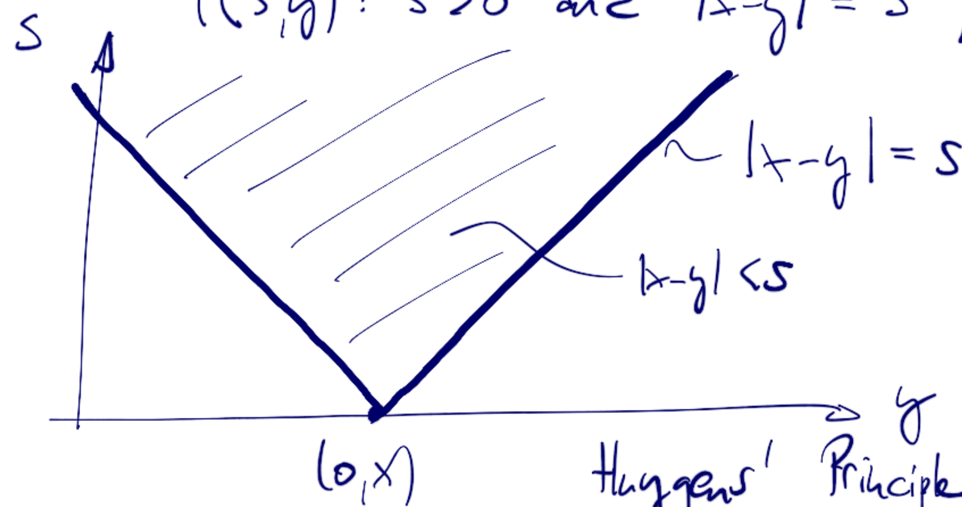
and since $\tilde{G}(x;t) = t \fint_{\partial B_t(x)} g(y)\,dy$,

$$\tilde{G}_t(x;t) = \fint_{\partial B_t(x)} g(y)\,dy + t \fint_{\partial B_t(x)} \nabla g(y) \cdot \frac{y-x}{t}\,dS(y)$$

indeed.

- Kirchhoff's formula exhibits again the finite propagation speed of the flow of the wave equation. In fact even more:
The value of the initial condition $g(x), h(x)$ for a fixed $x \in \mathbb{R}^3$ influence the solution $u(s,y)$ only on the <u>surface</u> of the <u>future lightcone</u>

$$\{(s,y) : s > 0 \text{ and } |x-y| = s\}$$



<u>Huygens'</u> Principle

- The derivation using the E-P-D equation fails if $n \neq 3$. The case $n=2$ can however be solved by considering it as a special case of a 3-D problem.
Let $u \in C^2((0,\infty) \times \mathbb{R}^2)$ be a solution of (WE^2), and let $\bar{u} : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\bar{u}(t, x_1, x_2, x_3) := u(t, x_1, x_2)$$

Extending the initial conditions similarly to $\bar{g}, \bar{h}$, $\bar{u}$ is a solution of

$$\begin{cases} \bar{u}_{tt}(t,x) - \Delta \bar{u}(t,x) = 0 & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ \bar{u}(0,x) = \bar{g}(x) \, ; \, \bar{u}_t(0,x) = \bar{h}(x) & x \in \mathbb{R}^3 \end{cases}$$

Hence $\bar{u}$ can be expressed using Kirchhoff's formula. For this, let $x = (x_1, x_2)$; $\bar{x} = (x_1, x_2, 0)$, so that

$$u(t,x) = \bar{u}(t,\bar{x}) = \fint_{\partial B_t(\bar{x})} \left(\bar{g}(y) + D\bar{g}(y)\cdot(y-\bar{x}) + t\bar{h}(y) \right) dS(y)$$

where the integral is over the sphere in 3-D.

We parametrize the sphere:

for $z \in B_t^{(2)}(x)$:

$$\gamma(z) = \sqrt{t^2 - |z-x|^2}, \quad \bar{\gamma}(z) = (z_1, z_2, \gamma(z))$$

$$\fint_{\partial B_t(\bar{x})} \bar{h}(y)\, dS(y) = \frac{2}{4\pi t^2} \int_{B_t^{(2)}(x)} h(z) |\partial_1 \bar{\gamma} \wedge \partial_2 \bar{\gamma}| \, dz = \frac{1}{2\pi t^2} \int_{B_t^{(2)}(x)} h(z) \left(1 + |D\gamma(z)|^2\right)^{1/2} dy$$

since $\bar{h}$ is independent of $\gamma(z)$.

Now: $$\left(1 + |D\gamma(z)|^2\right)^{1/2} = t\left(t^2 - |z-x|^2\right)^{-1/2}$$

which yields: $$\fint_{\partial B_t(\bar{x})} t\bar{h}(y)\, dS(y) = \frac{1}{2\pi} \int_{B_t^{(2)}(x)} \frac{h(z)}{\sqrt{t^2 - |z-x|^2}}\, dz$$

We thereby obtain <u>Poisson's formula</u> in $\mathbb{R}^2$:

$$u(t,x) = 2 \fint_{B_t(x)} \frac{tg(z) + tDg(z)\cdot(z-x) + t^2 h(z)}{\sqrt{t^2 - |z-x|^2}}\, dz \qquad (t,x) \in (0,\infty) \times \mathbb{R}^2$$

- Remarks: * In 2D, the value of the initial condition at x influences the solution not only on the surface of the future lightcone, but also in its <u>interior</u> $t > |y - x|$, unlike in 3D.
  * The solutions given by Kirchhoff and Poisson are in fact unique as see later in energy methods.
  * General dimensions.

  i) odd dimension: $n = 2k+1$. The method of spherical averaging can be adapted, namely with

$$\tilde{U} = \left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}\bar{U}\right)$$

i.e. u solves $(WE^n) \Rightarrow \bar{U}$ solves E-P-D equation

$\Rightarrow \tilde{U}$ solves the wave equation on $(0,\infty) \times [0,\infty)$

$$\Rightarrow u(t,x) = \frac{1}{\gamma_n}\left[\partial_t\left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2} \fint_{\partial B_t(x)} g(y)\,dS(y)\right) + \left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B_t(x)} h(y)\,dS(y)\right)\right]$$

is the solution of (WE^n), where $\gamma_n = (n-2)(n-4)\ldots 5\cdot 3$.

Moreover: $\left(g \in C^{n+3/2}(\mathbb{R}^n),\ h \in C^{\frac{n+1}{2}}(\mathbb{R}^n)\right) \Rightarrow u \in C^2([0,\infty)\times\mathbb{R}^n)$.

namely: if $n > 1$, the solution is not as smooth as the initial condition

ii) n even. By extending trivially to $n+1$ and using the odd-dimensional result:

$$u(t,x) = \frac{1}{\tilde{\gamma}_n}\left[\partial_t \left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}} \left(t^n ⨍_{B_t(x)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy \right) + \left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}} \left(t^n ⨍_{B_t(x)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy \right)\right]$$

with $\tilde{\gamma}_n = n(n-2)\dots 4\cdot 2$.

$\left(g \in C^{n/2+2}, h \in C^{\frac{n}{2}+1}\right) \Rightarrow u \in C^2([0,\infty)\times\mathbb{R}^n)$.

- In all dimensions, the wave propagation has finite velocity, in odd dimensions is the velocity exactly c or wave front. In even dimensions is it $\leq c$ no wave front and the interior of the light cone.

- Next we consider the inhomogeneous problem:

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = f(t,x) & (t,x) \in (0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x);\ u_t(0,x) = h(x) & x \in \mathbb{R}^n \end{cases} \quad \text{(IWE)}$$

If u^h is the solution for $f \equiv 0$, and u^i is the solution for $g = h \equiv 0$ then $u = u^h + u^i$ solves (IWE). Hence it remains to construct u^i. But Duhamel's principle asserts that

$$u^i(t,x) = \int_0^t \left(S(t-s) f_s \right)(x)\,ds$$

where $S(t)$ is the solution flow in the case $f \equiv 0$, $g \equiv 0$, see page 94. Hence,

• <u>Theorem</u>: Let $n \geq 2$, $f \in C^{[n/2]+1}([0,\infty)\times\mathbb{R}^n)$. Let

$$u(t,x) = \int_0^t (S(t-s)f_s)(x)\,ds \qquad (t,x) \in [0,\infty)\times\mathbb{R}^n.$$

Then: i) $u \in C^2([0,\infty)\times\mathbb{R}^n)$

ii) $u_{tt}(t,x) - \Delta u(t,x) = f(t,x) \quad \forall (t,x) \in (0,\infty)\times\mathbb{R}^n$

iii) $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u(t,x) = 0$; $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u_t(t,x) = 0 \quad \forall x_0 \in \mathbb{R}^n$

<u>Proof</u>: i) if n is odd, then $[\frac{n}{2}]+1 = \frac{n+1}{2}$

if n is even, then $[\frac{n}{2}]+1 = \frac{n}{2}+1$

hence by the remarks on the solution of the homogeneous problem $u \in C^2([0,\infty)\times\mathbb{R}^n)$. (see p. 106, 107)

ii) By direct computation:

$$u_t(t,x) = \underbrace{(S(0)f_t)(x)}_{=0} + \int_0^t \partial_t (S(t-s)f_s)(x)\,ds$$

$$\begin{aligned} u_{tt}(t,x) &= \partial_t (S(t-s)f_s)(x)\Big|_{s=t} + \int_0^t \partial_{tt}^2 (S(t-s)f_s)(x)\,ds \\ &= f(t,x) + \int_0^t \Delta (S(t-s)f_s)(x)\,ds \\ &= f(t,x) + \Delta u(t,x) \qquad \text{indeed} \end{aligned}$$

It also immediately follows that $u(0,x) = u_t(0,x) = 0$ $\square$

• As an example we exhibit the <u>retarded potential</u> in $n=3$.

Kirchhoff's formula implies that

$$u(t,x) = \int_0^t \fint_{\partial B_{t-s}(x)} (t-s) f(s,y)\, dS(y)\, ds$$

namely $$u(t,x) = \frac{1}{4\pi}\int_0^t \int_{\partial B_{t-s}(x)} \frac{f(s,y)}{(t-s)}\, dS(y)\, ds$$

letting $r = t-s$ and noting that in the y integral $r = |y-x|$:

$$u(t,x) = \frac{1}{4\pi}\int_0^t \int_{\partial B_r(x)} \frac{f(t-|y-x|,y)}{|y-x|}\, dS(y)\, dr$$

$$= \frac{1}{4\pi}\int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\, dy .$$

In other words, the effect of the source at (s,y) is given by the Coulomb potential at x at the retarded time $t = s + |y-x|$.
$\leadsto$ finite propagation speed again.

- We finally turn to "energy methods". First we show how energy conservation implies uniqueness of the solution.

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, $T > 0$.

Definition: Let $u \in C^2(\overline{\Omega_T})$ be a solution of the wave equation in the parabolic cylinder Ω_T. One defines:

* the kinetic energy
$$K(t) = \frac{1}{2}\int_\Omega |u_t(t,x)|^2\, dx \geq 0$$

* the potential energy
$$V(t) = \frac{1}{2}\int_\Omega |Du(t,x)|^2\, dx \geq 0$$

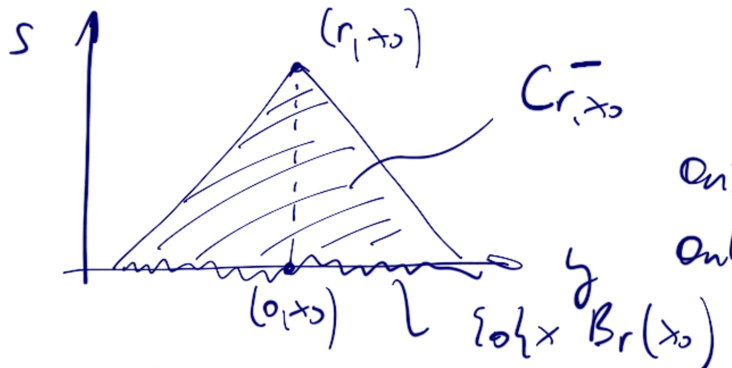
* the total energy of u: $E(t) := K(t) + V(t)$.

- We first derive yet another version of the finite propagation speed:

<u>Lemma</u>: Let $C^-_{t,x} := \{(s,y) \in [0,t] \times \mathbb{R}^n, \ |y-x| \leq t-s\}$

and let $u \in C^2([0,\infty) \times \mathbb{R}^n)$ be a solution of the homogeneous wave equation. Let $x_0 \in \mathbb{R}^n$, $r \geq 0$. If
$u(0,x) = 0$, $u_t(0,x) = 0$ for all $x \in B_r(x_0)$,
then
$$u(s,y) = 0 \quad \text{for all } (s,y) \in C^-_{r,x_0}$$

In a picture:



namely perturbation outside $B_r(x_0)$ can only reach x_0 after a time r.

<u>Proof</u>: Let $e : [0,r] \to \mathbb{R}$
$$t \mapsto e(t) := \frac{1}{2} \int_{B_{r-s}(x_0)} \left(|u_t(t,x)|^2 + |Du(t,x)|^2 \right) dx$$

Then:
$$\frac{d}{dt} e(t) = \int_{B_{r-t}(x_0)} \left(u_t u_{tt} + Du \cdot Du_t \right) - \int_{\partial B_{r-t}(x_0)} \left(|u_t|^2 + |Du|^2 \right) \frac{1}{2}$$

$$= \int_{B_{r-t}(x_0)} u_t (u_{tt} - \Delta u) + \int_{\partial B_{r-t}(x_0)} \left(u_t \frac{\partial u}{\partial \nu} - \frac{1}{2} |u_t|^2 - \frac{1}{2} |Du|^2 \right)$$

by Gauss–Green.

Estimating the first boundary term

$$\left|u_t\frac{\partial u}{\partial \nu}\right| = |u_t|\left|\frac{\partial u}{\partial \nu}\right| \le |u_t|\,|Du| \le \frac{1}{2}\left(|u_t|^2 + |Du|^2\right)$$

and noting that the integral over $B_{r-t}(x_0)$ vanishes by assumption, we get

$$\frac{d}{dt}e(t) \le \left|\int_{\partial B_{r-t}(x_0)} u_t\frac{\partial u}{\partial \nu}\right| - \frac{1}{2}\int_{\partial B_{r-t}(x_0)}\left(|u_t|^2 + |Du|^2\right) \le 0$$

for $0 \le t \le r$. Hence,

$$0 \le e(t) \le e(0) = \frac{1}{2}\int_{B_r(x_0)}\left(|u_t|^2 + |Du|^2\right) = 0$$

where the last equality is again by assumption, so that $e(t) = 0$, and since $u_t(t,\cdot), Du(t,\cdot)$ are continuous:

$$u_t(t,x) = 0 = Du(t,x) \quad \text{for all } (t,x) \in \overline{C_{r,x_0}},$$

and hence $u \equiv 0$ in $\overline{C_{r,x_0}}$, since $u(0,\cdot) = 0$ in $B_r(x_0)$.

$\square$

- With, we prove energy conservation (i) below and energy equipartition (ii) below.

Theorem: Let $u \in C^2([0,\infty) \times \mathbb{R}^n)$ be a solution of

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & (t,x) \in (0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x);\ u_t(0,x) = h(x) & x \in \mathbb{R}^n \end{cases}$$

where $g \in C^1_c(\mathbb{R}^n)$ and $h \in C^0_c(\mathbb{R}^n)$. Let

$$E_0 := \frac{1}{2}\int_\Omega |h(x)|^2\,dx + \frac{1}{2}\int_\Omega |Dg(x)|^2\,dx.$$

(i) $E(t) = E_0$ for all $t \geq 0$.

(ii) If n is odd, then there is $t_0 \geq 0$ such that $K(t) = V(t)$ for all $t \geq t_0$.

• Remarks: * E_0 thereby gives an a priori estimate controlling the solution $u(t,x)$ for all $t \geq 0$.

* Uniqueness also holds for the mixed problem on Ω_T, $\Omega \subset\subset \mathbb{R}^n$.

* Equipartition (ii) fails in even dimensions.

• Proof (only (i)): Let K be a compact set such that

$$\operatorname{supp} g \subset K, \quad \operatorname{supp} h \subset K.$$

For any $t \geq 0$, let $K_t := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq t\}$.

Then K_t is a bounded set, $K_t \subseteq K_{t'}$ if $t \leq t'$, and the lemma implies that

$$\operatorname{supp} u(t,\cdot) \subseteq K_t \quad \text{for all } t \geq 0.$$

Let $T \geq 0$ and $0 \leq t < T$. Then

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|u_t|^2 + |Du|^2\right) = \frac{1}{2}\int_{K_T} \left(|u_t|^2 + |Du|^2\right) < \infty.$$

and further:

$$\frac{d}{dt}E(t) = \int_{K_T} \left(u_t u_{tt} + Du \cdot Du_t\right)$$

$$= \int_{K_T} u_t(u_{tt} - \Delta u) + \int_{\partial K_T} u \frac{\partial u}{\partial \nu}\, dS = 0$$

since u solves the wave equation and by $u(t,\cdot)\restriction_{\partial K_T} = 0$, whenever $t < T$. Hence $E(t) = E(0) = E_0$ for all $0 \leq t < T$, from which (i) follows since $T > 0$ is arbitrary.

(ii) The case $n=1$ left as an exercise using d'Alembert's formula. For higher dimensions, standard proof uses Fourier analysis. $\square$

- Corollary: There is at most one solution $u \in C^2([0,\infty) \times \mathbb{R}^n)$ of the initial value problem for the (possibly inhomogeneous) wave equation.

Proof: Let u_1, u_2 be two solutions. Then $w := u_2 - u_1$ solves the homogeneous wave equation with $g = h = 0$. Hence $E_w(t) = E_w(0) = 0$ for all $t \geq 0$, and it follows again that $w_t = Dw = 0$ in $[0,\infty) \times \mathbb{R}^n$, hence $w \equiv 0$ $\square$.

5. On the method of characteristics

- We now turn to the method of characteristics, which is not restricted to linear equations and is particularly useful to obtain local existence results as well as to understand local singularities of solutions such as shocks.
- Basic strategy from an analytic point of view: transform the PDE into a system of ordinary differential equations
- Basic geometric intuition

A first order PDE in $\mathbb{R}^n$ is of the form

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = 0 \qquad (*)$$

The set $S = \{(x_1, \dots, x_n, u(x_1, \dots, x_n)) : (x_1, \dots, x_n) \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ is a surface in $\mathbb{R}^{n+1}$. Provided $(*)$ can be inverted to yield $u_{x_1}, \dots, u_{x_n}$, the surface S is described by its normal vector. Indeed: let

$$s \mapsto (x(s), u(x(s))) \qquad , x(s) \in \mathbb{R}^n$$

be a C^1 curve in S. Then

$$s \mapsto (v(s), Du(x(s)) \cdot v(s))$$

is the tangent vector along the curve, with $v(s) = \dot{x}(s)$. But

$$(Du, -1) \cdot (v, Du \cdot v) = 0$$

so that $(Du(x), -1)$, $x \in \mathbb{R}^n$ is the normal vector field to the surface S.

- This works for quasilinear equations. In general, the scheme must be extended to consider $u_{x_1}, \dots, u_{x_n}$ as further independent variables and a surface in $\mathbb{R}^{2n+1}$.
- We first illustrate the method with two simple examples in $\mathbb{R}^2$.

i) Linear, first order equation:

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c_0(x,y)u(x,y) + c_1(x,y)$$

The boundary condition is given along a curve

$$\gamma(s) = (x_0(s), y_0(s)) \qquad s \in (\alpha, \beta)$$

namely $u(\gamma(s)) = u_0(\gamma(s))$.

<u>Step 1</u>: Parametric representation of the initial condition in $\mathbb{R}^3$.

$$\Gamma(s) = (x_0(s), y_0(s), u_0(s))$$

<u>Step 2</u>: The equation reads

$$(a, b, c_0 u + c_1) \cdot (u_x, u_y, -1) = 0$$

namely $(a, b, c_0 u + c_1)$ is a tangential vector to the solution surface S.

Hence, if $x(t), y(t), u(t)$ solve

$$(**) \qquad \begin{cases} \dot{x}(t) = a(x(t), y(t)) \\ \dot{y}(t) = b(x(t), y(t)) \\ \dot{u}(t) = c_0(x(t), y(t))u(t) + c_1(x(t), y(t)) \end{cases}$$

with an initial condition

$$(x(0), y(0), u(0)) \in \Gamma,$$

then $t \mapsto (x(t), y(t), u(t))$ is a curve lying in S.

Step 3: Repeat for any point on Γ, namely solve the characteristic equations (**) for any initial condition on Γ, yielding a family
$\{(x(t,s), y(t,s), u(t,s)) ; s \in (\alpha, \beta)\}$
of solutions such that
$x(0,s) = x_0(s), \quad y(0,s) = y_0(s), \quad u(0,s) = u_0(s)$

Conclusion: the parametric surface
$(t,s) \mapsto (x(t,s), y(t,s), u(t,s))$
is the solution surface S.
If the transformation $(t,s) \mapsto (x,y)$ can be inverted, then the solution u is explicitly given by $u(t(x,y), s(x,y))$

(ii) Quasilinear, first order PDE:

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u), \qquad (\diamond)$$

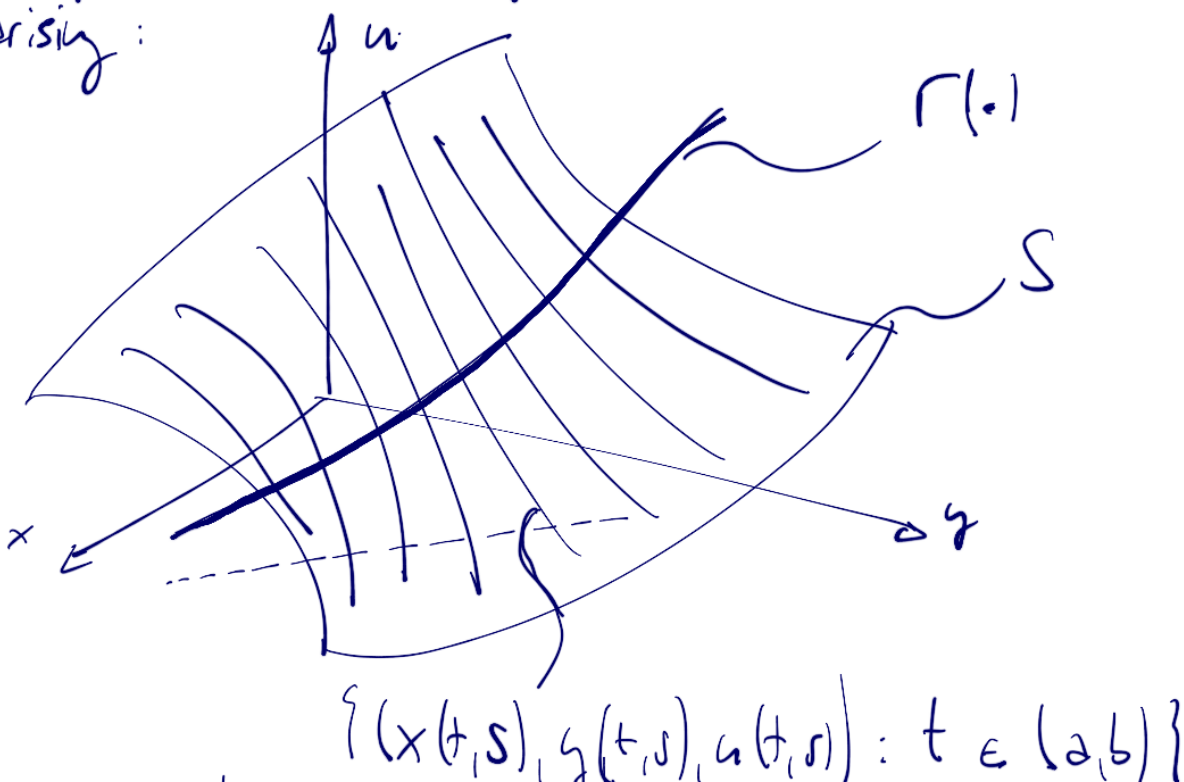
namely $(a,b,c)\cdot(u_x, u_y, -1) = 0$.

The characteristic equations now read

$$\begin{cases} \dot{x}(t) = a(x(t), y(t), u(t)) \\ \dot{y}(t) = b(x(t), y(t), u(t)) \\ \dot{u}(t) = c(x(t), y(t), u(t)) \end{cases}$$

Note: Here, the two first ones cannot be solved independently of the third one any more.

- Summarising:



And indeed: solving the PDE reduces to solving a system of ODEs (**)

- Example: $\begin{cases} u_x + u_y = 2 \\ u(x,0) = x^2 \end{cases}$

① : $\Gamma(s) = (s, 0, s^2)$

② : Characteristic equations : $\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \\ \dot{u} = 2 \end{cases}$

③ : Solutions : $\begin{cases} x(t,s) = t + s \\ y(t,s) = t \\ u(t,s) = 2t + s^2 \end{cases}$

④ : Inverting $(t,s) \mapsto (x,y)$:

$$t(x,y) = y$$
$$s(x,y) = x - y$$

$\Rightarrow$ Solution: $\quad u(x,y) = 2y + (x-y)^2$

- Remarks: * Under mild conditions on the functions a, b, c, the characteristic equations (**) have a unique solution in an interval around $t=0$ for any s, by the theorem of Picard-Lindelöf, Ex 4 Sheet 2.
  $\rightsquigarrow$ local existence of a solution around the initial curve $\gamma(s)$.

  * $(t,s) \mapsto (x,y)$ is invertible if its Jacobian does not vanish, namely $J(t,s) = x_t y_s - x_s y_t \neq 0$.
  Using the characteristic equations, this reads

$$a(x(t,s), y(t,s))\, y_s(t,s) - b(x(t,s), y(t,s))\, x_s(t,s) \neq 0.$$

or $\quad (a,b)$ is not parallel to (x_s, y_s)

This is ensured in a neighbourhood of the initial curve γ if γ is never tangential to the projection of the characteristic curve. One says that γ is <u>non-characteristic</u> or satisfies the <u>transversality condition</u> at s_0 if

$$\det \begin{pmatrix} a & b \\ x_0' & y_0' \end{pmatrix}(s_0) \neq 0 \qquad (T)$$

where $a(s) = a(x_0(s), y_0(s), u_0(s))$.

* Obstruction to global existence : If $\Gamma(\cdot)$ intersects some characteristics in more than 1 point : The initial value problem for the ODEs (**) is overdetermined.
* Also: in general ODEs have neither global existence nor global uniqueness.
* Finally, if the solution of (**) is unique, then it is a C^1 function of s under general conditions (differentiability of a, b, c)

- Altogether. For the quasilinear equation (Q) with initial curve $\{\Gamma(s), s\in(\alpha,\beta)\}$

Proposition. Assume that there exists a neighbourhood $\Omega \supset \Gamma$, $\Omega \subset \mathbb{R}^3$ s.t. $a, b, c \in C^1(\Omega; \mathbb{R})$. Let $s_0 \in (\alpha,\beta)$, $\delta > 0$: $(s_0 - 2\delta, s_0 + 2\delta) \subset (\alpha, \beta)$. If (T) holds for all $s \in (s_0 - 2\delta, s_0 + 2\delta)$, there is $\varepsilon > 0$ s.t. (**) has a unique solution

$$x, y, u \in C^1((-\varepsilon, \varepsilon) \times (s_0 - \delta, s_0 + \delta))$$

We will prove a more general theorem later. Here, we check that the proposition yields indeed a solution of the PDE.

Let $\quad \tilde{u}(x,y) = u(t(x,y), s(x,y))$

where the functions t, s are C^1 by (T) and the inverse function theorem. Then:

$$a\tilde{u}_x + b\tilde{u}_y = (at_x + bt_y)u_t + (as_x + bs_y)u_s$$

But

$$1 = t_t = t_x x_t + t_y y_t = at_x + bt_y$$

by the characteristic equation, and

$$0 = s_t = s_x x_t + s_y y_t = a s_x + b s_y .$$

Hence,

$$a\tilde{u}_x + b\tilde{u}_y = u_t = c$$

again by (**),

and $\tilde{u}$ is indeed a solution of the PDE.

- Remark: If (T) fails in an interval, then the equation either has no solution or a infinite number of solutions:
  Consider $u_x + u_y = 1$, with characteristic eq:
  $$\dot{x} = 1, \quad \dot{y} = 1, \quad \dot{u} = 1$$
  i) Γ given by $u(x,x) = 1$, namely
  $$x(s) = s, \quad y(s) = s, \quad u(s) = 1.$$
  here: $J(s) = \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$

  but Γ is not a characteristic (since $\dot{u} = 0$)

  $\Rightarrow$ no solution

  ii) Γ given by $u(x,x) = x$, namely
  $$x(s) = s, \quad y(s) = s, \quad u(s) = s$$
  and Γ is a characteristic

  $\Rightarrow$ infinitely many solutions,
  $$u(x,y) = y + f(y-x)$$
  for any $f \in C^1 : f(0) = 0.$

- We now turn to the fully non-linear term, where $u_{x_1}, \dots, u_{x_n}$ must be considered as independent variable. Hence, the PDE

$$F(x, u, Du) = 0 \qquad (\Delta)$$

can be seen to define a surface of codimension 1 in $\mathbb{R}^{2n+1}$.

- We derive the characteristic equation analytically. Let $u \in C^2$ be a solution of (Δ) and let

$$z(t) = u(x(t))$$
$$p(t) = Du(x(t)) \quad \text{with components } p^i(t) = u_{x^i}(x(t))$$

where $x(t)$ is a map to be constructed, so that (x, z, p) solve a closed system of ODEs. We compute:

  * $$\dot{p}^i = \sum_{j=1}^{n} u_{x^i x^j} \dot{x}^j$$
  * Taking ∂_{x^i} of (Δ):
$$\sum_{j=1}^{n} \left(F_{p^j} u_{x^i x^j} + F_z u_{x^i} + F_{x^i} \right) = 0$$
  * We now <u>choose</u> $\dot{x}(t)$ such that
$$\dot{x}^j =: F_{p^j}$$

so that the equations above imply

$$\dot{p}^i = \sum_{j=1}^{n} u_{x^i x^j} F_{p^j} = - \left(F_z u_{x^i} + F_{x^i} \right)$$

while

$$\dot{z}(t) = \sum_{j=1}^{n} u_{x_j} \dot{x}^j = \sum_{j=1}^{n} u_{x_j} F_{p_j}$$

and we have a closed system of $(2N+1)$ ODEs

$$\begin{cases} \dot{x}(t) = D_p F\big(x(t), z(t), p(t)\big) & \text{(a)} \\ \dot{z}(t) = p(t) \cdot D_p F\big(x(t), z(t), p(t)\big) & (**)\ \text{(b)} \\ \dot{p}(t) = -D_x F\big(x(t), z(t), p(t)\big) & \\ \qquad\qquad - p(t) F_z\big(x(t), z(t), p(t)\big) & \text{(c)} \end{cases}$$

to be solved for $x(t)$, $z(t)$, $p(t)$.

- $(**)$ are the <u>characteristic equations</u>, the curve
$$t \mapsto (x(t), z(t), p(t)) \in \mathbb{R}^{2n+1}$$
is a <u>characteristic</u>, while $t \mapsto x(t) \in \mathbb{R}^n$ is the <u>reduced characteristic</u>.
- We have just proved:

<u>Proposition</u>: Let $u \in C^2(\Omega)$ be a solution of (0) for $x \in \Omega$. If x solves $(**)(a)$ where $p(t) = Du(x(t))$ and $z(t) = u(x(t))$, then p solves $(**)(c)$ and z solves $(**)(b)$ for all t such that $x(t) \in \Omega$.

<u>Note</u>: We shall be more interested in using $(**)$ to construct a local solution along an initial curve

- Example: $n=2$ $\begin{cases} u_{x^1} u_{x^2} = u \\ u(0,x^2) = (x^2)^2 \end{cases}$

Here: $f(x^1, x^2, z, p^1, p^2) = p^1 p^2 - z$

and (**) read: $\begin{cases} \dot{x}^1 = p^2 \quad ; \quad \dot{x}^2 = p^1 \\ \dot{z} = 2p^1 p^2 \\ \dot{p}^1 = p^1 \quad ; \quad \dot{p}^2 = p^2 \end{cases}$

Initial conditions? $x^1(0,s) = 0$, $x^2(0,s) = s$

$z(0,s) = s^2$

We further need the compatibility condition:

$p^2(0,s) = u_{x^2}(x^1(0,s), x^2(0,s))$

$= z_s(0,s)\, s_{x^2} = 2s$, and

$p^1(0,s)\, p^2(0,s) = z(0,s)$ by the PDE itself, namely

$p^1(0,s) = s/2$.

Solution: * $p^1(t,s) = (s/2)e^t$, $p^2(t,s) = 2s\, e^t$

* $z(t,s) = z(0,s) + \int_0^t 2s^2 e^{2t'} dt'$

$= s^2 + s^2(e^{2t} - 1) = s^2 e^{2t}$

* $x^1(t,s) = 2s(e^t - 1)$

$x^2(t,s) = s + \frac{s}{2}(e^t - 1) = \frac{s}{2}(e^t + 1)$

We finally invert $(t,s) \mapsto (x^1, x^2)$

$$s = x^2 - \frac{1}{4}x^1$$

$$e^t = \frac{4x^2 + x^1}{4x^2 - x^1}$$

to obtain the solution:

$$u(x^1, x^2) = z\big(t(x^1,x^2), s(x^1,x^2)\big) = \frac{(4x^2 + x^1)^2}{16}$$

- In $\mathbb{R}^n$, the boundary condition is given along a surface which is given locally by a C^1-function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}^n$,

$$\Gamma := \left\{ \left(\gamma^1(s), \dots, \gamma^n(s)\right) : s \in B_r(0) \subset \mathbb{R}^{n-1} \right\}$$

The boundary condition for the PDE is given by

$$u(\gamma(s)) = g(s) \qquad \text{(BC)}$$

This translates into initial conditions for the characteristic equations (**):

$$x(0,s) = \gamma(s)$$
$$z(0,s) = g(s) \quad ;$$

by taking derivatives of (BC):

$$g_{s_i}(s) = Du(\gamma(s)) \cdot \gamma_{s_i}(s)$$

namely

$$p(s) \cdot \gamma_{s_i}(s) = g_{s_i}(s) \quad , \text{ since}$$

n equations for $p^1, \dots, p^n$ when complemented with the PDE itself:

$$F(\gamma(s), g(s), p(s)) = 0$$

<u>Note</u>: given a point $\sigma = \gamma(0) \in \Sigma$, then the value of $\zeta = z(0)$ is fixed by the (BC). However, a vector $\pi = p(0)$ solving the two eq at $s=0$ may not exist or may not be unique.

A triplet (σ, ζ, π) is <u>admissible</u> if

$$\begin{cases} \pi \cdot \gamma_{s_i}(0) = g_{s_i}(0) & (C1) \\ F(\sigma, \zeta, \pi) = 0 & (C2) \end{cases}$$

Now: in a neighbourhood of $s=0$, (C1) determines $n-1$ components of $p(s)$ with $p(0) = \pi$. It remains to solve (C2) for $p^n(s)$. Hence, by the implicit function theorem:

<u>Lemma</u>: If (σ, ζ, π) is admissible and if

$$D_p F(\sigma, \zeta, \pi) \cdot \nu(\sigma) \neq 0, \quad \left(\begin{matrix} \nu(\sigma): \text{normal vector} \\ \text{to } \Sigma \text{ at } \sigma \end{matrix}\right)$$

then there is $\pi : B_\rho(0) \to \mathbb{R}^n$ s.t. $\pi(0) = \pi$ and

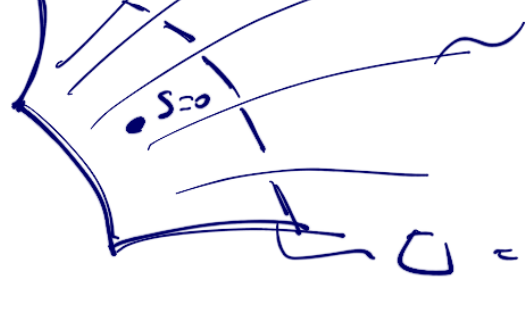
$$\begin{cases} \pi(s) \cdot \gamma_{s_i}(s) = g_{s_i}(s) \\ F(\gamma(s), g(s), \pi(s)) = 0 \end{cases}$$

for all $s \in B_\rho(0)$.

- An admissible triple (σ, ζ, π) is <u>noncharacteristic</u>
$$D_p F(\sigma, \zeta, \pi) \cdot \nu(\sigma) \neq 0.$$

Note: for a quasilinear equation in $\mathbb{R}^2$
$\gamma : \mathbb{R} \to \mathbb{R}^2$, $s \mapsto \gamma(s) = (x_0(s), y_0(s))$ is a curve
it normal vector is $(y_0'(s), -x_0'(s))$
the equation being $a u_x + b u_y = c$, the non-characteristic condition reads
$$(a, b) \cdot (y_0', -x_0') = \det \begin{pmatrix} a & b \\ x_0' & y_0' \end{pmatrix} \neq 0$$
which we called transversality condition above.

- We now solve the characteristic equations (**) with initial conditions
$$\begin{cases} x(0,s) = \gamma(s) \\ z(0,s) = \zeta(s) \\ p(0,s) = \pi(s) \end{cases}$$
where $\pi(s)$ is given by the lemma, to get
$$\{x(t,s), z(t,s), p(t,s)\}$$
in a neighbourhood of $(t=0, s=0) \subset \mathbb{R} \times \mathbb{R}^{n-1}$



$x(t,s)$: the projected characteristics.

$\Gamma = \{t = 0\} \subset \mathbb{R}^n$

Again, it remains to invert the map $(t,s) \mapsto x$.
This can be done in a neighbourhood of $(t=0, s=0)$ if $(\operatorname{Det} Dx)(0,0) \neq 0$ by the inverse function theorem. But

$$\det Dx = \det \left(\dot{x} \,\middle|\, D_s x \right) \overset{(**)}{=} \det \left(D_p F \,\middle|\, D_s x \right)$$

$$= D_p F(\sigma, g, \varnothing) \cdot \nu(\sigma)$$

which is $\neq 0$ under the noncharacteristic condition. Define

$$\begin{cases} u(x) = z(t(x), s(x)) \\ \tilde{p}(x) = p(t(x), s(x)) \end{cases}$$

- **Theorem**: There is a neighbourhood $V \subset \mathbb{R}^n$ of σ s.t. the function $u \in C^2(V; \mathbb{R})$ and solves

$$F(x, u(x), Du(x)) = 0 \qquad x \in V$$

with $u(x) = g(x)$ for all $x \in V \cap \Sigma$.

- **Proof**: Let $x(t,s)$, $z(t,s)$, $p(t,s)$ be the solution of $(**)$, and let

$$f(t,s) := F(x(t,s), z(t,s), p(t,s)) .$$

Then.

* $f(0,s) = F(\sigma(s), y(s), \varpi(s)) = 0$ by the lemma (compatibility conditions)
* $\frac{\partial}{\partial t} f(t,s) = D_x F \cdot \dot{x} + F_z \dot{z} + D_p F \cdot \dot{p}$ $\quad \left(\dot{} = \frac{\partial}{\partial t} \right)$.

$$= D_p F \left(D_x F - D_x F - p F_z \right) + F_z p \cdot D_p F$$
$$= 0$$

$\Rightarrow f(t,s) = 0$

Since $(t,s) \mapsto x$ is locally invertible and by the definition of $u(x)$, $\tilde{p}(x)$, this reads

$$F(x, u(x), \tilde{p}(x)) = 0$$

and it remains to prove that $\tilde{p}(x) = Du(x)$. For this, we note

$$\dot{z} = p \cdot D_p F = p \cdot \dot{x} \qquad (i)$$

by (**), and further claim that:

$$r^i := z_{s^i} - p \cdot x_{s^i} = 0 \qquad (ii)$$

Note that (ii) holds at $t=0$ by the compatibility condition. Moreover:

$$\dot{r}^i = \left(p_{s^i} \cdot \dot{x} + \cancel{p \cdot \dot{x}_{s^i}} \right) - \dot{p}\, x_{,i} - \cancel{p \cdot \dot{x}_{s^i}} = p_{s^i} \cdot \dot{x} - \dot{p} \cdot x_{s^i}$$

By the characteristic equations:

$$\dot{r}^i = p_{s^i} \cdot D_p F + D_x F \cdot x_{s^i} + p F_z \cdot x_{s^i}$$

But $D_p F \cdot p_{s^i} + D_x F \cdot x_{s^i} + F_z z_{s^i} = 0$ by the PDE,
so that

$$\dot{r}^i = F_z \left(p \cdot x_{s^i} - z_{s^i} \right) = -F_z r^i$$

Hence: r solves a linear ODE with initial condition $r = 0$,
this implies (ii).

Finally: $u_{x^i} = \dot{z} t_{x^i} + D_s z \cdot s_{x^i}$

$$= p \cdot \dot{x}\, t_{x^i} + p \cdot D_s x \cdot s_{x^i} \qquad (i)(ii)$$
$$= p \cdot x_{x^i} = p_{x^i}$$

which is what we had set to prove $\boxed{☺}$

- We now discuss a number of examples.

(1) Linear: $F(x, z, p) = b(x) \cdot p + c(x) z$

Noncharacteristic assumption at σ:

$$b(\sigma) \cdot \nu(\sigma) \neq 0$$

is indep. of z, p.

(*)(a) yields an independent equation for the reduced characteristics:

$$\dot{x}(t,s) = b(x(t,s)).$$

and $x(t,s)$ are integral curves of the vector field b.

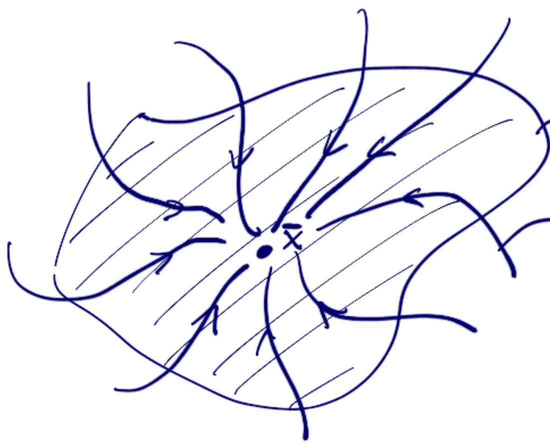
furthermore: $\dot{z} = b(x) \cdot p = - c(x) \cdot z$

by the PDE.

in particular: if $c(x) = 0$ for all $x \in \Omega$, z is constant along the reduced characteristics:

$$u(x(t,s)) = u(x(0,s))$$

i)



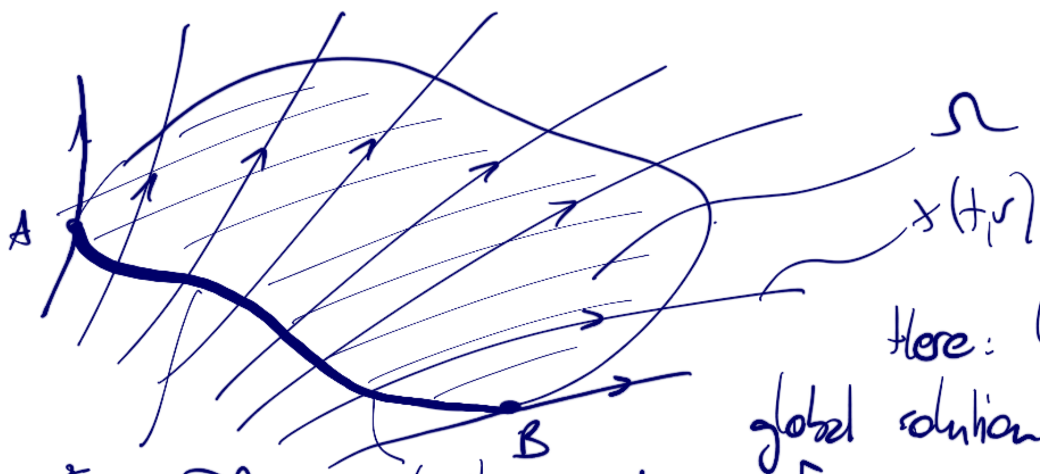
$\Sigma = \partial\Omega$

$x(t,s)$: $\dot{x} = b(x)$ and $b(\bar{x}) = 0$

$\rightsquigarrow$ the ODE can be solved locally near Σ.

Let $u\restriction_\Sigma = g$. Then u <u>cannot</u> be extended to all $x \in \Omega$ unless g is a constant function.

ii)



$\Sigma \subset \partial\Omega$

Here: Unique, global solution in Ω for $u\restriction_\Sigma = g$ by setting

$$u(x(t,s)) = g(x(0,s))$$

(2) Quasilinear: the case of conservation laws for $u(\tau, x)$:

$$G(\tau, x, u(\tau,x), u_\tau(\tau,x), Du(\tau,x))$$
$$= u_\tau(\tau,x) + \operatorname{div} F(u(\tau,x))$$

$$= u_\tau(\tau,x) + F'(u(\tau,x)) \cdot Du(\tau,x)$$

where $F \in C^1(\mathbb{R}; \mathbb{R}^n)$, and $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$.

Boundary condition: $u(0,x) = g(x)$ for all $x \in \mathbb{R}^n$.

Letting $y = (\tau, x)$; $q = (p^0, p)$

$$G(y, z, q) = p^0 + F'(z) \cdot p = 0$$

and $D_y G = 0$; $G_z = F''(z) \cdot p$; $D_q G = (1, F'(z))$

* Noncharacteristic assumption:

$$(1, F'(z)) \cdot (1, 0) = 1 \neq 0$$

$\Rightarrow$ local existence.

* Characteristic equations: $\begin{cases} \dot{\tau} = 1 \\ \dot{x} = F'(z) \end{cases}$ (a)

namely $\tau = t$. Further:

$$\dot{z} = p \cdot (1, F'(z)) = 0 \quad \text{by the PDE}$$

and the solution is again transported along the characteristic: $u(x(t,s)) = u(x(0,s)) = g(x(0,s))$

It follows that

$$x(t,s) = F'(g(x(0,s)))\, t + x(0,s)$$

The reduced characteristic $y(t,s) = (t, F'(g(x(0,s)))\, t + x(0,s))$

are straight lines along which the solution u is constant.
Identifying $x(0,s)$ with an arbitrary point $x \in \mathbb{R}^n$, and denoting $x(t) = x(t,s)$, we obtain

$$u(t, x(t)) = z(t)$$
$$= g(x) = g\big(x(t) - F'(g(x))t\big)$$

namely: $\quad u = g\big(x - tF'(u)\big)$
which is an implicit equation for the solution $u(t,x)$.

③ Nonlinear: Hamilton-Jacobi equation

$$u_\tau(\tau, x) + H\big(x, Du(\tau, x)\big) = 0$$

namely (with the notations above):

$$G(y, z, q) = p^0 + H(x, p).$$

Now: $D_y G = \big(0, D_x H(x,p)\big)$; $D_q G = \big(1, D_p H(x,p)\big)$
and $G_z = 0$.
Again, $(1, D_p H) \cdot (1, 0) \neq 0$ so that the initial value problem is noncharacteristic, $\Rightarrow$ local existence.
As for the characteristic equations:

$$\dot{\tau} = 1$$
$$\dot{x} = D_p H(x, p) \qquad \text{(H1)}$$

so that again $\tau = t$. Furthermore:

$$\dot{p}^0 = 0$$

$$\dot{p} = -D_x H(x,p) \qquad (H2)$$

Finally:
$$\dot{z} = p^0 + p \cdot D_p H(x,p)$$
$$= -H(x,p) + p \cdot D_p H(x,p)$$

(H1) & (H2) are called <u>Hamilton's equations</u>. Once $x(t), p(t)$ are obtained, $z(t)$ is given by a simple integration.

- We conclude the chapter with a discussion of <u>shocks</u> for conservation laws in $\mathbb{R}$:

$$\begin{cases} u_t(t,x) + \frac{\partial}{\partial x} F(u(t,x)) = 0 & (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = g(x) & x \in \mathbb{R} \end{cases}$$

- Example: Burger's equation:

$F(u) = \frac{1}{2} u^2$, namely

$$u_t(t,x) + u(t,x) u_x(t,x) = 0$$

The characteristic equations read

$$\begin{cases} \dot{t} = 1, \quad \dot{x} = z \\ \dot{z} = p^0 + F'(z)\, p = 0 \quad \text{by the PDE} \\ \dot{p}^0 = -p^0 p \,; \quad \dot{p} = p^2 \end{cases}$$

Again: $z(t,s) = z(0,s) =: z_0(s)$.

and further: $x(t,s) = z_0(s)\,t + x_0(s)$

Since $x(0,s) = x_0(s)$, we identify $x_0(s)$ with s to get:

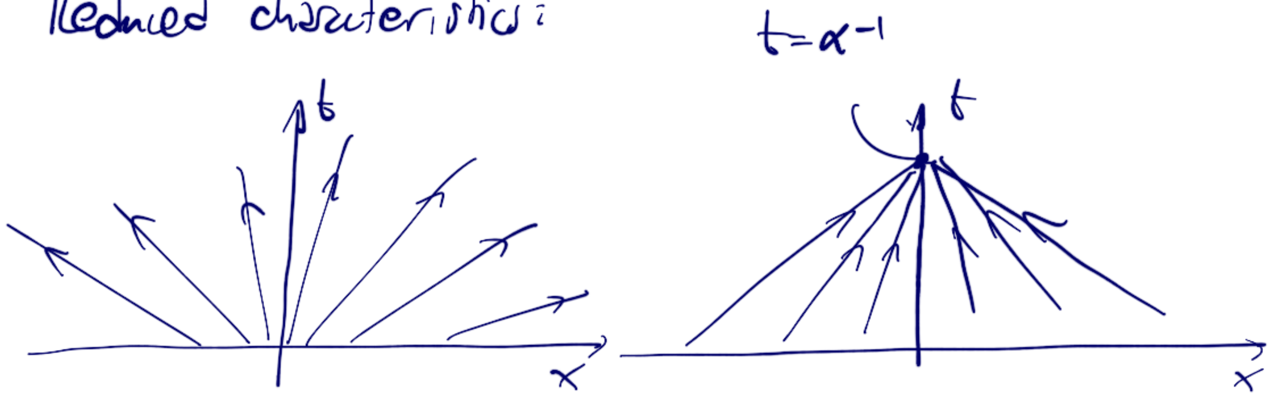
$$\begin{cases} z(t,s) = g(s) \\ x(t,s) = g(s)\,t + s \end{cases}$$

i) $g(s) = \alpha s \implies x = \alpha s t + s \to s = x/(1+\alpha t)$

$\implies$ Solution:

$$u(t,x) = g\left(\frac{x}{1+\alpha t}\right) = \frac{\alpha x}{1+\alpha t}$$

Reduced characteristics:



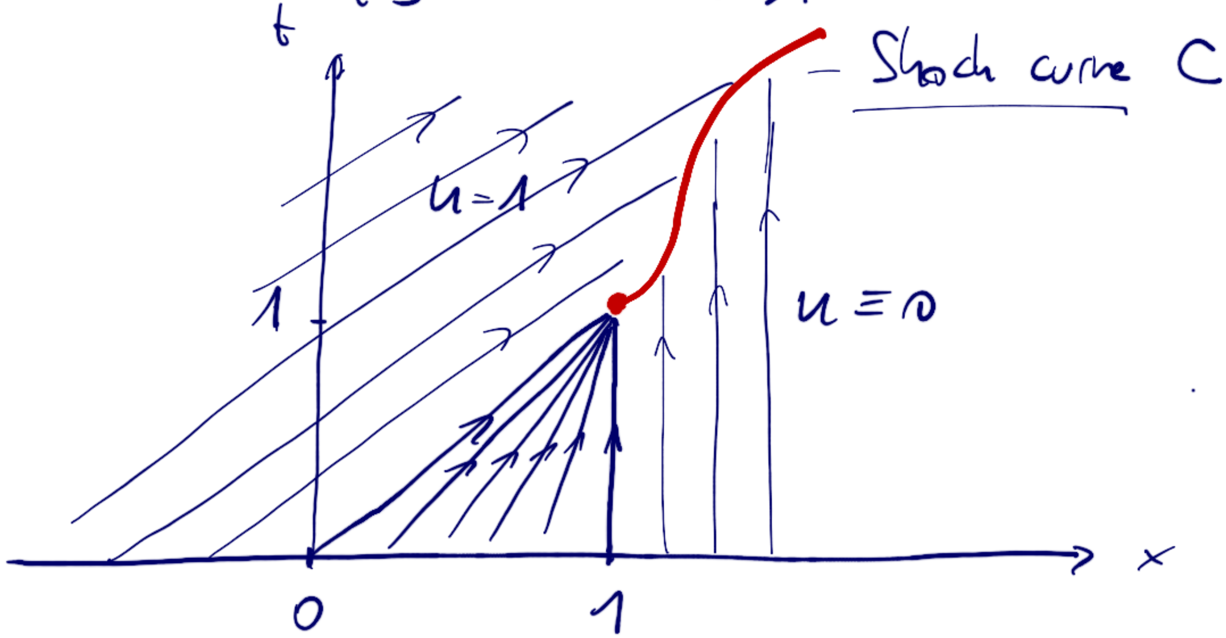
$\alpha \geq 0$ — "Rarefaction wave"

$\alpha < 0$ — "Shock" and $u(\alpha^{-1}, x)$ ceases to exist.

ii) $$g(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ 1 - s & \text{if } 0 < s \leq 1 \\ 0 & \text{if } s > 1 \end{cases}$$

The reduced characteristics exhibit a shock again:

$$x(t,s) = \begin{cases} s+t & s \le 0 \\ s+(1-s)t & 0 < s \le 1 \\ s & s > 1 \end{cases}$$



since u is constant along the reduced characteristics.

- Question: Clearly, $u(t,x)$ sketched above is discontinuous along C, so it is in particular not a classical solution of the PDE in $(0,\infty)\times\mathbb{R}$.
  In which sense is it a solution?
- Let $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$ be a <u>test function</u>. If u were a classical solution of

$$u_t(t,x) + \frac{\partial}{\partial x} F(u(t,x)) = 0 ,$$

then

$$0 = \int_0^\infty \int_{-\infty}^{+\infty} \left(u_t + \partial_x F(u)\right) \varphi \, dt\, dx$$

$$= -\int_0^\infty \int_{-\infty}^{+\infty} u \varphi_t \, dt\, dx - \int_{-\infty}^{+\infty} u \varphi \, dx \Big|_{t=0}$$

$$- \int_0^\infty \int_{-\infty}^{+\infty} F(u) \varphi_x \, dt\, dx$$

where we used the fact that φ is compactly supported.

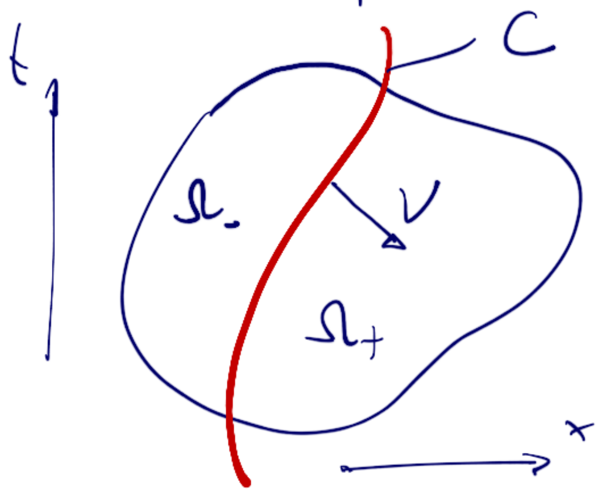
Now: this calculation only makes sense if $u \in C^1$, but the resulting equality

$$\int_{[0,\infty)\times\mathbb{R}} \left(u\varphi_t + F(u)\varphi_x\right) dt\, dx + \int_{\mathbb{R}} g(x)\varphi(0,x)\, dx = 0 \qquad (I)$$

makes sense even if u is only bounded.

A function $u \in L^\infty((0,\infty)\times\mathbb{R})$ satisfying (I) for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$ is called an <u>integral solution</u>

- We consider a simple case: $u : \Omega \to \mathbb{R}$ is so that $u|_{\Omega_\pm} \in C^1(\overline{\Omega_\pm})$ and C is a C^1 curve in $\mathbb{R}^2$



Let C be parametrised by

$$C = \{ (t,x) \in \mathbb{R}^2 : x = f(t) \text{ and } f \in C^1((0,\infty)) \}.$$

and denote $C_\Omega := C \cap \Omega$.

<u>Proposition</u>: If u is an integral solution of

$$u_t(t,x) + \partial_x F(u(t,x)) = 0 \quad , \quad (t,x) \in \Omega$$

then

$$F(u_+(t,x)) - F(u_-(t,x)) = \dot{f}(t)\big(u_+(t,x) - u_-(t,x)\big) \qquad (RH)$$

for all $(t,x) \in C_\Omega$, where $u_\pm$ denote the left and right limits of u at (t,x).

Equation (RH) is called the <u>Rankine–Hugoniot condition</u> along the shock curve C.

We note that by choosing $\varphi \in C_c^\infty(\Omega_+)$, the condition that u is an integral solution implies that

$$0 = \int_0^\infty \int_{-\infty}^{+\infty} u \varphi_t \, dt\,dx + F(u)\varphi \, dt\,dx$$

$$= \int_0^\infty \int_{-\infty}^{+\infty} \big(u_t + \partial_x F(u)\big) \varphi \, dt\,dx$$

and since φ is arbitrary : $u_t(t,x) + \partial_x F(u(t,x)) = 0$ for all $(t,x) \in \Omega_+$. Hence u is a classical solution of the PDE in $\Omega_\pm$.

- Proof: Let $\varphi \in C_c^\infty(\Omega)$ with $\operatorname{supp}\varphi \cap \Omega_\pm \neq \emptyset$. Since u is an integral solution,

$$0 = \int_0^\infty \int_{\mathbb{R}} \left(u\varphi_t + F(u)\varphi_x\right) dt\,dx = \left(\int_{\Omega_+} + \int_{\Omega_-}\right)(\dots) \qquad (*)$$

Since $u \in C^1(\overline{\Omega_+})$,

$$\int_{\Omega_+} \left(u\varphi_t + F(u)\varphi_x\right) dt\,dx = \int_{\Omega_+} (u, F(u)) \cdot D\varphi \, dt\,dx$$
$$= - \int_{\Omega_+} \left(u_t + \partial_x F(u)\right)\varphi \, dt\,dx + \int_C \varphi\,(u_+, F(u_+)) \cdot \nu_+ \, dl$$
$$= \int_C (u_+, F(u_+)) \cdot \nu_+ \varphi \, dt\,dx$$

since u is a classical solution in Ω_+.

Repeating the argument for Ω_- and noting that $\nu_- = -\nu_+$, $(*)$ yields

$$\int_C \left(u_+ - u_-, F(u_+) - F(u_-)\right) \cdot \nu_+ \varphi \, dl = 0 .$$

Hence $\left(u_+ - u_-, F(u_+) - F(u_-)\right)$ is tangential to C at every point $(t,x) \in C_\Omega$:

$$\left(u_+ - u_-, F(u_+) - F(u_-)\right) \propto (1, j)$$

namely: $$\frac{F(u_+) - F(u_-)}{u_+ - u_-} = j$$ $\square$

- Back to the example on page 135. If u is to be an integral solution, we look for the shock curve C that satisfies the (RH) condition.

Setting $u_+(t,x) = 1$, $u_-(t,x) = 0$, and recalling that $F(u) = \frac{1}{2}u^2$, we have

$$\frac{F(u_+) - F(u_-)}{u_+ - u_-} = \frac{1}{2}$$

It remains to note that $(1,1) \in C$ to conclude that

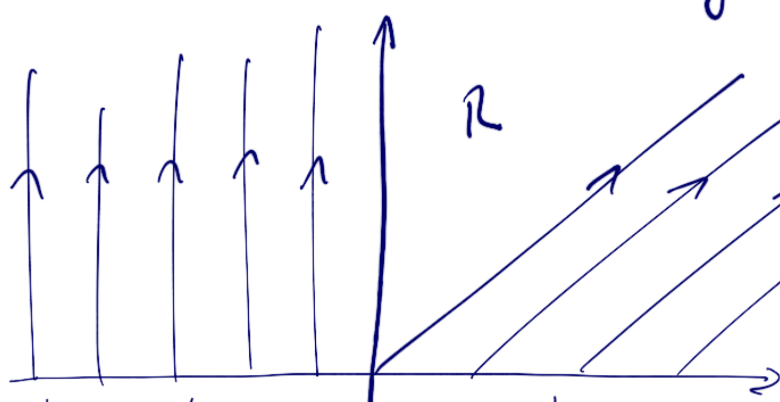
$$\gamma(t) = \frac{1}{2}(1+t)$$

- As a further example, we consider again Burger's equation

$$\begin{cases} u_t(t,x) + u(t,x)u_x(t,x) = 0 & (t,x) \in (0,\infty)\times\mathbb{R} \\ u(0,x) = g(x) & x \in \mathbb{R} \end{cases}$$

where: $g(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$

and projected characteristics $x(t,s) = g(s)t + s$, namely



The characteristics do not cross, but they are not determined by the initial condition & the region

$$\Omega = \{(t,x) \in (0,\infty)\times(0,\infty) \, : \, t > x\}$$

→ No uniqueness.

Recalling that the solution u must be transported along the reduced characteristics we propose two choices:

$$u_1(t,x) = \begin{cases} 0 & x \leq t/2 \\ 1 & x > t/2 \end{cases}$$

$$u_2(t,x) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t} & 0 < x \leq t \\ 1 & x > t \end{cases}$$

Now: both are integral solutions and indeed the (RH) condition holds for u_1

In other words: the (RH) condition is not sufficient to determine a unique integral solution.

One possible additional condition: <u>entropy condition</u>

- For a general conservation law

$$u_t(t,x) + \partial_x F(u(t,x)) = 0 \,,$$

namely

$$u_t(t,x) + F'(u(t,x))\, u_x(t,x) = 0$$

the characteristic equations read:

$$\dot{t}(\tau, s) = 1 \quad , \quad \dot{x}(\tau, s) = F'(z(\tau, s))$$
$$\dot{z}(\tau, s) = p^0 + pF'(z) = 0 \quad \text{by the PDE.}$$

and the "speed" of the characteristic is given by

$$\frac{dx}{dt}(s) = F'(u(x(s)))$$

For Burger's equation: $\frac{dx}{dt} = u$ so that "taller waves move faster than shorter waves" creating the shock.

⇝ We shall say that a curve C of discontinuity of an integral solution satisfies the <u>entropy condition</u> if

$$F'(u_-) > \dot{j} > F'(u_+)$$

The solution u_1 has

$$F'(u_{1,-}) = 0 \quad , \quad \dot{j} = \frac{1}{2} \quad , \quad F'(u_{1,+}) = 1$$

and so it violates the entropy condition.

Note: if $F \in C^2$ and uniformly convex, $F'' \geq \kappa > 0$, then F' is a strictly increasing function and the entropy condition is equivalent to

$$u_- > u_+$$

along any shock curve (such as on p. 135).

6. Weak solutions and Sobolev spaces: an overview

- Introductory example: a boundary value problem on the interval $[0,1]$:

$$\begin{cases} -u''(x)+u(x) = f(x) & , x \in (0,1). \\ u(0) = u(1) = 0 \end{cases} \qquad (P)$$

with $f \in C^0([0,1])$.

Recall: a classical solution of (P) is a function $u \in C^2([0,1];\mathbb{R})$, which solves the equation pointwise everywhere.

Now: let $\varphi \in C_c^\infty((0,1))$. We multiply the equation with φ and integrate by parts to obtain

$$\int_0^1 u'(x)\varphi'(x)\,dx + \int_0^1 u(x)\varphi(x)\,dx = \int_0^1 f(x)\varphi(x)\,dx$$

and note that this equation makes perfect sense if $u \in C^1((0,1))$ only.

In fact if $u \in L^1((0,1))$, and if the concept of a derivative can be extended and $u' \in L^1((0,1))$, then the equation still makes sense $\rightsquigarrow$ "weak solution"

- The general programme: i) Define the concept of weak solution $\rightsquigarrow$ introduce Sobolev spaces

  ii) Prove existence and uniqueness of weak solutions $\rightsquigarrow$ theorem of Lax-Milgram

iii) Study the actual regularity of weak solutions
$\rightsquigarrow$ Sobolev embedding theorems

iv) Prove that a weak solution that is sufficiently differentiable is a classical solution.

- Remark: (iv) is elementary in the case of (P):
  If $u \in C^2([0,1])$ is a weak solution with $u(0) = u(1) = 0$, then
  $$\int_0^1 (-u'' + u - f)\varphi = 0$$
  for any $\varphi \in C_c^\infty((0,1))$ by integration by parts, and the fundamental lemma of the calculus of variations implies that
  $$-u'' + u - f = 0$$
  holds almost everywhere. Since $u \in C^0([0,1])$, the equation holds pointwise everywhere. $\square$

① Weak derivatives $\quad \Omega \subset \mathbb{R}^n$ open

- Def: A function $u \in L^1_{loc}(\Omega)$ is weakly differentiable in x^i, if there is a function $g_i \in L^1_{loc}(\Omega)$ s.t.:
  $$\int_\Omega u \frac{\partial \varphi}{\partial x^i} = -\int_\Omega g_i \varphi \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

g_i is called the weak derivative of u in the direction i and often denoted $\frac{\partial u}{\partial x^i}$.

- Rem: - here the "integration by parts" formula is taken as a formula *defining* the weak derivative.
  - the existence of a weak derivative is *not* equivalent to the existence of the usual derivative almost everywhere.
  - in general: if α is a multiindex, the weak derivative is defined by

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g^\alpha \varphi$$

for all $\varphi \in C_c^\infty(\Omega)$, with $g^\alpha \in L^1_{loc}(\Omega)$.

- Examples: 1) $u(x) = |x|$ on $\Omega = (-1,1)$. We have:

$$\int_{-1}^1 |x| \varphi'(x)\, dx = \int_{-1}^0 (-x)\varphi'(x)\, dx + \int_0^1 x\varphi'(x)\, dx$$

$$= \int_{-1}^0 \varphi(x)\, dx - \int_0^1 \varphi(x)\, dx$$

by integration by parts. Hence, the weak derivative of u is given by

$$g(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$$

as could have been expected

2) $u(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$ is <u>not</u> weakly differentiable.

Indeed, if $\varphi \in C_c^\infty((-1,1))$, then

$$\int_{-1}^{1} u(x)\varphi'(x)\,dx = -\int_{-1}^{0}\varphi'(x)\,dx + \int_0^1 \varphi'(x)\,dx = -2\varphi(0)$$

so that if there is a weak derivative g of u, it must satisfy
$$\int g\varphi = 2\varphi(0)$$

In particular, $\int g\varphi = 0 \quad \forall \varphi \in C_c^\infty((-1,0)\cup(0,1))$ implying that $g = 0$ almost everywhere, and so
$$\int g\varphi = 0 \quad \forall \varphi \in C_c^\infty((-1,1)),$$ contradiction.

$\leadsto$ the condition $g \in L^1_{loc}(\Omega)$ is important. The u above is differentiable in the <u>sense of distributions</u>, and its derivative is the distribution
$$\delta(\varphi) = 2\varphi(0)$$

(remember the heat equation)

- <u>Properties</u>:
  * Uniqueness: A weak α^{th}-partial derivative, if it exists, is unique up to a set of measure zero.
  * Approximation: The following are equivalent:

a) $u \in L^1_{loc}(\Omega)$ has a weak α^{th} derivative in Ω

b) $\exists \, (u_n)_{n\in\mathbb{N}} : u_n \in C^\infty(\Omega)$ s.t.

$u_n \to u$ and $D^\alpha u_n \to g$

in $L^1_{loc}(\Omega)$.

In this case, $g = D^\alpha u \in L^1_{loc}(\Omega)$.

<u>Note</u>: Proof by mollifying.

- Commutation: if $D^\alpha u$, $D^\beta u$ exist, and if one of $D^{\alpha+\beta} u$, $D^\alpha D^\beta u$, $D^\beta D^\alpha u$ exists, then all three exist and are equal.

- Product & chain rules: if $u \in L^1_{loc}(\Omega)$ is weakly diff:

a) If $\Psi \in C^1(\Omega)$, then Ψu is weakly diff. and
$$\partial_i(\Psi u) = (\partial_i \Psi) u + \Psi(\partial_i u)$$

b) If $\zeta \in C^1(\mathbb{R})$ with $\zeta' \in L^\infty(\mathbb{R})$, then $v = \zeta \circ u$ is weakly diff. and
$$\partial_i v = \zeta'(u)\, \partial_i u$$

- <u>Sobolev spaces</u>

<u>Definition</u>: Let $\Omega \subset \mathbb{R}^n$ be open and $\Omega \neq \emptyset$, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define
$$W^{k,p}(\Omega) := \{ u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega), \; 0 \leq |\alpha| \leq k \}$$

That is: $u \in W^{k,p}(\Omega)$ iff u and all its (weak) derivatives of are in $L^1(\Omega)$ (and by definition also $L^1_{loc}(\Omega)$).

- Notation: Since $L^2(\Omega)$ is a Hilbert space:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

- Norm on $W^{k,p}(\Omega)$:

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p} \qquad (1 \leq p < \infty)$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u| \qquad (p = \infty)$$

$\leadsto$ <u>Theorem</u>: For any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

- Example: $\Omega = B_1(0) \in \mathbb{R}^n$ and $u(x) = |x|^{-\alpha}$, $\alpha > 0$.

  $\triangleright$ If $x \neq 0$,
  $$u_{x^i}(x) = \frac{-\alpha x^i}{|x|^{\alpha+2}}$$
  so that $$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \quad \text{for all } x \in B_1(0) \setminus \{0\}.$$

  $\triangleright$ Let $\varphi \in C_c^\infty(B_1(0))$ and $\varepsilon > 0$. Then

$$\int_{\Omega \setminus B_\varepsilon(0)} u \varphi_{x^i} = -\int_{\Omega \setminus B_\varepsilon(0)} u_{x^i} \varphi + \int_{\partial B_\varepsilon(0)} u \varphi \nu^i \, dS$$

ν^i inward normal

Now, if $\alpha+1<n$ then $|Du(x)| \in L^1(\Omega)$.
Under the same condition,

$$\left| \int_{\partial B_\varepsilon(0)} u \varphi \nu^i \, dS \right| \leq \|\varphi\|_\infty \int_{\partial B_\varepsilon(0)} \varepsilon^{-\alpha} dS = C \varepsilon^{-\alpha+n-1} \longrightarrow 0$$

as $\varepsilon \searrow 0$. Hence, if $0 \leq \alpha < n-1$, then

$$\int_\Omega u \varphi_{x^i} = - \int_\Omega u_{x^i} \varphi$$

for all $\varphi \in C_c^\infty(\Omega)$, and $u_{x^i}(x) = -\alpha x^i / |x|^{\alpha+2}$ is the weak derivative of u in Ω. Moreover, $u \in L^p(\Omega)$ iff

$$\alpha p < n,$$

and $Du \in L^p(\Omega)$ iff

$$(\alpha+1)p < n.$$

Consequently:

$$|\cdot|^{-\alpha} \in W^{1,p}(B_1(0)) \Leftrightarrow \alpha < \frac{n-p}{p}.$$

For $n=3$, the Newton potential $|\cdot|^{-1}$ belongs to $W^{1,1}(B_1(0))$

- <u>Remark</u>: The following spaces are useful in the study of the Dirichlet problem:

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{k,p}}$$

namely: $u \in W_0^{k,p}(\Omega) \Leftrightarrow \exists (u_n)_{n\in\mathbb{N}} : u_n \in C_c^\infty(\Omega)$ and $\|u_n - u\|_{k,p} \longrightarrow 0 \quad (n \to \infty)$.

- <u>Proposition</u>: $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$

  ... but the equality does <u>not</u> hold for general $\Omega \subset \mathbb{R}^n$.

- <u>Remarks</u>: * If $u, v \in H^1(\Omega)$, then

$$\langle u, v \rangle := \langle u, v \rangle_{L^2} + \sum_{j=1}^{n} \langle \partial_j u, \partial_j v \rangle_{L^2}$$

$$= \int_\Omega \left(u(x)v(x) + \sum_{j=1}^{n} u_{x_j}(x) v_{x_j}(x) \right) dx$$

defines a scalar product on $H^1(\Omega)$.

  * $H^1(\mathbb{R}^n)$ has an <u>equivalent</u> characterisation in terms of the Fourier transform:

$$H^1(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) ,\ (1 + |\cdot|)\, \hat{u} \in L^2(\mathbb{R}^n) \right\}.$$

- For the study of (elliptic) regularity, the following theorem (Sobolev inequalities) is crucial:

<u>Theorem</u>: $\Omega \subset \mathbb{R}^n$ bounded open with C^1-boundary and let $u \in W^{k,p}(\Omega)$.

(i) If $k > n/p$, then

$$u \in C^{k - \lfloor n/p \rfloor - 1, \gamma}(\overline{\Omega})$$

where

$$\gamma = \begin{cases} \text{any positive number} < 1 & \text{if } \frac{n}{p} \in \mathbb{N} \\ \lfloor n/p \rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N}. \end{cases}$$

and $\exists\, C \geq 0$ s.t.

$$\|u\|_{C^{k-[n/p]-1,\gamma}(\bar\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

(ii) If $k < n/p$, then $u \in L^q(\Omega)$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

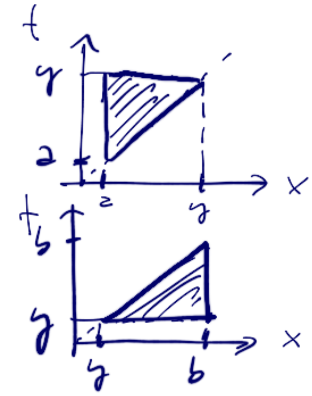
- The space $C^{m,\gamma}(\bar\Omega)$ is the space of all functions $u \in C^m(\bar\Omega)$ and such that $D^\alpha u \in C^{0,\gamma}(\bar\Omega)$ for all $|\alpha| = m$. The γ^{th} Hölder norm is given by

$$\|u\|_{C^{0,\gamma}(\bar\Omega)} := \sup_{x\in\Omega}|u(x)| + \sup_{\substack{x,y\in\Omega\\x\neq y}}\left\{\frac{|u(x)-u(y)|}{|x-y|^\gamma}\right\}.$$

- Remarks: * Hence, if u is in a $W^{k,p}(\Omega)$ for a sufficiently large k, it is in fact $k-[n/p]-1$ times continuously (strongly) differentiable.
  In particular, if $u \in W^{k,p}(\Omega)$ for all $k \in \mathbb{N}$, then $u \in C^\infty(\Omega)$.
  * One could also say that $W^{k,p}(\Omega)$ is embedded in $C^{m,\gamma}(\bar\Omega)$ if $kp > n$, or in $L^q(\Omega)$ otherwise.
  $\rightsquigarrow$ "Sobolev embeddings".
  * Special case: $n = 1$, $k = 1$. Then for any $1 \leq p \leq \infty$,

$$u \in W^{1,1}((a,b)) \Rightarrow u \in C^0((a,b)), \text{ and}$$
$$u(x) - u(y) = \int_x^y u'(t)\,dt \qquad x, y \in (a,b).$$

In this case, the theorem is easy to prove:
Let $y \in (a,b)$ and let
$$\tilde{u}(x) := \int_y^x u'(t)\,dt$$
Since $u' \in L^1_{loc}((a,b))$ by definition, $\tilde{u} \in C^0((a,b))$.
Furthermore,



$$\int_a^b \tilde{u}\varphi' = -\int_a^y \left(\int_x^y u'(t)\,dt\right)\varphi'(x)\,dx + \int_y^b \left(\int_y^x u'(t)\,dt\right)\varphi'(x)\,dx$$

$$\overset{\text{Fubini}}{=} -\int_a^y u'(t)\left(\int_a^t \varphi'(x)\,dx\right)dt + \int_y^b u'(t)\left(\int_t^b \varphi'(x)\,dx\right)dt$$

$$= -\int_a^b u'(t)\varphi(t)\,dt \qquad \text{since } \varphi(a) = \varphi(b) = 0$$

Hence: $\displaystyle\int_a^b (\tilde{u} - u)\varphi' = 0$

so that $u = \tilde{u} + \text{constant}$ is continuous indeed. $\square$.

- We now use this in connection with the boundary value problem
$$\begin{cases} (Lu)(x) = f(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$
with : * Ω : open and bounded
* L is a second order elliptic differential operator:
$$(Lu)(x) = -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j}(x) + \sum_{j=1}^{n} b^j(x) u_{x^j}(x) + c(x) u(x)$$

and $\exists\, \delta > 0$ s.t. $\langle \xi, A(x)\xi \rangle \geq \delta |\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, and $(A(x))_{ij} = a^{ij}(x)$

Finally, we assume that
$$a^{ij}, b^j, c \in L^\infty(\Omega) \quad (1 \leq i,j \leq n).$$
$$f \in L^2(\Omega).$$

- Central tool: Lax-Milgram's theorem

Theorem: Let $\mathcal{H}$ be a real Hilbert space, and let
$$B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$
be a bilinear mapping such that

i) $\exists\, \alpha > 0$:
$$|B(u,v)| \leq \alpha \|u\| \|v\|$$

ii) $\exists\, \beta > 0$: $B(u,u) \geq \beta \|u\|^2$ (coercivity)

If $F : \mathcal{H} \to \mathbb{R}$ is a bounded linear functional, namely $|F(u)| \leq C\|u\| \quad \forall u \in \mathcal{H}$, then there exists a unique $v \in \mathcal{H}$ s.t.

$$B(v,u) = F(u) \qquad \text{for all } u \in \mathcal{H}$$

- Note: If B were symmetric, then this would follow from Riesz representation theorem. Lax-Milgram's theorem does not require symmetry.
- In order to apply L-M-theorem successfully to the PDE problem, we shall finally need Poincaré's inequality

Let Ω be a bounded open set and let $u \in W_0^{1,p}(\Omega)$ for $1 \leq p < n$. Then

$$\|u\|_{L^q(\Omega)} \leq C \|Du\|_{L^p(\Omega)}$$

for each $q \in \left[1, \frac{np}{n-p}\right]$

Sobolev conjugate p^*: $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Note: that $u \in W_0^{1,p}(\Omega)$ and not $W^{1,p}(\Omega)$ is clearly essential: in $W^{1,p}(\Omega)$ one could add a constant function to increase $\|u\|$ without changing $\|Du\|$.

- Let $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$B(u,v) = \int_\Omega \Big(\sum_{i,j=1}^n a^{ij} u_i v_j + \sum_{j=1}^n b^i u_i v + c u v \Big) dx$$

<u>Proposition</u>: $\exists\, \alpha, \beta, \gamma > 0$:

i) $|B(u,v)| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$

ii) $\beta \|u\|^2_{H_0^1(\Omega)} \leq B(u,u) + \gamma \|u\|^2_{L^2(\Omega)}$

for all $u, v \in H_0^1(\Omega)$

<u>Proof</u>:

i) $$|B(u,v)| \leq \sum_{i,j} \|a^{ij}\|_{L^\infty(\Omega)} \int_\Omega |Du||Dv| + \sum_j \|b^j\|_{L^\infty(\Omega)} \int_\Omega |Du||v| + \|c\|_{L^\infty(\Omega)} \int_\Omega |u||v|$$

$$\leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

where we used the Cauchy-Schwarz inequality.

ii) Since L is uniformly elliptic:

$$\delta \int_\Omega |Du|^2 \leq \int_\Omega \sum_{i,j} a^{ij} u_{x_i} u_{x_j}$$

which implies that

$$\delta \|Du\|_{L^2}^2 \le B(u,u) + \sum_j \|b^j\|_{L^\infty} \int_\Omega |Du||u| + \|c\|_{L^\infty} \|u\|_{L^2}^2$$

Now: $\displaystyle\int_\Omega |Du||u| \le \varepsilon \|Du\|_{L^2}^2 + \frac{1}{4\varepsilon}\|u\|_{L^2}^2$

for any $\varepsilon > 0$. Choosing ε so that $\varepsilon \sum_j \|b^j\|_{L^\infty} < \frac{\delta}{2}$:

$$\frac{\delta}{2}\|Du\|_{L^2}^2 \le B(u,u) + C\|u\|_{L^2}^2$$

By Poincaré's inequality, $\|u\|_{L^2(\Omega)} \le \kappa \|Du\|_{L^2(\Omega)}$,

$$\frac{\delta}{2}\|Du\|_{L^2} = \left(\frac{\delta}{2} - \nu\right)\|Du\|_{L^2} + \nu\|Du\|_{L^2} \ge \tilde{\kappa}\, \|u\|_{H_0^1(\Omega)}$$

with the choice $\nu = \left(1 + \frac{1}{\kappa}\right)\frac{\delta}{2}$, which proves (ii) $\square$

- <u>Theorem</u>: There exists $\gamma \ge 0$ such that for any $\mu \ge \gamma$ and any $f \in L^2(\Omega)$, the problem

$$(WP) \quad \begin{cases} (Lu)(x) + \mu u(x) = f(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

<u>Proof</u>: Let $\gamma \ge 0$ be the constant provided by the proposition, and let, for any $\mu \ge \gamma$:

$$B_\mu(u,v) := B(u,v) + \mu\langle u, v\rangle_{L^2(\Omega)}$$

Then B_μ satisfie the assumptions of Lax-Milgram's theorem.

Now: if $f \in L^2(\Omega)$, then

$$\left|\langle f, u\rangle_{L^2(\Omega)}\right| \leq \|f\|_{L^2} \|u\|_{L^2} \leq C\|f\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)}$$

so that $u \mapsto \langle f, u\rangle$ defines a bounded linear functional on $H_0^1(\Omega)$. Hence, there exists a unique function $u \in H_0^1(\Omega)$ s.t.

$$B_\mu(u, v) = \langle f, v\rangle_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$, namely, u is the unique weak solution of (WP) $\square$

- It remains to study regularity of the solution, which is a priori only in $H_0^1(\Omega)$.

As an example, we consider the one-dimensional case

$$-u'' + u = f \qquad \text{on} \quad (a, b)$$

The solution satisfies

$$\int_a^b u'v' = \int_a^b (f-u)v \qquad \text{for all } v \in H_0^1((a,b))$$

In particular, this holds for all $v \in C_c^\infty((a,b))$:
since $(f-u) \in L^2((a,b)) \subset L^1_{loc}((a,b))$, this means that u' is weakly differentiable and $(u')' = (u-f) \in L^2((a,b))$, i.e. $u \in H^2((a,b))$.

We assume further that $f \in C^0([a,b])$. Since u is continuous, $u'' = (f-u)$ is continuous and by the remark after the Sobolev inequalities $u' \in C^1([a,b])$ and further $u \in C^2([a,b])$.

- In general, under the condition of uniform ellipticity:

<u>Theorem</u>: Assume that $a^{ij} \in C^1(\Omega)$, $b^i, c \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω, then

$$u \in H^2(K)$$

and

$$\|u\|_{H^2(K)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

for all compact sets $K \subset \Omega$.

<u>Remarks</u>: * We do not assume $u \in H^1_0(\Omega)$, so that the thm also applies to non-vanishing B.C.

* The fact that $u \in H^2(K)$ implies that u solves the PDE strongly almost everywhere. Indeed: $\forall \varphi \in C_c^\infty(\Omega)$:

$$B(u,\varphi) = \langle f, \varphi \rangle$$

By the properties of the weak derivative:

$$B(u,\varphi) = \langle Lu, \varphi \rangle_{L^2} \qquad \text{if } u \in H^2$$

hence
$$\langle Lu - f, \varphi \rangle_{L^2} = 0 \qquad \forall \varphi \in C_c^\infty(\Omega)$$

which implies that

$$Lu - f = 0 \qquad \text{almost everywhere.}$$

By induction, this can be extended to the following:
If $a^{ij}, b^{i}, c \in C^{m+1}(\Omega)$ and

$$f \in C^{m}(\Omega)$$

then a weak solution $u \in H^1(\Omega)$ of $Lu = f$ is in fact

$$u \in H^{m+2}(K)$$

for any compact $K \subset \Omega$. By the Sobolev inequalities:

- **Theorem**: Assume uniform ellipticity, $a^{ij}, b^{i}, c \in C^{\infty}(\Omega)$ and $f \in C^{\infty}(\Omega)$. Assume that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, namely

$$B(u,v) = \langle f, v \rangle \qquad \forall v \in H^1(\Omega)$$

Then $u \in C^{\infty}(\Omega)$