TORSION ORDERS OF FANO HYPERSURFACES

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Abstract. We find new lower bounds on the torsion orders of very general Fano hypersurfaces over (uncountable) fields of arbitrary characteristic. Our results imply that unirational parametrizations of most Fano hypersurfaces need to have very large degree. Our results also hold in characteristic two, where they solve the rationality problem for hypersurfaces under a logarithmic degree bound, thereby extending a previous result of the author from characteristic different from two to arbitrary characteristic.

1. Introduction

Let X be a projective variety over a field k. The torsion order $\mathrm{Tor}(X)$ of X is the smallest positive integer e, such that e times the diagonal of X admits a decomposition in the Chow group of $X \times X$, that is,

$$e\Delta_X = [z \times X] + B \in \mathrm{CH}_{\dim X}(X \times X),$$

where $z \in \mathrm{CH}_0(X)$ is a zero-cycle of degree e and B is a cycle on $X \times X$ that does not dominate the second factor. If no such decomposition exists, we put $\mathrm{Tor}(X) = \infty$. If k is algebraically closed, then $\mathrm{Tor}(X)$ is the smallest positive integer e such that for any field extension L of k, the kernel of the degree map $\mathrm{CH}_0(X_L) \to \mathbb{Z}$ is e-torsion, and $\mathrm{Tor}(X) = \infty$ if no such integer exists. This notion goes back to Bloch [Blo79] (using an idea of Colliot-Thélène) and Bloch–Srinivas [BS83], and has for instance been studied in [ACTP13] and [Voi15], and in the above form by Chatzistamatiou–Levine [CL17] and Kahn [Kah17].

The torsion order is a stable birational invariant of smooth projective varieties; it is finite if X is rationally connected and it is 1 if X is stably rational. Moreover, if $f : Y \to X$ is a generically finite morphism, then $\mathrm{Tor}(X)$ divides $\deg(f) \cdot \mathrm{Tor}(Y)$. In particular, the degree of any unirational parametrization of X is divisible by $\mathrm{Tor}(X)$.

The torsion order is a powerful invariant of rationally connected varieties, which we would like to compute for interesting classes of varieties. In particular, it is desirable to

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do so for smooth hypersurfaces $X_d \subset \mathbb{P}^{N+1}_k$ of degree $d \leq N+1$. By a result of Roitman [Roi72] and Chatzistamatiou–Levine [CL17, Proposition 5.2], we have

$$\operatorname{Tor}(X_d) \mid d!\,. \tag{1}$$

This yields an upper bound which holds over any field k.

Finding lower bounds for $\operatorname{Tor}(X_d)$ over algebraically closed fields is in general a difficult problem. By a result of Chatzistamatiou–Levine [CL17, Theorem 8.2], building on earlier work of Totaro [Tot16] and Kollár [Kol95], the torsion order of a very general complex hypersurface $X_d \subset \mathbb{P}^{N+1}_{\mathbb{C}}$ of degree $d \geq p^j \lceil \frac{N+2}{p^j+1} \rceil$ is divisible by p^j, where p denotes a prime number. This yields non-trivial lower bounds roughly in degrees $d > \frac{2}{3}N$. In [Sch19b], the author dealt with lower degrees by showing that the torsion order of a very general hypersurface $X_d \subset \mathbb{P}^{N+1}_{\mathbb{C}}$ of degree $d \geq \log_2 N + 2$ and dimension $N \geq 3$ is divisible by 2. This paper generalizes that result as follows.

Theorem 1.1. *Let k be an uncountable field. Then the torsion order of a very general Fano hypersurface $X_d \subset \mathbb{P}^{N+1}_k$ of degree $d \geq 4$ is divisible by every integer $m \leq d - \log_2 N$ that is invertible in k.*

The first new case concerns very general quintic fourfolds $X_5 \subset \mathbb{P}^5_k$ over (algebraically closed) fields of characteristic different from 3, for which we get $3 \mid \operatorname{Tor}(X_5)$. If $\operatorname{char} k = 0$, then $\operatorname{Tor}(X_5)$ is also divisible by 2 and 5 (see [Tot16] and [CL17, Theorem 8.2]) and so our result determines all prime factors of $\operatorname{Tor}(X_5)$ by (1).

The strength of Theorem 1.1 lies in its asymptotic behaviour for large N. To illustrate this, let $X_{100} \subset \mathbb{P}^{100}_{\mathbb{C}}$ be a very general complex hypersurface of degree 100. Then $\operatorname{Tor}(X_{100})$ is divisible by

$$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \prod_{\substack{p \leq 89 \\ p \text{ prime}}} p = 718\,766\,754\,945\,489\,455\,304\,472\,257\,065\,075\,294\,400,$$

while it was previously only known to be divisible by $2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 = 138\,600$.

Even though smooth hypersurfaces $X_d \subset \mathbb{P}^{N+1}_{\mathbb{C}}$ of degree d with $d! \leq \log_2(N+1)$ are known to be unirational [HMP98, BR19], very general Fano hypersurfaces of large degree are conjecturally not unirational. While this paper does not solve this problem, it does show that for most Fano hypersurfaces, unirational parametrizations need to have enormously large degree, strengthening previous bounds on this problem: In [Kol95, Theorem 4.3], Kollár gave lower bounds on the degree of a uniruled parametrization of high-degree Fano hypersurfaces, and, relying on [Tot16], Chatzistamatiou–Levine produced slightly better bounds for unirational parametrizations in [CL17, Theorem 8.2].

In [Sch19b] it was shown that very general hypersurfaces of dimension $N \geq 3$ and degree $d \geq \log_2 N + 2$ are stably irrational over any uncountable field of characteristic different from two. This improved [Kol95, Tot16] in characteristic $\neq 2$, but the Kollár–Totaro bound $d \geq 2\lceil \frac{N+2}{3} \rceil$ remained the best known result in characteristic two.

Applying Theorem 1.1 to $m = 3$, this paper solves the rationality problem for hypersurfaces in characteristic two under a logarithmic degree bound.

Corollary 1.2. *Let k be an uncountable field of characteristic two. Then a very general hypersurface $X \subset \mathbb{P}_k^{N+1}$ of degree $d \geq \log_2 N + 3$ is stably irrational.*

The method of this paper is flexible and applies to other types of varieties as well. To illustrate this, we include here the example of cyclic covers of projective space.

Theorem 1.3. *Let k be an uncountable field and let $m \geq 2$ be an integer that is invertible in k. Then the torsion order of a cyclic $m : 1$ cover $X \to \mathbb{P}_k^N$ branched along a very general hypersurface of degree $d \geq \log_2 N + 2m - 2$ (with $m \mid d$) is divisible by m.*

In particular, under the above degree bound, very general cyclic $m : 1$ covers are stably irrational. Even for $k = \mathbb{C}$, this extends previous results on this problem substantially, see [Kol96, CTP16b, Oka19].

The above results are proven via a version of the degeneration technique that the author developed in [Sch19a, Sch19b] and which improved the method of Voisin [Voi15] and Colliot-Thélène–Pirutka [CTP16a]. An essential ingredient in this approach is the construction of varieties that have nontrivial unramified cohomology.

Constructing rationally connected varieties with nontrivial unramified cohomology is a subtle problem. In degree two, the first examples are due to Saltman [Sal84]. Building on [AM72], the first examples in degree three and with $\mathbb{Z}/2$-coefficients have been constructed in [CTO89]. This has later been generalized to arbitrary degrees and $\mathbb{Z}/\ell$-coefficients for any prime ℓ in [Pey93, Aso13]. Starting with [CTO89], all these constructions rely on norm varieties attached to symbols in Milnor K-theory mod ℓ. Norm varieties attached to symbols of length two are Brauer–Severi varieties. For $\ell \neq 2$, such varieties have large degree, compared to their dimensions, which hints that they are not useful for our purposes. Moreover, for symbols of length at least three, norm varieties for $\ell \neq 2$ are very intricate objects, whose construction, due to Rost, relies inductively on the Bloch–Kato conjecture in lower degrees, see [SJ06]. The situation is special for $\ell = 2$, where norm varieties are Pfister quadrics, which are much simpler objects. Pfister quadrics are used in [Sch19b], which explains the restriction to the prime 2.

This paper introduces for any integer m large classes of hypersurfaces with unramified $\mathbb{Z}/m$-cohomology, see Theorem 5.3 below. As in [Sch19b], an important ingredient is a quite flexible degeneration argument which allows to prove nontriviality of certain classes without any deep result from K-theory, see item (3) in Theorem 5.3. Besides the ideas from [Sch19b], the main new ingredient of this paper is the definition and usage of universal relations in Milnor K-theory, see Definition 3.1 below. Our approach is elementary, works for any positive integer m (not necessarily prime) and does not rely on norm varieties, nor on Voevodsky's proof of the Bloch–Kato conjectures. As an important example, the concept of universal relations in Milnor K-theory allows us to prove the following result of independent interest, which generalizes a famous vanishing result in the theory of Pfister forms to analogous forms of higher degree.

Corollary 1.4. *Let $\mu_1, \dots, \mu_n \in L$ be nonzero elements of a field L. Consider the hypersurface $X_{\mu_1,\dots,\mu_n} \subset \mathbb{P}_L^{2^n-1}$ of degree m, given by*

$$\sum_{\epsilon \in \{0,1\}^n} (-\mu_1)^{\epsilon_1}(-\mu_2)^{\epsilon_2} \dots (-\mu_n)^{\epsilon_n} \cdot y_{\phi(\epsilon)}^m = 0,$$

where $\phi(\epsilon) = \sum_{i=1}^n \epsilon_i \cdot 2^{i-1}$. If $X_{\mu_1,\dots,\mu_n}$ is integral (e.g. if $\frac{1}{m} \in L$), then

$$(\mu_1, \dots, \mu_n) \in \ker(K_n^M(L)/m \longrightarrow K_n^M(L(X_{\mu_1,\dots,\mu_n}))/m).$$

Asok's examples [Aso13] with nontrivial unramified $\mathbb{Z}/\ell$-cohomology in degree n have dimension $N \gg \ell^n$, which grows rapidly with n. For any given prime ℓ and integer $N \geq 3$, this led Asok [Aso13, Question 4.5] to ask for general restrictions on the possible degrees in which rationally connected complex varieties of dimension N can have nontrivial unramified $\mathbb{Z}/\ell$-cohomology. This is a quite subtle problem already for rationally connected threefolds, where by a result of Voisin [Voi06] and Colliot-Thélène–Voisin [CTV12, Théorème 1.2], it boils down to understanding the possible Brauer groups, see [Aso13, Remarks 4.7, 4.8 and 4.10].

As a consequence of our proof of Theorem 1.1, we obtain the following uniform result in arbitrary dimension; the case $m = 2$ is due to [Sch19b, Theorem 1.5].

Theorem 1.5. *Let $m, n \geq 2$ and $N \geq 3$ be integers with $\log_2(m+1) \leq n \leq N+1-m$. Then there is a rationally connected smooth complex projective variety X of dimension N such that the n-th unramified cohomology $H^n_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/m)$ of X contains an element of order m.*

It is natural to wonder whether the upper bound in the above theorem is sharp. For instance, is the unramified $\mathbb{Z}/2$-cohomology of a rationally connected smooth complex projective variety X trivial in degree $n = \dim X$?

Remark 1.6. *The main results of this paper are formulated over uncountable fields. However, our proofs show that for any field k with $\mathbb{Q}(t) \subset k$ or $\mathbb{F}_p(t,s) \subset k$, there are hypersurfaces as in Theorems 1.1 and 1.3 that are defined over k, and it is easy to extract explicit equations for these examples. Moreover, the varieties in Theorem 1.5 may be chosen to be defined over $\mathbb{Q}$.*

2. Preliminaries

2.1. **Conventions.** A variety is an integral separated scheme of finite type over a field. For a scheme X, we denote its codimension one points by $X^{(1)}$. A property holds for a very general point of a scheme if it holds at all closed points inside some countable intersection of open dense subsets.

2.2. **Degenerations.** Let R be a discrete valuation ring with fraction field K and algebraically closed residue field k. Let $\mathcal{X} \to \operatorname{Spec} R$ be a proper flat morphism with generic fibre X and special fibre Y. Then we say that X degenerates to Y. We also say that the base change of X to any larger field degenerates (or specializes) to Y. For instance, if $\mathcal{X} \to B$ is a proper flat morphism of varieties over an algebraically closed uncountable field, then the fibre X_t over a very general point $t \in B$ degenerates to the fibre X_0 for any closed point $0 \in B$ in the above sense, see e.g. [Sch19a, §2.2]. In particular, a very general hypersurface $X \subset \mathbb{P}^{n+1}_k$ over an algebraically closed uncountable field k specializes to any given hypersurface of the same dimension and degree over k.

2.3. **Alterations.** Let Y be a variety over an algebraically closed field k. An alteration of Y is a proper generically finite surjective morphism $\tau : Y' \to Y$, where Y' is a nonsingular variety over k. De Jong [deJ96] proved that alterations always exist. Later, Gabber showed that one can additionally require that $\deg(\tau)$ is prime to any given prime $\ell \neq \operatorname{char}(k)$. Temkin [Tem17, Theorem 1.2.5] generalized this further, ensuring that $\deg(\tau)$ is a power of the characteristic of k (or one if $\operatorname{char}(k) = 0$).

2.4. **Milnor K-theory.** Let L be a field. Recall that Milnor K-theory $K^M_n(L)$ of L in degree $n \geq 2$ is defined as the quotient of $(L^*)^{\otimes n}$, where L^* denotes the multiplicative group of units in L, by the subgroup generated by tensors of the form $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_{i+1} = 1$ for some $1 \leq i \leq n-1$. Moreover, $K^M_0(L) = \mathbb{Z}$ and $K^M_1(L) = L^*$. The image of a tensor $a_1 \otimes \cdots \otimes a_n$ in $K^M_n(L)$ is denoted by $(a_1, \dots, a_n)$. The direct sum $K^M_*(L) := \bigoplus_{n\geq 0} K^M_n(L)$ has a natural product structure, induced by the tensor product.

For an integer $m \geq 2$, the defining relation for Milnor K-theory implies the following basic relation in Milnor K-theory mod m, see [Mil70, Lemma 1.3].

Lemma 2.1. *Let L be a field and let $b_1, \dots, b_n \in L^*$ such that $\sum b_i = c^m$ for some $c \in L$. Then $(b_1, \dots, b_n) = 0 \in K^M_n(L)/m$.*

Let A be a ring and let A^* be the multiplicative group of units in A. We define $K^m_n(A)$ as the quotient of $(A^*)^{\otimes n}$ by the subgroup generated by $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_{i+1} = 1$ for some $1 \leq i \leq n-1$. If A is a field, then this definition coincides with the one above. If $A \to B$ is a homomorphism of rings, then we obtain an induced homomorphism $K^M_n(A) \to K^M_n(B)$. In particular, if A is an integral domain with fraction field L, then there is a natural map $\psi_L : K_n^M(A) \to K_n^M(L)$, and if A is local with residue field κ, then there is a natural map $\psi_\kappa : K_n^M(A) \to K_n^M(\kappa)$.

Lemma 2.2. *Let A be a regular local ring with fraction field L and residue field κ. Then for any integer $m \geq 1$, we have*

$$\ker\left(\psi_L : K_n^M(A)/m \longrightarrow K_n^M(L)/m\right) \subseteq \ker\left(\psi_\kappa : K_n^M(A)/m \longrightarrow K_n^M(\kappa)/m\right).$$

Proof. By [CT95, Lemma 2.1.5(a)], it suffices to prove the lemma in the case where A is a complete discrete valuation ring. In this case, let $\pi \in A$ be a uniformizer. This induces a residue homomoprhism $\partial_\pi : K_{n+1}^M(L)/m \to K_n^M(\kappa)/m$, such that for $\alpha \in K^M(A)/m$,

$$\psi_\kappa(\alpha) = \partial_\pi((\pi) \otimes \psi_L(\alpha)),$$

where $(\pi) \in K_1^M(L)/m = L^*/(L^*)^m$, see [Mil70, Lemma 2.1]. This immediately shows $\ker(\psi_L) \subset \ker(\psi_\kappa)$, as we want. $\square$

2.5. **Galois cohomology and unramified cohomology.** Let L be a field. Let m be an integer that is invertible in L and assume for simplicity that L contains a primitive m-th root of unity. We denote by $H^i(L, \mathbb{Z}/m)$ the Galois cohomology of the absolute Galois group of L with the trivial action on $\mathbb{Z}/m$. Kummer theory induces an isomorphism $H^1(L, \mathbb{Z}/m) \simeq L^*/(L^*)^m = K_1^M(L)/m$ which is canonical up to the choice of a primitive m-th root of unity of L. By [BT73], this induces via cup products a morphism of graded rings

$$K_*^M(L)/m \longrightarrow H^*(L, \mathbb{Z}/m). \tag{2}$$

(In fact, this map is an isomorphism by the Bloch–Kato conjecture, proven by Voevodsky, but we will not use this fact in this paper.) By slight abuse of notation, we denote the image of a class $(a_1, \dots, a_n) \in K_*^M(L)/m$ in $H^*(L, \mathbb{Z}/m)$ by the same symbol.

Let A be an integral domain in which m is invertible and let $L := \operatorname{Frac} A$ be its fraction field. Since $H^i(L, \mathbb{Z}/m)$ coincides with the étale cohomology of $\operatorname{Spec} L$ with values in $\mathbb{Z}/m$, there is a natural pullback map $H^i_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/m) \to H^i(L, \mathbb{Z}/m)$. If A is a regular local ring, this map is injective, see e.g. [CT95, Lemma 2.1.5(b) and §3.6].

Lemma 2.3. *Let A be a regular local ring and let m be an integer that is invertible in A. Then the natural map*

$$H^i_{ét}(\operatorname{Spec} A, \mathbb{Z}/m) \longrightarrow H^i(\operatorname{Frac} A, \mathbb{Z}/m) \tag{3}$$

is injective.

For any discrete valuation ν on L, such that m is invertible in the residue field $\kappa(\nu)$, there is a residue map

$$\partial_\nu : H^n(L, \mathbb{Z}/m) \longrightarrow H^{n-1}(\kappa(\nu), \mathbb{Z}/m),$$

which is compatible with the aforementioned residue map in Milnor K-theory. If $A := \mathcal{O}_\nu$ denotes the valuation ring of ν, then the kernel of ∂_ν is given by the image of the injective map (3) from Lemma 2.3, see e.g. [CT95, §3.6].

Assume now that $L = k(X)$ is the function field of a k-variety X and let $\mathrm{Val}(L/k)$ be the set of all valuations on L that are induced by a prime divisor on some normal birational model of X. The unramified $\mathbb{Z}/m$-cohomology of X in degree n is defined as

$$H^n_{nr}(k(X)/k, \mathbb{Z}/m) := \{\alpha \in H^n(k(X), \mathbb{Z}/m) \mid \partial_\nu \alpha = 0 \ \ \forall \nu \in \mathrm{Val}(k(X)/k)\}.$$

This subgroup of $H^n(k(X), \mathbb{Z}/m)$ is a stable birational invariant of X, see [CTO89].

Let $\gamma \in H^n_{nr}(k(X)/k, \mathbb{Z}/m)$ be unramified and let $E \subset X$ be a subvariety whose generic point x lies in the smooth locus of X. Then γ lifts uniquely to a class in the cohomology of $\operatorname{Spec} \mathcal{O}_{X,x}$ (see Lemma 2.3) and so it can be restricted to the closed point to give a class in $H^n(\kappa(x), \mathbb{Z}/m) = H^n(k(E), \mathbb{Z}/m)$ that we denote by $\gamma|_x$ or $\gamma|_E$.

3. Universal relations in Milnor K-theory modulo m

Fix a base field k and a natural number $m \geq 2$. For integers $n, s \geq 1$, let

$$R_{n,s} := k[x_1, x_2, \ldots, x_n, y_1, \ldots, y_s]$$

be the polynomial ring over k in $n + s$ variables and let $L_{n,s} := \operatorname{Frac} R_{n,s}$ be its field of fractions.

Definition 3.1. *A universal relation in Milnor K-theory modulo m over the field k is an identity*

$$(x_1, \ldots, x_n) = \lambda \cdot (a_1, \ldots, a_n) \in K^M_n(L_{n,s})/m, \tag{4}$$

for some polynomials $a_1, \ldots, a_n \in R_{n,s}$ and $\lambda \in (\mathbb{Z}/m)^*$.

3.1. **General properties.** The terminology in Definition 3.1 is due to the following property.

Lemma 3.2. *Let (4) be a universal relation in Milnor K-theory modulo m over the field k. Let L be a field extension of k and let $\phi : R_{n,s} \to L$ be a morphism of k-algebras such that $\phi(x_i)$ and $\phi(a_i)$ are invertible in L for all $i = 1, \ldots, n$. Then,*

$$(\phi(x_1), \ldots, \phi(x_n)) = \lambda \cdot (\phi(a_1), \ldots, \phi(a_n)) \in K^M_n(L)/m.$$

Proof. The morphism ϕ yields a morphism of schemes $\varphi : \operatorname{Spec} L \to \operatorname{Spec} R_{n,s} = \mathbb{A}^{n+s}_k$. Let $x \in \mathbb{A}^{n+s}_k$ be the image of φ. Then the field L is an extension of the residue field $\kappa(x)$ of x and so there is a natural homomorphism $K^M_n(\kappa(x))/m \to K^M_n(L)/m$. In order to prove the lemma, we may thus without loss of generality assume $L = \kappa(x)$ and so φ denotes the inclusion of the scheme-point $x \in \mathbb{A}^{n+s}_k$.

Let A be the local ring of $\mathbb{A}_k^{n+s}$ at x. Since $\phi(x_i), \phi(a_i) \in L^*$, we get $x_i, a_i \in A^*$ and so

$$(x_1, \dots, x_n) - \lambda(a_1, \dots, a_n) \in K_n^M(A)/m. \tag{5}$$

This element lies in the kernel of $K_n^M(A)/m \to K_n^M(\operatorname{Frac} A)/m$, because $\operatorname{Frac} A = L_{n,s}$ and (4) is a universal relation. It thus follows from Lemma 2.2 that (5) lies in the kernel of $K_n^M(A)/m \to K_n^M(L)/m$, because $L = \kappa(x)$ is the residue field of the local ring A by assumption. This concludes the lemma. □

The following proposition shows that universal relations allow to construct varieties whose function fields kill a given symbol in Milnor K-theory modulo m.

Proposition 3.3. *Let (4) be a universal relation in Milnor K-theory modulo m over the field k in degree $n \geq 1$. Let L be a field extension of k and let $\mu_1, \dots, \mu_n \in L^*$. Let s' be a positive integer and let $\phi : R_{n,s} \to L[y_1, \dots, y_{s'}]$ a homomorphism of k-algebras with $\phi(x_i) = \mu_i$ and $\phi(a_i) \neq 0$ for all $i = 1, \dots, n$. Let $c \in L[y_1, \dots, y_{s'}]$ be nonzero such that*

$$F := c^m - \sum_{i=1}^{n} \phi(a_i) \in L[y_1, \dots, y_{s'}]$$

is irreducible and let W be a projective model of $\{F = 0\} \subset \mathbb{A}_L^{s'}$. Then

(a) $(\mu_1, \dots, \mu_n) \in \ker\left(K_n^M(L)/m \longrightarrow K_n^M(L(W))/m\right)$.

(b) Let Y be a variety over L and let $\iota : Y \to W$ be a morphism of L-varieties such that the image $\iota(\eta_Y)$ of the generic point of Y lies in the regular locus of W. Then

$$(\mu_1, \dots, \mu_n) \in \ker\left(K_n^M(L)/m \longrightarrow K_n^M(L(Y))/m\right).$$

Proof. Since F is irreducible, W is integral and so it is regular at the generic point. In particular, item (a) is a special case of (b). Nonetheless, we will prove (a) first. For this, we denote by $\overline{\phi(a_i)}$ the image of $a_i \in R_{n,s}$ in $L(W)$. Since $c \neq 0$ and $\phi(a_i) \neq 0$ for all i, we find $\overline{\phi(a_i)} \neq 0$ for all i. Hence,

$$(\phi(x_1), \dots, \phi(x_n)) = \lambda \cdot (\overline{\phi(a_1)}, \dots, \overline{\phi(a_n)}) \in K_n^M(L(W))/m$$

by Lemma 3.2, and so this class vanishes by Lemma 2.1 because $\sum_i \overline{\phi(a_i)}$ is an m-th power in $L(W)$ by the definition of F. This proves item (a) because $\phi(x_i) = \mu_i$ for all i.

To prove item (b), let $w = \iota(\eta_Y) \in W$ be the image of the generic point of Y. Let $A = \mathcal{O}_{W,w}$ be the local ring of W at w. By assumption, A is a regular local ring. Since $\mu_i \in L^* \subset A^*$ for all i,

$$(\mu_1, \dots, \mu_n) \in \ker(K_n^M(A)/m \longrightarrow K_n^M(L(W))/m)$$

by item (a) proven above. Applying Lemma 2.2, we then find

$$(\mu_1, \dots, \mu_n) \in \ker(K_n^M(A)/m \longrightarrow K_n^M(\kappa(w))/m).$$

Item (b) stated in the proposition follows from this because $L(Y)$ is a field extension of $\kappa(w)$ and so the natural map $K_n^M(L)/m \longrightarrow K_n^M(L(Y))/m$ factors through $K_n^M(\kappa(w))/m$. This concludes the proof of the proposition. $\square$

3.2. **Examples.** The simplest example of a universal relation modulo m is given by

$$(x_1) = (x_1 y_1^m) \in K_1^M(L_{1,1})/m.$$

The next lemma allows to produce universal relations in Milnor K-theory mod m in arbitrary degree by starting with a single relation in low degree.

Lemma 3.4. *Let $(x_1, \dots, x_n) = \lambda \cdot (a_1, \dots, a_n) \in K_n^M(L_{n,s})/m$ be a universal relation in degree n. Then*

$$(x_1, \dots, x_n, x_{n+1}) = \lambda \cdot \left(a_1, \dots, a_n, x_{n+1}\left(y_{s+1}^m - \sum_{i=1}^n a_i'\right)\right) \in K_{n+1}^M(L_{n+1,2s+1})/m,$$

is a universal relation in degree $n+1$, where $a_i' := a_i(x_1, \dots, x_n, y_{s+2}, \dots, y_{2s+1})$.

Proof. Since $(x_1, \dots, x_n) = \lambda \cdot (a_1, \dots, a_n) \in K_n^M(L_{n,s})/m$, we have

$$\left(x_1, \dots, x_n, x_{n+1}\left(y_{s+1}^m - \sum_{i=1}^n a_i'\right)\right) = \lambda \cdot \left(a_1, \dots, a_n, x_{n+1}\left(y_{s+1}^m - \sum_{i=1}^n a_i'\right)\right)$$

in $K_n^M(L_{n+1,2s+1})/m$. The claim in the lemma is thus equivalent to

$$\left(x_1, \dots, x_n, y_{s+1}^m - \sum_{i=1}^n a_i'\right) = 0 \in K_n^M(L_{n+1,2s+1})/m. \tag{6}$$

Relabelling the y-coordinates in the universal relation $(x_1, \dots, x_n) = \lambda \cdot (a_1, \dots, a_n)$ shows by Lemma 3.2 that $(x_1, \dots, x_n) = \lambda \cdot (a_1', \dots, a_n') \in K_n^M(L_{n+1,2s+1})/m$ and so (6) is equivalent to

$$\lambda \cdot \left(a_1', \dots, a_n', y_{s+1}^m - \sum_{i=1}^n a_i'\right) = 0 \in K_n^M(L_{n+1,2s+1})/m,$$

which holds by Lemma 2.1. This concludes the proof of the lemma. $\square$

To illustrate the above result, start with the trivial relation $(x_1) = (x_1 y_1^m)$ in degree one. Applying the lemma, we arrive at the relation

$$(x_1, x_2) = (x_1 y_1^m, x_2 y_2^m - x_1 x_2 y_3^m) \in K_2^M(L_{2,3})/m$$

in degree two. Applying the lemma once again, we get the universal relation

$$(x_1, x_2, x_3) = (x_1 y_1^m, x_2 y_2^m - x_1 x_2 y_3^m, x_3 y_4^m - x_1 x_3 y_5^m - x_2 x_3 y_6^m + x_1 x_2 x_3 y_7^m)$$

in $K_3^M(L_{3,7})/m$. Repeating this process inductively, we are led to the universal relation in degree n from Proposition 4.1 below.

4. Fermat–Pfister forms

Let k be a field and $m \geq 2$ an integer. For $n \geq 1$, we define the n-th Fermat–Pfister form of degree m (and with coefficients in the polynomial ring $k[x_1, \ldots, x_n]$) as

$$\mathrm{Pf}_{m,n}(y_0, \ldots, y_{2^n-1}) := \sum_{\epsilon \in \{0,1\}^n} (-x_1)^{\epsilon_1}(-x_2)^{\epsilon_2} \ldots (-x_n)^{\epsilon_n} \cdot y^m_{\phi(\epsilon)}, \tag{7}$$

where $\phi : \{0,1\}^n \to \{0, 1, \ldots, 2^n - 1\}$ denotes the bijection given by

$$\phi(\epsilon) = \sum_{i=1}^{n} \epsilon_i \cdot 2^{i-1}.$$

Our definition generalizes the famous quadratic forms of Pfister [Pfi65] to higher degrees. We denote the coefficient in front of y_i by c_i and get

$$\mathrm{Pf}_{m,n}(y_0, \ldots, y_{2^n-1}) = \sum_{i=0}^{2^n-1} c_i y_i^m. \tag{8}$$

By definition, $c_0 = 1$, $c_1 = -x_1$ and $c_{2^n-1} = (-1)^n x_1 \cdots x_n$.

For $n \geq 1$, we have

$$\mathrm{Pf}_{m,n}(y_0, \ldots, y_{2^n-1}) = \mathrm{Pf}_{m,n-1}(y_0, \ldots, y_{2^{n-1}-1}) - x_n \cdot \mathrm{Pf}_{m,n-1}(y_{2^{n-1}}, \ldots, y_{2^n-1}),$$

where we set $\mathrm{Pf}_{m,0}(y_0) := y_0^m$. Inductively, this yields

$$\mathrm{Pf}_{m,n}(y_0, \ldots, y_{2^n-1}) = y_0^m - \sum_{i=1}^{n} a_i, \tag{9}$$

where

$$a_i := x_i \cdot \mathrm{Pf}_{m,i-1}(y_{2^{i-1}}, \ldots, y_{2^i-1}). \tag{10}$$

Proposition 4.1. *Let k be a field and let $a_i \in k[x_1, \ldots, x_i, y_1, \ldots, y_{2^i-1}]$ be as in (10). Then,*

$$(x_1, \ldots, x_n) = (a_1, \ldots, a_n) \in K_n^M(L_{n,2^n-1})/m,$$

is a universal relation in Milnor K-theory modulo m over k.

Proof. We aim to prove the proposition by induction on n. For $n = 1$, the proposition is saying that $(x_1) = (x_1 y_1^m)$, which is clear. We now assume that the proposition is proven for some $n \geq 1$ and we aim to prove it for $n + 1$. Applying Lemma 3.4 to the given universal relation in degree n, we obtain

$$(x_1, \ldots, x_n, x_{n+1}) = \left(a_1, \ldots, a_n, x_{n+1}\left(y_{2^n}^m - \sum_{i=1}^{n} a_i'\right)\right), \tag{11}$$

in $K_n^M(L_{n+1,2^{n+1}-1})/m$, where

$$a_i' = a_i(x_1, \ldots, x_n, y_{2^n+1}, \ldots, y_{2^{n+1}-1}) = x_i \cdot \mathrm{Pf}_{m,i-1}(y_{2^n+2^{i-1}}, \ldots, y_{2^n+2^i-1}).$$

The recursive relation (9) implies

$$y_{2^n}^m - \sum_{i=1}^n a_i' = \mathrm{Pf}_{m,n}(y_{2^n}, y_{2^n+1}, \ldots, y_{2^{n+1}-1})$$

and so

$$x_{n+1}\left(y_{2^n}^m - \sum_{i=1}^n a_i'\right) = x_{n+1}\,\mathrm{Pf}_{m,n}(y_{2^n}, y_{2^n+1}, \ldots, y_{2^{n+1}-1}) = a_{n+1}$$

by (10). Hence, (11) simplifies to

$$(x_1, \ldots, x_n, x_{n+1}) = (a_1, \ldots, a_n, a_{n+1}) \in K_n^M(L_{n+1,2^{n+1}-1})/m,$$

as we want. This concludes the proposition. □

We are now in the position to prove Corollary 1.4 stated in the introduction.

Proof of Corollary 1.4. Let k be the prime field of L and consider the polynomial ring $R_{n,2^n-1} = k[x_1, \ldots, x_n, y_1, \ldots, y_{2^n-1}]$ from Section 3. Let $\phi : R_{n,s} \to L[y_0, y_1, \ldots, y_{2^n-1}]$ be the morphism of k-algebras, given by $\phi(x_i) = \mu_i$ and $\phi(y_j) = y_j$ for all i and j. Let further $a_i \in R_{n,2^n-1}$ be as in (10), so that the universal relation $(x_1, \ldots, x_n) = (a_1, \ldots, a_n) \in K_n^M(L_{n,2^n-1})/m$ holds by Proposition 4.1. By (9), the hypersurface $X_{\mu_1,\ldots,\mu_n}$ from Corollary 1.4 is given by

$$y_0^m - \sum_{i=1}^n \phi(a_i) = 0.$$

By assumption, $X_{\mu_1,\ldots,\mu_n}$ is integral and so $y_0^m - \sum_{i=1}^n \phi(a_i)$ is irreducible. Since $\phi(x_i) = \mu_i \in L^*$ and $\phi(a_i) \neq 0$, it thus follows from item (a) in Proposition 3.3 that

$$(\mu_1, \ldots, \mu_n) \in \ker(K_n^M(L)/m \longrightarrow K_n^M(L(X_{\mu_1,\ldots,\mu_n}))/m).$$

This proves Corollary 1.4. □

Note that in Corollary 1.4, the integer m is not assumed to be invertible in L. Adding this assumption, $X_{\mu_1,\ldots,\mu_n}$ is automatically integral and in fact smooth over L and we obtain the following stronger statement.

Corollary 4.2. *Let L be a field in which m is invertible and let $\mu_1, \ldots, \mu_n \in L^*$. Consider the smooth hypersurface $X_{\mu_1,\ldots,\mu_n} \subset \mathbb{P}_L^{2^n-1}$ of degree m, given by*

$$\sum_{\epsilon \in \{0,1\}^n} (-\mu_1)^{\epsilon_1}(-\mu_2)^{\epsilon_2} \ldots (-\mu_n)^{\epsilon_n} \cdot y_{\phi(\epsilon)}^m = 0,$$

where $\phi(\epsilon)=\sum_{i=1}^n \epsilon_i\cdot 2^{i-1}$. Let Y be a variety over L which admits a morphism $\iota: Y\to X_{\mu_1,\dots,\mu_n}$ of L-varieties. Then

$$(\mu_1,\dots,\mu_n)\in\ker\left(K_n^M(L)/m\longrightarrow K_n^M(L(Y))/m\right).$$

Proof. Let k be the prime field of L and recall $R_{n,s}=k[x_1,\dots,x_n,y_1,\dots,y_s]$ from section 3. Let $\phi: R_{n,2^n-1}\to L[y_0,\dots,y_{2^n-1}]$ be the morphism of k-algebras with $\phi(x_i)=\mu_i$ and $\phi(y_j)=y_j$ for all $i=1,\dots,n$ and $j=1,\dots,2^n-1$.

Note that $W:=X_{\mu_1,\dots,\mu_n}$ is defined by the Fermat–Pfister form $\mathrm{Pf}_{m,n}(y_0,\dots,y_{2^n-1})$ of degree m from (7), where x_i is replaced by μ_i for $i=1,\dots,n$. Hence, by (9),

$$W=\left\{y_0^m-\sum_{i=1}^n\phi(a_i)=0\right\}\subset\mathbb{P}_L^{2^n-1},$$

where $a_i=x_i\,\mathrm{Pf}_{m,n}(y_{2^{i-1}},\dots,y_{2^i-1})$. Recall also that $(x_1,\dots,x_n)=(a_1,\dots,a_n)\in K_n^M(L_{n,2^n-1})/m$ is a universal relation in Milnor K-theory modulo m by Proposition 4.1.

Since $\mu_i\neq 0$ for all i and m is invertible in L, $W=X_{\mu_1,\dots,\mu_n}$ is smooth over L by the Jacobi criterion. In particular, the image $\iota(\eta_Y)\in W$ of the generic point of Y lies in the regular locus of W and so Corollary 4.2 follows from item (b) in Proposition 3.3. ☐

5. Unramified cohomology via universal relations

Definition 5.1. *Let k be a field. A homogeneous polynomial $g\in k[x_0,x_1,\dots,x_n]$ is of twisting type modulo m if for all $i=0,1,\dots,n$:*

- *g contains the monomials $x_i^{\deg g}$ nontrivially;*
- *g is an m-th power modulo x_i.*

An inhomogeneous polynomial $b\in k[x_1,\dots,x_n]$ is of twisting type *modulo m if its homogenization in $k[x_0,x_1,\dots,x_n]$ has this property.*

Note that the degree of a polynomial which is of twisting type modulo m must be a multiple of m. The following slightly technical lemma will be crucial.

Lemma 5.2. *Let $b\in k[x_1,\dots,x_n]$ be an inhomogeneous polynomial of twisting type modulo m. Let $x\in S^{(1)}$ be a codimension one point of some normal birational model S of $\mathbb{P}_k^n$. Let $z\in\mathbb{P}_k^n$ be the image of x under the birational map $S\dashrightarrow\mathbb{P}_k^n$ and assume that z is the generic point of the intersection of $c\geq 1$ coordinate hyperplanes $\{x_{i_1}=\dots=x_{i_c}=0\}$ with $0\leq i_1<\dots<i_c\leq n$. Then b becomes an m-th power in the fraction field of the completion $\widehat{\mathcal{O}_{S,x}}$ of the local ring of S at x.*

Proof. Let $g\in k[x_0,x_1,\dots,x_n]$ be the homogeneous polynomial of twisting type given by homogenization of b. The existence of z implies $c\leq n$ and so there is some index

$0 \leq i_0 \leq n$ with $x_{i_0}(z) \neq 0$. Let b' be the inhomogeneous polynomial, given by setting $x_{i_0} = 1$ in g. Then

$$\begin{aligned} b\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) &= g\left(\frac{x_0}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= \left(\frac{x_{i_0}}{x_0}\right)^{\deg g} \cdot g\left(\frac{x_0}{x_{i_0}}, \frac{x_1}{x_{i_0}}, \dots, \frac{x_n}{x_{i_0}}\right) \\ &= \left(\frac{x_{i_0}}{x_0}\right)^{\deg g} \cdot b'\left(\frac{x_0}{x_{i_0}}, \frac{x_1}{x_{i_0}}, \dots, \widehat{\frac{x_{i_0}}{x_{i_0}}}, \dots, \frac{x_n}{x_{i_0}}\right). \end{aligned}$$

Since g is of twisting type modulo m, $\deg(g)$ is divisible by m and so b becomes an m-th power in $\operatorname{Frac} \widehat{\mathcal{O}_{S,x}}$ if and only if this holds for b'. For this reason we may without loss of generality assume that $i_0 = 0$. In particular, the inhomogenization b given by setting $x_0 = 1$ in g will be defined at z. Since g contains the monomials $x_i^{\deg g}$ nontrivially for all $i = 0, \dots, n$, it follows that the image $\bar{b}$ of b in $\kappa(z)$ is nontrivial. Moreover, $\bar{b}$ is an m-th power, as g is an m-th power modulo x_i for all i and $c \geq 1$ by assumption. The result thus follows from Hensel's lemma, applied to $\widehat{\mathcal{O}_{S,x}}$. □

For $n = 2$ and $m = 2$, the equation of a conic tangent to the three coordinate lines in $\mathbb{P}^2$ is of twisting type, see [HPT18]. An instructive example for arbitrary m and n is given by

$$g = G^m + x_0^{em-n} x_1 \cdots x_n, \tag{12}$$

where G is homogeneous of degree e with $em > n$ and G contains x_i^e nontrivially for all $i = 0, 1, \dots, n$. For $m = 2$, this simple but flexible example was used very succesfully in [Sch19b]. The general idea of tangentially meeting degeneracy loci goes back to Artin–Mumford [AM72] and has since then been used by many authors, see e.g. [CTO89, Pir18, Sch19a].

Theorem 5.3. *Let $m \geq 2$, $n, s \geq 1$ be integers and let k be an algebraically closed field in which m is invertible. Let $(x_1, \dots, x_n) = \lambda \cdot (a_1, \dots, a_n) \in K_n^M(L_{n,s})/m$ be a universal relation in Milnor K-theory modulo m over k and let $b \in k[x_1, \dots, x_n]$ be an inhomogeneous polynomial of twisting type modulo m, see Definitions 3.1 and 5.1. Assume that the polynomial*

$$F := b - \sum_{i=1}^{n} a_i \in R_{n,s} = k[x_1, \dots, x_n, y_1, \dots, y_s] \tag{13}$$

is irreducible and let W be a projective model of $\{F = 0\} \subset \mathbb{A}_k^{n+s}$ such that projection to the x_i-coordinates induces a morphism $h : W \to \mathbb{P}_k^n$. Let Y be a projective variety over k together with a morphism $\iota : Y \to W$, such that

- *the image $\iota(\eta_Y)$ of the generic point of Y lies in the smooth locus of W;*

- *the composition* $f := h \circ \iota : Y \longrightarrow \mathbb{P}^n_k$ *is surjective.*

Then

$$\alpha := (x_1, \dots, x_n) \in H^n(k(\mathbb{P}^n), \mathbb{Z}/m)$$

has the following properties.

(1) *The pullback* $f^*\alpha \in H^n_{nr}(k(Y)/k, \mathbb{Z}/m)$ *is unramified over* k.

(2) *For any generically finite morphism of* k*-varieties* $\tau : Y' \to Y$ *and any subvariety* $E \subset Y'$ *which meets the smooth locus of* Y' *and which does not dominate* $\mathbb{P}^n$ *via* $f \circ \tau$, *we have*

$$(\tau^* f^* \alpha)|_E = 0 \in H^n(k(E), \mathbb{Z}/m).$$

(3) *Assume that there is a discrete valuation ring* $R \subset k$ *with residue field* κ *and a proper flat* R*-scheme* $\mathcal{Y} \to \operatorname{Spec} R$ *with* $Y \simeq \mathcal{Y} \times_R k$. *Assume further that* $f : Y \to \mathbb{P}^n_k$ *extends to a morphism* $\mathcal{Y} \to \mathbb{P}^n_R$ *whose base change* $f_0 : Y_0 := \mathcal{Y} \times_R \kappa \to \mathbb{P}^n_\kappa$ *to the special point of* $\operatorname{Spec} R$ *admits a rational section* $\xi : \mathbb{P}^n_\kappa \dashrightarrow Y_0$ *whose image lies generically in the smooth locus of* Y_0. *Then* $f^*\alpha \in H^n_{nr}(k(Y)/k, \mathbb{Z}/m)$ *has order* m, *i.e.,* $e \cdot f^*\alpha \neq 0$ *for all* $e = 1, 2, \dots, m-1$.

Proof. Since $E \subset Y'$ in item (2) meets the smooth locus of Y', we may without loss of generality assume that Y' is normal. Replacing Y by its normalization, we may then assume that Y is normal as well (because $\tau : Y' \to Y$ factors through the normalization of Y, once Y' is normal). By the same argument as at the beginning of the proof of [Sch19b, Proposition 5.1], item (1) and (2) follow if we can show that for any codimension one point $y \in Y^{(1)}$, which does not map to the generic point of $\mathbb{P}^n_k$,

$$\partial_y(f^*\alpha) = 0 \in H^{n-1}(\kappa(y), \mathbb{Z}/m) \quad \text{and} \quad (f^*\alpha)|_y = 0 \in H^n(\kappa(y), \mathbb{Z}/m). \tag{14}$$

To prove (14), let us fix $y \in Y^{(1)}$ as above and let c denote the number of coordinate hyperplanes $\{x_i = 0\} \subset \mathbb{P}^n_k$ which contain the point $f(y)$. By [Mer08, Proposition 1.6], we may also choose a normal birational model S of $\mathbb{P}^n_k$, such that y maps via the induced rational map $Y \dashrightarrow S$ to a codimension one point $x \in S^{(1)}$ on S.

Let us first assume that $f(y) \in \mathbb{P}^n_k$ has codimension c. Then $f(y)$ must be the generic point of an intersection of c coordinate hyperplanes. (In particular, we have $c \geq 1$, because $f(y)$ is not the generic point of $\mathbb{P}^n_k$.) Since b is of twisting type modulo m, it follows from Lemma 5.2 that b becomes an m-th power in the fraction field $L := \operatorname{Frac} \widehat{\mathcal{O}_{S,x}}$ of the completion $\widehat{\mathcal{O}_{S,x}}$ of the local ring of S at x.

Let Y_η and W_η be the generic fibres of $f : Y \to \mathbb{P}^n_k$ and $h : W \to \mathbb{P}^n_k$, respectively. These are varieties over the field $k(\mathbb{P}^n)$. Since L is a field extension of $k(\mathbb{P}^n)$, we can consider the L-varieties

$$(Y_\eta)_L := Y_\eta \times_{k(\mathbb{P}^n)} L \quad \text{and} \quad (W_\eta)_L := W_\eta \times_{k(\mathbb{P}^n)} L.$$

Since $b = c^m$ for some $c \in L$, we find that $(W_\eta)_L$ is birational to

$$\left\{ c^m - \sum_{i=1}^{n} a_i = 0 \right\} \subset \mathbb{A}^s_L.$$

The morphism $\iota : Y \to W$ induces a morphism $(\iota_\eta)_L : (Y_\eta)_L \to (W_\eta)_L$ and the image of the generic point of $(Y_\eta)_L$ lies in the smooth locus of $(W_\eta)_L$, by assumption. Since $(x_1, \ldots, x_n) = \lambda \cdot (a_1, \ldots, a_n) \in K^M_n(L_{n,s})/m$ is a universal relation modulo m over k, we thus deduce from Proposition 3.3, applied to the natural morphism $\phi : R_{n,s} \to L[Y_\eta]$, induced by $k[x_1, \ldots, x_n] \subset L$, that

$$(x_1, \ldots, x_n) \in \ker \left(K^M_n(L)/m \longrightarrow K^M_n(L(Y_\eta))/m \right). \tag{15}$$

Let now $\widehat{\mathcal{O}_{Y,y}}$ be the completion of Y at the codimension one point y. Then the fraction field $\operatorname{Frac} \widehat{\mathcal{O}_{Y,y}}$ is a field extension of $L(Y_\eta)$ and so (15) implies

$$(x_1, \ldots, x_n) \in \ker \left(K^M_n(L)/m \longrightarrow K^M_n \left(\operatorname{Frac} \widehat{\mathcal{O}_{Y,y}} \right) /m \right).$$

Mapping this identity to cohomology via (2), we find that $f^*\alpha$ lies in the kernel of the natural map

$$\varphi : H^n(k(Y), \mathbb{Z}/m) \longrightarrow H^n(\operatorname{Frac} \widehat{\mathcal{O}_{Y,y}}, \mathbb{Z}/m).$$

The residue of $f^*\alpha$ at y factors through φ, and so $\partial_y f^*\alpha = 0$. This implies

$$\varphi(f^*\alpha) = 0 \in H^n_{\text{ét}}(\operatorname{Spec} \widehat{\mathcal{O}_{Y,y}}, \mathbb{Z}/m) \subset H^n(\operatorname{Frac} \widehat{\mathcal{O}_{Y,y}}, \mathbb{Z}/m),$$

where the latter inclusion follows from Lemma 2.3. Hence, the restriction $(f^*\alpha)|_y$ factors through φ as well and so $(f^*\alpha)|_y = 0$, which concludes (14) in this case.

It remains to deal with the case where $f(y) \in \mathbb{P}^n_k$ has codimension less than c (e.g. this happens if $c = 0$). Using homogeneous coordinates, we have $\alpha = \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right)$. Fix some $j \in \{1, \ldots, n\}$. Multiplying each entry of α by $(x_0/x_j)^m$, we find

$$\alpha = \left(\frac{x_0^{m-1} x_1}{x_j^m}, \ldots, \frac{x_0^{m-1}}{x_j^{m-1}}, \ldots, \frac{x_0^{m-1} x_n}{x_j^m} \right).$$

Since k is algebraically closed, $(-1) \in (K^*)^m$ and so $(a, a) = 0$ for any $a \in k(\mathbb{P}^n)^*$, see Lemma 2.1. Applying this to $a = (x_0/x_j)^{m-1}$, the above identity simplifies to

$$\alpha = \left(\frac{x_1}{x_j}, \ldots, \frac{x_0^{m-1}}{x_j^{m-1}}, \ldots, \frac{x_n}{x_j} \right) = - \left(\frac{x_1}{x_j}, \ldots, \frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j} \right).$$

Since it suffices to prove (14) after changing the sign of α, we may thus, up to relabelling the coordinates, without loss of generality assume that $x_1, \ldots, x_c$ vanish at $f(y)$, while $x_0, x_{c+1}, \ldots, x_n$ do not vanish at $f(y)$.

Now the same argument as in Case 2 of the proof of [Sch19b, Proposition 5.1] applies; we repeat it for convenience of the reader.

First recall the normal birational model S of $\mathbb{P}^n_k$, such that y maps to a codimension one point $x \in S^{(1)}$ on S. Since $x_0, x_{c+1}, \dots, x_n$ do not vanish at $f(y)$, we get

$$\partial_x \alpha = (\partial_x (x_1, \dots, x_c)) \cup (x_{c+1}, \dots, x_n) \in H^{n-1}(\kappa(x), \mathbb{Z}/m),$$

see e.g. [Sch19b, Lemma 2.1]. Since $f(y)$ has codimension less than c and k is algebraically closed, $H^{n-c}(\kappa(f(y)), \mathbb{Z}/m) = 0$. Hence, $(x_{c+1}, \dots, x_n) = 0 \in H^{n-c}(\kappa(f(y)), \mathbb{Z}/m)$ and so $\partial_x \alpha = 0$ by the above formula. Since $\partial_y \alpha$ is up to a multiple given by the pullback of $\partial_x \alpha$ (see e.g. [CT95, Proposition 3.3.1]), we find that $\partial_y f^*\alpha = 0$. Moreover, the restriction $f^*\alpha|_y$ is given by pulling back the restriction $\alpha|_x \in H^n(\kappa(x), \mathbb{Z}/m)$, which vanishes because $\kappa(x)$ has cohomological dimension less than n, since k is algebraically closed. This proves (14), which establishes items (1) and (2) of Theorem 5.3.

To prove (3), we assume for a contradiction that for some $e \in \{1, 2, \dots, m-1\}$,

$$e \cdot f^*\alpha = 0 \in H^n(k(Y), \mathbb{Z}/m). \tag{16}$$

Since k is algebraically closed, $H^n(k(Y), \mathbb{Z}/n) \to H^n(K(Y), \mathbb{Z}/n)$ is injective for any algebraically closed field extension K of k. We may thus without loss of generality assume that k is the algebraic closure of $\operatorname{Frac} R$. Note also that the assumptions in (3) are stable under base change via an extension of discrete valuation rings $R \subset R'$. Replacing R by its completion $\hat{R}$, $\mathcal{Y} \to \operatorname{Spec} R$ by the corresponding base change and k by the algebraic closure of $\hat{R}$, we may thus assume that R is complete. Since $H^n(k(Y), \mathbb{Z}/m)$ is the direct limit $\lim_L H^n(L(Y), \mathbb{Z}/m)$, where L runs through all finitely generated extensions of $\operatorname{Frac} R$, there is a finite field extension L of $\operatorname{Frac} R$ such that

$$e \cdot f^*\alpha = 0 \in H^n(L(Y), \mathbb{Z}/m). \tag{17}$$

Replacing R by its integral closure in L (which is again a discrete valuation ring because R is complete, see [EGAIV, Théorème 23.1.5 and Corollaire 23.1.6]), $\mathcal{Y} \to \operatorname{Spec} R$ by the corresponding base change and κ by the induced finite field extension, we may finally assume that $L = \operatorname{Frac} R$ in (17).

By assumption, there is a rational section $\xi : \mathbb{P}^n_\kappa \dashrightarrow Y_0$ such that the image $y_0 = \xi(\eta_{\mathbb{P}^n_\kappa})$ of the generic point of $\mathbb{P}^n_\kappa$ is contained in the smooth locus of Y_0. Since R is a discrete valuation ring and Y_0 is the special fibre of the proper flat morphism $\mathcal{Y} \to \operatorname{Spec} R$, we find that y_0 is contained in a unique irreducible component Y_0' of Y_0 and Y_0 must be generically reduced along Y_0'. In particular, the local ring $A := \mathcal{O}_{\mathcal{Y}, \eta_{Y_0'}}$ of $\mathcal{Y}$ at the generic point of Y_0' is a discrete valuation ring with fraction field $L(Y)$. This implies that the natural map

$$H^n_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/m) \longrightarrow H^n(L(Y), \mathbb{Z}/m)$$

is injective, see Lemma 2.3. Since $e \cdot f^*\alpha$ lies in the image of the above map, (17) implies $0 = e \cdot f^*\alpha \in H^n_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/m)$. Restricting this to the special point of $\operatorname{Spec} A$, we get

$$e \cdot f_0^*\alpha = 0 \in H^n(\kappa(Y_0'), \mathbb{Z}/m).$$

Let B be the local ring of Y_0 at the generic point of the image of the section $\xi : \mathbb{P}^n_\kappa \dashrightarrow Y_0$. Since the image of ξ is generically contained in the component Y_0' of Y_0 and in the smooth locus of Y_0, B is a regular local ring with fraction field $\kappa(Y_0')$ and so the natural map

$$H^n_{\text{ét}}(\operatorname{Spec} B, \mathbb{Z}/m) \longrightarrow H^n(\kappa(Y_0'), \mathbb{Z}/m)$$

is injective, see Lemma 2.3. Moreover, $e \cdot f_0^*\alpha$ is contained in the image of the above map and so

$$e \cdot f_0^*\alpha = 0 \in H^n_{\text{ét}}(\operatorname{Spec} B, \mathbb{Z}/m).$$

After restriction to the closed point of $\operatorname{Spec} B$ and pulling this back to $\kappa(\mathbb{P}^n)$ via the rational section $\xi : \mathbb{P}^n_\kappa \dashrightarrow Y_0$, we find

$$e \cdot \alpha = 0 \in H^n(\kappa(\mathbb{P}^n), \mathbb{Z}/m),$$

because $e \cdot \xi^* f_0^*\alpha = e \cdot \alpha$. Since $\alpha = (x_1, \dots, x_n)$, this statement is false, as one shows by induction on n by taking the residue along $x_n = 0$. This contradicts (16), which completes the proof of the theorem. □

Remark 5.4. *Starting with any universal relation $(x_1, \dots, x_n) = \lambda \cdot (a_1, \dots, a_n) \in K^M_n(L_{n,s})/m$, Theorem 5.3 produces hypersurfaces in $\mathbb{A}^{n+s}_k$ with nontrivial unramified $\mathbb{Z}/m$-cohomology whose degree is roughly the maximum of $m\lfloor \frac{n+1}{m} \rfloor$ (the degree of g in (12)) and the degrees of the a_i. One source of examples for universal relations is given by Lemma 3.4, but the notion is much more general than that. For instance, if a_i for $i = 1, \dots, n$ is as in (10), then a similar argument as in Lemma 3.4 shows that for any $1 \leq i \leq n$:*

$$(x_1, \dots, x_n) = \left(a_1, \dots, a_{i-1}, a_i \cdot \left(y_{2^n}^m - \sum_{j=1}^{i-1} a_j'\right), a_{i+1}, \dots, a_n\right) \in K^M_n(L_{n,2^n+2^i-1})/m$$

where $a_j' = a_j(x_1, \dots, x_n, y_{2^n+2^{j-1}}, \dots, y_{2^n+2^j-1})$. Since $\deg a_i = i + m$, the maximum of the degrees of the entries in the above relation coincides with those of $(x_1, \dots, x_n) = (a_1, \dots, a_n)$ from Proposition 4.1 as long as $i \leq (n - m + 1)/2$, but the number of y-variables involved in the above relation is larger. We will however not be able to use such relations in the proof of Theorem 1.1, because the hypersurface in $\mathbb{P}^{2^n+2^i-1}_K$ over $K = k(x_1, \dots, x_n)$ given by the projective closure of $F = 0$ with F as in (13) is not smooth over K.

6. Degeneration

The following proposition generalizes [Sch19b, Proposition 3.1] to degenerations with reducible special fibres. The result is a variant of the author's improvement [Sch19a] of the method of Voisin [Voi15] and Colliot-Thélène–Pirutka [CTP16a]. The original method of Voisin and Colliot-Thélène–Pirutka had been generalized to degenerations with reducible special fibres by Totaro [Tot16].

Proposition 6.1. *Let R be a discrete valuation ring with fraction field K and algebraically closed residue field k. Let $\mathcal{X} \to \operatorname{Spec} R$ be a proper flat R-scheme with geometric generic fibre $X_{\overline{\eta}} = \mathcal{X} \times_R \overline{K}$ and special fibre $X_0 = \mathcal{X} \times_R k$. Assume that $X_{\overline{\eta}}$ is integral. Let $Y \subset X_0^{\mathrm{red}}$ be an irreducible component of the reduction of X_0 and assume that X_0 is reduced at the generic point of Y. Let $m \geq 2$ be an integer that is invertible in k and let $\tau : Y' \to Y$ be an alteration whose degree is coprime to m. Suppose that for some $n \geq 1$ there is a class $\gamma \in H^n_{nr}(k(Y)/k, \mathbb{Z}/m)$ of order m such that*

$$(\tau^*\gamma)|_E = 0 \in H^n(k(E), \mathbb{Z}/m) \quad \text{for any subvariety } E \subset \tau^{-1}(Y \cap X_0^{\mathrm{sing}}).$$

Then the torsion order of $X_{\overline{\eta}}$ is divisible by m.

Proof. We may assume that $e := \operatorname{Tor}(X_{\overline{\eta}})$ is finite. Since torsion orders remain unchanged under passage from an algebraically closed field to a bigger field (see [CL17, Lemma 1.11]), we may after replacing R by its completion assume that R is complete. The decomposition of $e \cdot \Delta_{X_{\overline{\eta}}}$ in the Chow group of $X_{\overline{\eta}} \times X_{\overline{\eta}}$ holds already over a finite field extension L of $\operatorname{Frac}(R)$, and so $X_\eta \times L$ has torsion order e for some finite extension L of $\operatorname{Frac}(R)$, where $X_\eta = \mathcal{X} \times_R K$ denotes the generic fibre. Since R is a complete discrete valuation ring, the integral closure R' of R in L is again a complete discrete valuation ring, see [EGAIV, Théorème 23.1.5 and Corollaire 23.1.6]. Replacing R by the base change to R', we may thus assume that the generic fibre X_η has torsion order e (note that this does not change the special fibre).

Let $A := \mathcal{O}_{\mathcal{X},y}$ be the local ring of $\mathcal{X}$ at the generic point $y \in \mathcal{X}$ of Y. Since X_0 is a Cartier divisor on $\mathcal{X}$ which is reduced at y, it follows that $\mathcal{X}$ is regular at y. Hence, A is a discrete valuation ring with fraction field $K(X_\eta)$. Let $\delta_{X_\eta} \in \mathrm{CH}_0(X_\eta \times K(X_\eta))$ be the class induced by the diagonal. By assumption,

$$e \cdot \delta_{X_\eta} = z \times K(X_\eta) \in \mathrm{CH}_0(X_\eta \times K(X_\eta))$$

for a zero-cycle $z \in \mathrm{CH}_0(X_\eta)$ of degree e. Applying Fulton's specialization map on Chow groups [Ful98, §20.3] to the proper flat family $\mathcal{X}_A \to \operatorname{Spec} A$, given by base change of $\mathcal{X} \to \operatorname{Spec} R$, we find that

$$e \cdot \delta_Y = z_0 \times k(Y) \in \mathrm{CH}_0(X_0 \times k(Y)) \tag{18}$$

for some zero-cycle $z_0 \in \mathrm{CH}_0(X_0)$ of degree e, where δ_Y denotes the class of the diagonal of Y. Let $U \subset Y$ be the complement of $Y \cap X_0^{\mathrm{sing}}$. Since X_0 is reduced at the generic point of Y, U is a non-empty open subset of Y. Let $U' := \tau^{-1}(U) \subset Y'$. Note that U is smooth by construction, and so we can pullback cycle classes (modulo rational equivalence) via $U' \to U$ (see [Ful98, §8]). Since $U \to X_0$ is an open embedding, it is flat and so we can pullback cycles (resp. cycle classes) via this map as well. Altogether, we can pullback (18) to $U' \times k(Y)$ via the natural map $U' \times k(Y) \to X_0 \times k(Y)$. Applying the localization exact sequence associated to $U' \subset Y'$, we get

$$e \cdot \tau^* \delta_Y = z_{Y'} \times k(Y) + z' \in \mathrm{CH}_0(Y' \times k(Y)), \tag{19}$$

for some zero-cycle $z_{Y'} \in \mathrm{CH}_0(Y')$ (not necessarily of degree e anymore) and a zero-cycle $z' \in \mathrm{CH}_0(Y' \times k(Y))$ which is supported on

$$(Y' \setminus U') \times k(Y) = \tau^{-1}(Y \cap X_0^{\mathrm{sing}}) \times k(Y).$$

The end of the proof is now as in [Sch19b, Proposition 3.1]: the pairing (see [Mer08, §2.4]) of the unramified cohomology class $\tau^*\gamma \in H^n_{nr}(k(Y')/k, \mathbb{Z}/m)$ with the right hand side of (19) vanishes, because $\tau^*\gamma$ vanishes when restricted to closed points of Y' (because $k = \overline{k}$) or to subvarieties of $\tau^{-1}(Y \cap X_0^{\mathrm{sing}})$ (by assumption), while the left hand side evaluates to

$$e \cdot \langle \tau^* \delta_Y, \tau^* \gamma \rangle = e \cdot \langle \tau_* \tau^* \delta_Y, \gamma \rangle = e \cdot \deg(\tau) \cdot \gamma \in H^n(k(Y), \mathbb{Z}/m).$$

Hence, $e \cdot \deg(\tau) \cdot \gamma = 0$. Since γ has order m and $\deg(\tau)$ is coprime to m, this is only possible if e is divisible by m, as we want. This completes the proof. □

7. Proof of main results

Theorem 1.1 stated in the introduction follows from the following slightly stronger statement.

Theorem 7.1. *Let k be an uncountable field and let $m \geq 2$ be an integer that is invertible in k. Let $N \geq 3$ be an integer and write $N = n + r$ with $2^{n-1} - 2 \leq r \leq 2^n - 2$. Then the torsion order of a very general Fano hypersurface $X_d \subset \mathbb{P}^{N+1}_k$ of degree $d \geq m + n$ is divisible by m.*

Remark 7.2. *The bounds on r in Theorem 7.1 ensure that any integer $N \geq 3$ can be written uniquely as a sum $N = n + r$ as in the theorem. In the proof of Theorem 7.1 below, only the upper bound on r will be used, while the lower bound only appears for convenience as it yields the strongest results on the divisibility of the respective torsion orders in fixed dimension N.*

Proof. Replacing k by its algebraic closure, we may assume that k is algebraically closed. Denote by $x_0,\dots,x_n,y_1,\dots,y_{r+1}$ homogeneous coordinates of $\mathbb{P}^{N+1}_k$. Let $k'\subset k$ be the algebraic closure of the prime field of k and let $G\in k'[x_0,\dots,x_n]$ be a general homogeneous polynomial of degree $\lceil\frac{n+1}{m}\rceil$. Let $t\in k$ be transcendental over k' (if $\mathrm{char}(k)=0$, we may also take t to be a prime number coprime to m), and consider

$$g(x_0,\dots,x_n):=t^m\cdot G^m-(-1)^n x_0^{m\lceil\frac{n+1}{m}\rceil-n}x_1x_2\dots x_n, \tag{20}$$

which is a homogeneous polynomial of degree $\deg(g)=m\lceil\frac{n+1}{m}\rceil\leq m+n$ in $k'[x_0,\dots,x_n]$. Since $G\in k'[x_0,\dots,x_n]$ is general, g is of twisting type, see Definition 5.1.

We first deal with the case $d=m+n$. Consider the hypersurface $Z:=\{F=0\}\subset\mathbb{P}^{N+1}_k$ of degree $m+n$, given by

$$F:=g(x_0,\dots,x_n)\cdot x_0^{m+n-\deg(g)}+\sum_{i=1}^{r}x_0^{n-\deg c_i}c_i(x_1,\dots,x_n)y_i^m+(-1)^n x_1x_2\cdots x_n y_{r+1}^m,$$

where $c_i(x_1,\dots,x_n)\in k[x_1,\dots,x_n]$ denote the coefficients of the Fermat–Pfister form (8). The hypersurface Z is integral, because g is not divisible by x_i for any i. Consider the r-plane $P:=\{x_0=x_1=\dots=x_n=0\}\subset\mathbb{P}^{N+1}$ and let $Y:=Bl_PZ$. This blow-up is a hypersurface in $Bl_P(\mathbb{P}^{N+1})\simeq\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(-1)\oplus\mathcal{O}_{\mathbb{P}^n}^{\oplus(r+1)})$, given by the equation

$$g(x_0,\dots,x_n)\cdot x_0^{m+n-\deg(g)}z_0^m+\sum_{i=1}^{r}x_0^{n-\deg c_i}c_i(x_1,\dots,x_n)z_i^m+(-1)^n x_1x_2\cdots x_n z_{r+1}^m=0, \tag{21}$$

where z_0 is a local coordinate that trivializes $\mathcal{O}_{\mathbb{P}^n}(-1)$ and $z_1,\dots,z_{r+1}$ trivialize $\mathcal{O}_{\mathbb{P}^n}^{\oplus(r+1)}$. In the above coordinates, the exceptional divisor $D\subset Bl_PZ$ is given by $z_0=0$. Projection to the x_i-coordinates yields a morphism $f:Y\to\mathbb{P}^n_k$ whose generic fibre Y_η is the smooth hypersurface of degree m and dimension $r+1$ over $K=k(x_1,\dots,x_n)$, given by setting $x_0=1$ in (21).

To emphasize the dependence on the integers n and r, we write $Y=Y_{n,r}$ for the projective variety given by (21). Then $Y_{n,r}\subseteq Y_{n,2^n-2}$ because $r\leq 2^n-2$. We claim that Theorem 5.3 applies to $Y=Y_{n,r}$ and $W=Y_{n,2^n-2}$.

Recall that $\mathrm{Pf}_{m,n}(y_0,\dots,y_{2^n-1})=\sum_{j=0}^{2^n-1}c_jy_j^m$ by (8) with $c_{2^n-1}=(-1)^nx_1\dots x_n$. Setting $x_0=z_0=1$ in (21), we thus see that $W=Y_{n,2^n-2}$ is birational to the affine hypersurface, given by

$$g(1,x_1,\dots,x_n)+\mathrm{Pf}_{m,n}(0,y_1,y_2,\dots,y_{2^n-1})=0.$$

By (9), the above equation can be rewritten as

$$g(1,x_1,\dots,x_n)-\sum_{i=1}^{n}a_i=0$$

where $a_i = x_i \operatorname{Pf}_{m,i-1}(y_{2^{i-1}}, \dots, y_{2^i-1})$ is as in (10). By Proposition 4.1, we have the universal relation

$$(x_1, \dots, x_n) = (a_1, \dots, a_n) \in K_n^M(L_{n,2^n-1})/m.$$

Since the generic fibre of $W \to \mathbb{P}^n_k$ is smooth and contains the image of the generic point of $Y \hookrightarrow W$, we conclude that item (1) and (2) of Theorem 5.3 apply to $Y = Y_{n,r}$ and $W = Y_{n,2^n-2}$. To see that item (3) in Theorem 5.3 applies as well, consider a discrete valuation ring $R = k'[t]_{(t)} \subset k$ with parameter t and residue field $\kappa = k'$. Since Y is defined by the equation (21) whose coefficients are all contained in R and whose reduction modulo t is nonzero, it is immediate that Y extents to a proper flat R-scheme $\mathcal{Y} \to \operatorname{Spec} R$, where $\mathcal{Y}$ is the hypersurface defined by (21) inside the projective bundle $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^n_R}(-1) \oplus \mathcal{O}_{\mathbb{P}^n_R}^{\oplus(r+1)}\right)$ over $\mathbb{P}^n_R$. Since the morphism $f : Y \to \mathbb{P}^n_k$ is induced by projection to the x_i-coordinates, f extends to a morphism of R-schemes $\mathcal{Y} \to \mathbb{P}^n_R$. The reduction $Y_0 := \mathcal{Y} \times_R \kappa$ with morphism $f_0 : Y_0 \to \mathbb{P}^n_\kappa$ given by projection to the x_i-coordinates admits a rational section $\xi : \mathbb{P}^n_\kappa \dashrightarrow Y_0$, defined by setting $y_0 = y_{r+1} = 1$ and $y_j = 0$ for $1 \leq j \leq r$. Since m is invertible in k, the generic fibre of f_0 is smooth and so $\xi(\eta_{\mathbb{P}^n_\kappa})$ is contained in the smooth locus of Y_0. Hence, the assumptions of item (3) in Theorem 5.3 are satisfied as well and we conclude that

$$f^*(x_1, \dots, x_n) \in H^n_{nr}(k(Y)/k, \mathbb{Z}/m)$$

has order m.

Recall that $Y = Bl_P Z$ is birational to the hypersurface $Z := \{F = 0\} \subset \mathbb{P}^{N+1}_k$ from above and so the above unramified class yields a class

$$\gamma \in H^n_{nr}(k(Z)/k, \mathbb{Z}/m)$$

of order m. Let $\tau' : Y' \to Y$ be an alteration of degree coprime to m (which exists by [Tem17, Theorem 1.2.5] because m is invertible in k) and let $\tau : Y' \to Z$ be the induced alteration of Z (which has the same degree as τ'). Let $E \subset Y'$ be a closed subvariety with $\tau(E) \subset Z^{\text{sing}}$. If $\tau'(E) \subset Y$ does not dominate $\mathbb{P}^n_k$ via $f : Y \to \mathbb{P}^n_k$, then

$$\tau^*\gamma|_E = 0 \in H^n(k(E), \mathbb{Z}/m)$$

by item (2) in Theorem 5.3. Otherwise, the natural map $E \to \mathbb{P}^n_k$ induced by $f \circ \tau'$ is surjective and we denote its generic fibre by E_η. The alteration τ' induces a morphism

$$\tau'_\eta : E_\eta \longrightarrow Y_\eta,$$

where Y_η denotes the generic fibre of $f : Y \to \mathbb{P}^n_k$. Since Y_η is smooth (because m is invertible in k), $\tau(E) \subset Z^{\text{sing}}$ implies that

$$\tau'_\eta(E_\eta) \subset D_\eta,$$

where D_η denotes the generic fibre of $f|_D : D \to \mathbb{P}^n_k$, and where we recall that $D \subset Y$ denotes the exceptional divisor of the blow-up $Y = Bl_P Z \to Z$. As explained above, D is given by setting $z_0 = 0$ in (21) and so $D_\eta \subset \mathbb{P}^r_{k(\mathbb{P}^n)}$ is the hypersurface over $k(\mathbb{P}^n)$, given by

$$\sum_{i=1}^{r} c_i(x_1, \ldots, x_n) z_i^m + (-1)^n x_1 x_2 \cdots x_n z_{r+1}^m = 0.$$

Hence, D_η is a subvariety of the hypersurface $X_{x_1,\ldots,x_n} \subset \mathbb{P}^{2^n-1}_L$ over $L = k(\mathbb{P}^n)$ from Corollary 4.2 and so the natural map $E_\eta \to D_\eta$ induced by τ'_η induces a morphism $\iota : E_\eta \to X_{x_1,\ldots,x_n}$ of L-varieties. It thus follows from Corollary 4.2 that

$$(x_1, \ldots, x_n) \in \ker(K^M_n(k(\mathbb{P}^n))/m \longrightarrow K^M_n(k(\mathbb{P}^n)(E_\eta))/m).$$

Since $k(\mathbb{P}^n)(E_\eta) = k(E)$, we conclude by mapping this via (2) to cohomology that

$$\tau^*\gamma|_E = 0 \in H^n(k(E), \mathbb{Z}/m).$$

Altogether, this shows that the hypersurface $Z \subset \mathbb{P}^{N+1}_k$ of degree d satisfies the assumption on the special fibre in the degeneration technique of Proposition 6.1 and so any integral hypersurface which degenerates to Z (in the sense of Section 2.2) has torsion order divisible by m. This applies in particular to very general hypersurfaces of degree $d = m + n$ in $\mathbb{P}^{N+1}_k$ (see Section 2.2), which concludes the proof in the case where $d = m + n$.

If $d > n + m$, then a very general hypersurface of degree d in $\mathbb{P}^{N+1}_k$ degenerates to the union of Z from above with $\{x_0^{d-m-n} = 0\}$. Note that the preimage $f^{-1}\{x_0 = 0\} \subset Y$ does not dominate $\mathbb{P}^n_k$. Using item (2) in Theorem 5.3, we thus conclude as before that for any subvariety $E \subset \tau^{-1}(Z^{\text{sing}} \cup \{x_0 = 0\})$, $\tau^*\gamma|_E = 0$. Hence, Proposition 6.1 applies and we get $m \mid e$ as before. This concludes the proof of the theorem. ☐

Proof of Theorem 1.5. Let us now assume that $k = \mathbb{C}$ and let $m, n \geq 2$ and $N \geq 3$ be integers with $\log_2(m+1) \leq n \leq N + 1 - m$. We aim to construct a rationally connected smooth complex projective variety X such that $H^n_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/m)$ contains an element of order m. Replacing X by a product with projective space, we see that it suffices to deal with the case where

$$N = n - 1 + m \quad \text{and} \quad 2^n \geq m + 1.$$

Let $r := m-1$. Then $r \leq 2^n - 2$ and so we may consider the projective variety $Y = Y_{n,r}$ such that $H^n_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/m)$ contains an element of order m from the proof of Theorem 1.1. There is a morphism $f : Y \to \mathbb{P}^n$ whose generic fibre is a smooth hypersurface of degree m in $\mathbb{P}^{r+1}_{\mathbb{C}(\mathbb{P}^n)}$, given by the equation (21). Since $m = r + 1$, a general fibre of f is Fano and so it is rationally chain connected, see [Cam92, KMM92] or [Kol96, Theorem V.2.1], and hence rationally connected because $k = \mathbb{C}$, see [KMM92] or [Kol96, Theorem IV.3.10]. It thus follows from the Graber–Harris–Starr theorem [GHS02] that any resolution X of Y is a rationally connected variety of dimension $N = n + r$. Since X is birational to Y, $H^n_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/m) = H^n_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/m)$ contains an element of order m, as we want. This concludes the proof of Theorem 1.5. □

8. Cyclic covers

Theorem 1.3 follows from the following slightly more general result.

Theorem 8.1. *Let k be an uncountable field and let m be an integer that is invertible in k. Let $N \geq 3$ be an integer and write $N = n + r$ with $2^{n-1} - 2 \leq r \leq 2^n - 2$. Consider a cyclic $m : 1$ cover $X \to \mathbb{P}^N_k$ branched along a very general hypersurface of degree d with $m \mid d$. If $d \geq n + 2m - 2$, then the torsion order of X is divisible by m.*

Proof. Replacing k by its algebraic closure, we may assume that k is algebraically closed. Let $x_0, \dots, x_n, y_2, \dots, y_{r+1}$ be homogeneous coordinates of $\mathbb{P}^N$ (note that we left out y_1). Let $d \geq n + 2m - 2$ be an integer that is divisible by m. Let $g \in k[x_0, \dots, x_n]$ be the polynomial from (20) and consider the cyclic $m : 1$ cover $Z \to \mathbb{P}^N$ branched along the hypersurface $\{F = 0\} \subset \mathbb{P}^N$ given by

$$F := x_1^{m-1} g \cdot x_0^{d-\deg(g)-m+1} + x_1^{m-1} \sum_{i=2}^{r} x_0^{d-2m+1-\deg c_i} c_i y_i^m + (-1)^n x_0^{d-m-n+1} x_2 \dots x_n y_{r+1}^m,$$

where $c_i \in k[x_1, \dots, x_n]$ is as in (8). (Since $\deg(g) = m\lceil \frac{n+1}{m} \rceil$ and $\deg(c_i) < n$ for all $i \geq r$, the condition $d \geq n + 2m - 2$ ensures that the exponents of x_0 are non-negative.) The cyclic cover $Z \to \mathbb{P}^N$ is given by the equation $y_1^m = F$, where y_1 is a new variable.

Note that Z contains the $(r-1)$-dimensional hyperplane $P := \{x_0 = \dots = x_n = y_1 = 0\} \subset Z$. The blow-up $Y := Bl_P Z$ admits a morphism $f : Y \to \mathbb{P}^n$ given by projection to the x_i-coordinates. In suitable weighted projective space, Y is given by the global equation

$$y_1^m = x_1^{m-1} g \cdot x_0^{d-\deg(g)-m+1} y_0^m + x_1^{m-1} \sum_{i=2}^{r} x_0^{d-2m+1-\deg c_i} c_i y_i^m + (-1)^n x_0^{d-m-n+1} x_2 \dots x_n y_{r+1}^m,$$

and the exceptional divisor of the blow-up $Y \to Z$ is given by $y_0 = 0$. Multiplying the above equation with x_1 and absorbing x_1^m into the y_i variables whenever possible, we find after setting $x_0 = y_0 = 1$ that Y is birational to the affine hypersurface in $\mathbb{A}^{N+1}$, given by

$$x_1 y_1^m = g + \sum_{i=2}^{r} c_i y_i^m + (-1)^n x_1 x_2 \dots x_n y_{r+1}^m.$$

Since $c_1 = -x_1$ by (8), this is exactly the hypersurface used in the proof of Theorem 1.1. The same argument as in the proof of that result now shows that

$$\gamma := f^*(x_1, \dots, x_n) \in H^n_{nr}(k(Y)/k, \mathbb{Z}/m)$$

is an unramified class of order m. Moreover, for any alteration $\tau' : Y' \to Y$, the induced alteration $\tau : Y' \to Z$ has the property that for any subvariety $E \subset \tau^{-1}(Z^{\mathrm{sing}})$, $\tau^*\gamma|_E = 0$. Hence, Proposition 6.1 implies that m divides the torsion order of a cyclic $m : 1$ cover of $\mathbb{P}^N_k$, branched along a very general hypersurface of degree d, as the latter specializes to Z above, see Section 2.2. This concludes the proof. ☐

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