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Seminar *Logarithmic Algebraic Geometry*

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Introduction and philosophy. Many moduli spaces of smooth objects in algebraic geometry are not proper, which has to do with the fact that many interesting objects can degenerate. Probably the most familiar example is the (coarse) moduli space of elliptic curves over a field k, which is the affine line $\mathbb{A}^1_k$. It is also well-known that this moduli space can be compactified to $\mathbb{P}^1_k$ with a non-smooth curve that has a node as singularity. Quite generally, when looking for compactifications of moduli spaces of smooth objects, one wants to compactify with objects that are necessarily singular, but whose singularities can be controlled.

Here is the classical framework: let R be a local and complete DVR with field of fractions K and let X be a smooth and proper variety over K. Then, it is expected that after possibly replacing K by a finite extension and R by its integral closure in this extension, then X admits a proper and flat model over R, whose special fiber is *semi-stable*. If X is a curve or an Abelian variety or if R is of equal characteristic zero, then this expectation is known to be true, by theorems on *semi-stable reduction*.

Here, a scheme Y over a field k is said to be *semi-stable* of dimension d if it is equal to a finite union $Y = \bigcup Y_i$, such that each Y_i is a smooth and irreducible scheme of dimension d over k and such that étale locally around every closed point Y looks like the spectrum of

$$k[t_1, ..., t_d]/(t_1 \cdots t_r)$$

for some $0 \leq r \leq d$. (There is more than one definition of semi-stable in the literature, but this one will do for the moment.) If $r \geq 1$ at some closed point $y \in Y$, then the sheaf of Kähler differentials $\Omega_{Y/k}$ is not locally free $\mathcal{O}_Y$-module of rank d at y, that is, Y is not smooth over k. Now, one can define an $\mathcal{O}_Y$-module $\Omega^{\log}_{Y/k}$ that contains $\Omega_{Y/k}$ and that is generated étale locally by rational differential forms

$$d\log t_i := \frac{dt_i}{t_i} \quad \text{if } i \leq r \quad \text{and} \quad dt_i \quad \text{if } i > r$$

in the above notation. It turns out that $\Omega^{\log}_{Y/k}$ is locally free of rank d, that is, with respect to these *log differentials*, Y looks as if it was smooth over k.

The crucial notion of this seminar is that of a *log structure*, which is short for *logarithmic structure*: by definition, a log structure on a scheme $(Y, \mathcal{O}_Y)$ is a morphism of sheaves of commutative monoids $\alpha : M \to \mathcal{O}_Y$, where the monoid structure on $\mathcal{O}_Y$ is multiplication, and where $\alpha^{-1}(\mathcal{O}_Y^\times) \to \mathcal{O}_Y^\times$ is assumed to be an isomorphism. This gives rise to the category of *log schemes*. Once set up properly, there exist log versions of morphisms, Kähler differentials, smoothness, deformation theory, moduli spaces, etc.

The crucial observation is the following: if $Y = \bigcup_i Y_i$ is a semi-stable scheme over k as above, then there exist quite natural log structures on Y and $\operatorname{Spec} k$, such that the structure morphism $Y \to \operatorname{Spec} k$ is a log morphism that is *log smooth* with respect to these log structures. (In fact, the sheaf of log differentials with respect to this log structure coincides with $\Omega^{\log}_{Y/k}$ mentioned above.)

To quote Kato from the introduction of [FKato00]: (...) The motivating philosophy is that, since log smoothness includes some degenerating objects like semi-stable reductions, etc., the moduli space of log smooth objects should be already compactified, once its existence has been established. (...)

Here is an example that illustrates this philosophy: for integers $g \geq 2$ and $n \geq 0$, there exists a moduli space $\mathcal{M}_{g,n}$ for n-marked points of smooth curves of genus g. By a theorem of Deligne and Mumford [D-M69], this space can be compactified using semi-stable curves. As shown by Kato [FKato00], this compactification $\overline{\mathcal{M}}_{g,n}$ can also be interpreted as a moduli space of log smooth curves and there even exists a log structure on $\overline{\mathcal{M}}_{g,n}$ with respect to which this moduli space is log smooth.

Prerequisites. Algebraic geometry at the level of Hartshorne's textbook and the basics of deformation theory at the level of Sernesi's textbook. Ideally, a little bit of background in Grothendieck topologies, such as the étale site, and it would help if one has seen some moduli spaces before. Knowing a little bit about Deligne-Mumford stacks is not necessary, but might be helpful for the last talks.

Contents of the seminar. In the first part of the seminar, we will set up and develop the machinery of log geometry, such as log structures, log morphisms, log differentials, and log deformation theory. Here, we will follow [LOG] and [Ogus19] - the former gives a good overview (mostly without proofs) and the latter gives all the details (and much more). The lecture notes [Abr14] might be a good addition.

In the second part, we will discuss two applications of log geometry:

(1) After recalling semi-stable reduction of curves and examples following [Liu02] and [H-M98], we will rephrase these results via log geometry. Then, we will discuss Kato's interpretation [FKato00] of the Deligne-Mumford compactification [D-M69] of the moduli space $\mathcal{M}_{g,n}$ of n-marked curves of genus g via log geometry.
(2) As a second application of log geometry, we will discuss separable rational connectedness of general Fano complete intersections via log degenerations following Chen and Zhu [C-Z14].

TALK 0: Introduction and survey
(Christian Liedtke)

This talk s a small introduction into the topics of the seminar, which surveys and motivates the main objects. Note that this talk will not take place in Raitenhaslach, but at the first meeting, where the talks will be distributed.

TALK 1: Monoids, log structures, and log schemes
(Yukihide Nakada)

Introduce the basic notions of log geometry: monoids, prelog structures, log structures, charts of log structures, and some special log structures, such as fine, integral, saturated log structures. Introduce the notions of [LOG, Section 2]. It is very important to discuss Example 2.7, Example 2.8, and Example 2.11 of [LOG].

TALK 2: Logarithmic smoothness
(Claudia Stadlmayr)

Introduce strict log morphisms, infinitesimal extensions, and log differentials as in [LOG, Section 3]. Although important (but maybe not surprising), [LOG, Proposition 3.4] should be mentioned only briefly. On the other hand, [LOG, Example 3.7] is central and should be discussed in some detail.

Introduce log smoothness and state Kato's theorem [LOG, Theorem 3.11]. It would be nice to say a couple of words about its proof, which is not in [LOG], but which can be found in [KKato89, Theorem 3.5] or [Ogus19, Chapter IV]. It would be very nice to explain [LOG, Remark 3.12.(b)] (maybe with some more details), which links logarithmic geometry to toric geometry.

Finally, [LOG, Proposition 3.14] should be explained including maybe a couple of words concerning the proof.

TALK 3: Logarithmic deformation theory
(Matthias Paulsen)

Introduce integral log morphisms, state and sketch the proof of [LOG, Proposition 3.18] - the proof is not in [LOG], but can be found in [KKato89, Section 4.5] or [Ogus19, Chapter IV]. The main result of this talk is [LOG, Theorem 3.20] and it would be nice to say a couple of words about the proof, which can be found in [KKato89, Section 3.14].

If time permits, one could also briefly discuss the logarithmic cotangent complex from [Ols05].

TALK 4: Semi-stable reduction of curves
(Feng Hao)

Give an overview over (potential) (semi-)stable reduction of curves. For definitions, statement, and proofs, one could follow [Liu02, Chapter 10] or the original sources, which can be found in loc.cit. These results should be illustrated by examples, which can be found, for example, in [H-M98, Chapter 3, Section C]. One could also briefly (!) mention that at least in characteristic zero, there are very general semi-stable redution theorems available, such as [KKMSD] and [A-K00]. Note: in this talk, there is no log geometry.

TALK 5: Log smooth curves
(Daniel Boada)

Introduce log smooth curves following [LOG, Section 4] or [FKato00]. Discuss [LOG, Example 4.6], [LOG, Theorem 4.7], and [LOG, Proposition 4.9]. It would be very interesting to sketch as much as possible from the proof of [LOG, Theorem 4.7], which is not in [LOG], but which can be found in [FKato00].

TALK 6: Log structures on stable curves
(Roberto Laface)

The goal of this talk is to explain [LOG, Theorem 4.11] and possibly a little bit of the proof: roughly speaking, it says that on a marked stable curve, there exists a "minimal" log structure that makes this curve log smooth.

TALK 7: Moduli of curves
(Stefan Schreieder)

Given a survey over the existence of a moduli space $\mathcal{M}_{g,n}$ of n-marked smooth curves of genus g and the compactification $\overline{\mathcal{M}}_{g,n}$ of Deligne and Mumford [D-M69]. To do so, discuss deformations of n-marked and smooth (resp. stable) curves of genus g. Mention representability of the deformation functor by a smooth Deligne-Mumford stack $\mathcal{M}_{g,n}$ (resp. $\overline{\mathcal{M}}_{g,n}$). This talk is a survey of [D-M69] and it might be a good idea to consult [H-M98, Chapter 4] for intuition and pictures. Note: in this talk, there is no log geometry.

TALK 8: Moduli of log smooth curves
(Oliver Gregory)

This talk might be a little bit technical and might be more of a survey: introduce log structures on stacks. Discuss the stack of basic stable log curves of type (g, n). Shortly introduce the natural log structure on the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. Construct the morphism F and discuss [LOG, Theorem 4.13]. It might be a good idea to consult [FKato00].

TALK 9: Very free curves on Fano varieties
(Frank Gounelas)

This final talk gives another application of log geometry: in characteristic zero, smooth Fano varieties are rationally connected, see, for example, [Kol96, Chapter IV and V]. Using log degenerations, Chen and Zhu [C-Z14] proved that a general Fano complete intersection in $\mathbb{P}^N$ and in arbitrary characteristic contains a very free rational curve and is thus separably rationally connected.

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