

THE KUGA–SATAKE CONSTRUCTION UNDER DEGENERATION

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ABSTRACT. We extend the Kuga–Satake construction to the case of limit mixed Hodge structures of K3 type. We use this to study the geometry and Hodge theory of degenerations of Kuga–Satake abelian varieties associated to polarized variations of K3 type Hodge structures over the punctured disc.

1. INTRODUCTION

The Kuga–Satake construction [KS] associates to any polarized rational weight two Hodge structure V of K3 type (i.e. with $V^{2,0} \cong \mathbb{C}$) an abelian variety A , well-defined up to isogeny, with an embedding of Hodge structures

$$\text{ks} : V(1) \hookrightarrow \text{End}(H^1(A, \mathbb{Q})) \subset H^2(A \times A, \mathbb{Q})(1).$$

Here the rational vector space $H^1(A, \mathbb{Q})$ is given by the Clifford algebra $\mathcal{Cl}(V, q)$, associated to the polarization q of V . If $V \subset H^2(X, \mathbb{Q})$ for some smooth projective variety X (e.g. a K3 surface or, more generally, a projective hyperkähler manifold), then the above embedding corresponds to a Hodge class on $X \times A \times A$. Even though algebraicity of that class is known only in very few cases, the Kuga–Satake construction is expected to give a close relation between the geometry of X and the associated Kuga–Satake abelian variety A . The Kuga–Satake construction has been generalized by Voisin [V] and by Kurnosov, Verbitsky and the second author [KSV].

The Kuga–Satake abelian varieties may be seen as analogues of intermediate Jacobians for K3 type Hodge structures. If we apply this construction to families of K3 surfaces, it is important to understand its behaviour near the points where the surfaces become singular. For intermediate Jacobians this is a classical and well-studied subject, see e.g. [Cl], [Sa], [Zu]. This motivates the study of the Kuga–Satake construction under degeneration. We start from a polarized variation of Hodge structures (VHS) of K3 type over the punctured disc $\Delta^* = \Delta \setminus \{0\}$; up to a finite base change, we may assume that the monodromy of the underlying local system is unipotent, see [Sch, Lemma 4.5]. Geometrically such a VHS comes from a flat projective family over the unit disc $\pi : \mathcal{X} \rightarrow \Delta$, smooth over Δ^* and with general fibre for instance a projective hyperkähler manifold or an abelian surface. Applying the Kuga–Satake construction to the VHS over Δ^* we get a polarized variation of weight one Hodge structures. Using a result of Borel [Bor] and the semi-stable reduction theorem [KKMSD], we obtain (up to a finite base change) a semi-stable family $\alpha : \mathcal{A} \rightarrow \Delta$ of abelian varieties. If the fibre \mathcal{X}_0 is singular, then we expect \mathcal{A}_0 to be singular as well, and it is natural to wonder how to describe such singular fibres. When \mathcal{X} is a family of K3 surfaces, this question is well understood since the work of Kulikov [Ku]: there are essentially three types of fibres \mathcal{X}_0 that can appear. The analogous question for the associated Kuga–Satake varieties is however much more subtle.

Date: March 27, 2019.

2010 *Mathematics Subject Classification.* primary 14D06, 14D07; secondary 14D05.

The authors are supported by the SFB/TR 45 ‘Periods, Moduli Spaces and Arithmetic of Algebraic Varieties’ of the DFG (German Research Foundation).

As a first approximation, one can try to describe the mixed Hodge structure of \mathcal{A}_0 . Via the Clemens–Schmid exact sequence, this is essentially governed by the limit mixed Hodge structure on a smooth fibre \mathcal{A}_t , where $t \in \Delta^*$ is some base point. The limit mixed Hodge structure $H_{lim}^1(\mathcal{A}_t, \mathbb{Q})$ has as underlying vector space $H^1(\mathcal{A}_t, \mathbb{Q})$ and is given by two filtrations: the Hodge filtration F_{lim}^\bullet on $H^1(\mathcal{A}_t, \mathbb{C})$ and the weight filtration W_\bullet on $H^1(\mathcal{A}_t, \mathbb{Q})$. The weight filtration is induced by the monodromy operator given by parallel transport along a loop in Δ^* . The limit Hodge filtration is determined by the germ of the variation of Hodge structures on \mathcal{A}_s near zero. This filtration is not canonical and depends on the choice of the local coordinate on Δ^* ; in what follows we fix the local coordinate and do not consider this dependence.

In this paper we show that the limit mixed Hodge structure $H_{lim}^1(\mathcal{A}_t, \mathbb{Q})$ depends only on the limit mixed Hodge structure attached to the initial VHS of K3 type. Roughly speaking, this says that the limit of the Hodge structures on the Kuga–Satake side does not depend on the individual Hodge structures $H^1(\mathcal{A}_s, \mathbb{Q})$ for $s \in \Delta^*$, but only on the limit of the Hodge structures on the K3 side. In fact, we show more generally that the Kuga–Satake construction extends from the case of pure Hodge structures of K3 type to the case of limit mixed Hodge structures of K3 type (cf. Section 2.1.3 below) and this construction is compatible with the geometric situation described above.

All Hodge structures that we consider in this paper are rational and have level ≤ 2 . We consider the following categories, for precise definitions see Section 2 below:

- (PVHS_{K3}) = category of polarized VHS of K3 type and with unipotent monodromy over Δ^* ;
- (VHS_{Ab}) = category of polarizable VHS of weight one and with unipotent monodromy over Δ^* ;
- (PMHS_{K3}) = category of polarized MHS of K3 type;
- (MHS_{Ab}) = category of polarizable MHS of weight one.

The polarized MHS here are in the sense of [CKS, Definition 2.26], see also Section 2.1.3 below. For us it will be important that polarizations are fixed for only half of the above categories, see Remark 2.1 below. The above categories are related by the following diagram of functors:

$$(1.1) \quad \begin{array}{ccc} (\text{PVHS}_{K3}) & \xrightarrow{\text{KS}} & (\text{VHS}_{Ab}) \\ \downarrow \text{Lim}_{K3} & & \downarrow \text{Lim}_{Ab} \\ (\text{PMHS}_{K3}) & & (\text{MHS}_{Ab}) \end{array}$$

where KS denotes the Kuga–Satake functor described above, and Lim_{K3} and Lim_{Ab} denote the functors that compute the corresponding limit mixed Hodge structures. We will denote by (PMHS_{K3}^{lim}) the essential image of Lim_{K3} , i.e. the full subcategory of (PMHS_{K3}) whose objects are in the image of Lim_{K3} .

Theorem 1.1. *There exists a functor $\text{KS}^{lim} : (\text{PMHS}_{K3}^{lim}) \rightarrow (\text{MHS}_{Ab})$ which makes the diagram (1.1) commutative; that is,*

$$\text{Lim}_{Ab} \circ \text{KS} = \text{KS}^{lim} \circ \text{Lim}_{K3}.$$

Moreover, for any polarized limit mixed Hodge structure $\bar{V} = (V, q, F^\bullet, N) \in (\text{PMHS}_{K3}^{lim})$ of K3 type, there exists an embedding of mixed Hodge structures

$$\text{ks} : \bar{V}(1) \hookrightarrow \text{End}(\bar{H}), \quad \text{where } \bar{H} = \text{KS}^{lim}(\bar{V}).$$

Let us emphasize that the functor KS^{lim} is defined only on the essential image of Lim_{K3} . We do not claim that it extends in any natural way to the whole category (PMHS_{K3}) .

For any polarized limit mixed Hodge structure $\bar{V} = (V, q, F^\bullet, N) \in (\text{PMHS}_{K3}^{lim})$ of K3 type, the limit mixed Hodge structure $\text{KS}^{lim}(\bar{V}) = (H, F_{KS}^\bullet, N_{KS})$ of abelian type in the above theorem has as underlying

\mathbb{Q} -vector space the Clifford algebra $H := Cl(V, q)$. The Hodge filtration on H is determined by the half-dimensional subspace

$$F_{KS}^1 H_{\mathbb{C}} := F^2 V_{\mathbb{C}} \cdot H_{\mathbb{C}},$$

where $F^2 V_{\mathbb{C}} \cdot H$ denotes the right ideal in the Clifford algebra $H_{\mathbb{C}}$, generated by the one-dimensional subspace $F^2 V_{\mathbb{C}}$, cf. Section 3.2 below. This description of the Hodge filtration relies on a simple description of the usual Kuga–Satake construction, which might be of independent interest, see Lemma 3.4 below. Finally, the nilpotent operator N_{KS} is zero if $N = 0$ and it is given by left multiplication with the element $f_1 f_2 \in H$, where $f_1, f_2 \in V$ form a basis of $\text{im}(N : V \rightarrow V)$, which turns out to be two-dimensional whenever $N \neq 0$, cf. Proposition 4.1 below. As usual, the weight filtration W_{\bullet} on H is then given by $W_0 H = \text{im}(N_{KS})$, $W_1 H = \ker(N_{KS})$ and $W_2 H = H$, see e.g. [Mo].

Together with the Clemens–Schmid sequence, the above result allows us to classify completely the first cohomology groups of degenerations of Kuga–Satake varieties. Consider a polarized VHS $\bar{\mathcal{V}} = (\mathcal{V}, q, \mathcal{F}^{\bullet}) \in (\text{PVHS}_{K3})$ and let $T = e^N$ be the monodromy transformation of the local system \mathcal{V} . It is known that $N^3 = 0$. We will say that $\bar{\mathcal{V}}$ is of type I if $N = 0$, of type II if $N \neq 0$, $N^2 = 0$ and of type III if $N^2 \neq 0$. The rank of $\bar{\mathcal{V}}$ is by definition the rank of the local system \mathcal{V} .

Theorem 1.2. *Let $\bar{\mathcal{V}} = (\mathcal{V}, q, \mathcal{F}^{\bullet}) \in (\text{PVHS}_{K3})$ be a polarized VHS of K3 type of rank r , with associated semi-stable family of Kuga–Satake varieties $\alpha : \mathcal{A} \rightarrow \Delta$, smooth over the punctured disc Δ^* and with central fibre \mathcal{A}_0 . Then one of the following holds.*

- (1) *If $\bar{\mathcal{V}}$ is of type I, then α is birational to a smooth projective family of abelian varieties over Δ and the mixed Hodge structure on $H^1(\mathcal{A}_0, \mathbb{Q})$ is pure with Hodge numbers*

$$\begin{array}{cc} 0 & \\ 2^{r-1} & 2^{r-1} \\ 0 & \end{array}$$

- (2) *If $\bar{\mathcal{V}}$ is of type II, then the mixed Hodge structure on $H^1(\mathcal{A}_0, \mathbb{Q})$ has Hodge numbers*

$$\begin{array}{cc} 0 & \\ 2^{r-2} & 2^{r-2} \\ 2^{r-2} & \end{array}$$

- (3) *If $\bar{\mathcal{V}}$ is of type III, then the mixed Hodge structure on $H^1(\mathcal{A}_0, \mathbb{Q})$ is of weight zero with Hodge numbers*

$$\begin{array}{ccc} 0 & & \\ 0 & & 0 \\ & 2^{r-1} & \end{array}$$

We remark that the semi-stable family α in the above theorem exists after base change and its restriction to Δ^* is unique up to isogeny, see Section 2.3 below.

A natural invariant associated to any semi-stable family $\alpha : \mathcal{A} \rightarrow \Delta$ of Kuga–Satake abelian varieties is the dual complex Σ of the central fibre \mathcal{A}_0 . By [ABW], the homotopy type of Σ depends only on the restriction of \mathcal{A} to the punctured disc Δ^* and so it does not depend on the chosen semi-stable model.

As a consequence of our results, we are able to compute the rational cohomology algebra of Σ explicitly.

Corollary 1.3. *In the notation of Theorem 1.2, let Σ be the dual complex of the central fibre \mathcal{A}_0 .*

- (1) *If $\bar{\mathcal{V}}$ is of type I, then Σ is homotopy equivalent to a point.*

- (2) If $\bar{\mathcal{V}}$ is of type II, then $H^*(\Sigma, \mathbb{Q}) \simeq H^*(T, \mathbb{Q})$, where $T = (S^1)^w$ is a real torus of real dimension $w = 2^{r-2}$. In particular, the central fibre \mathcal{A}_0 has at least 2^{r-2} components.
- (3) If $\bar{\mathcal{V}}$ is of type III, then $H^*(\Sigma, \mathbb{Q}) \simeq H^*(T, \mathbb{Q})$, where $T = (S^1)^w$ is a real torus of real dimension $w = 2^{r-1}$. In particular, the central fibre \mathcal{A}_0 has at least 2^{r-1} components.

As a consequence of our construction, the semi-stable family $\alpha: \mathcal{A} \rightarrow \Delta$ of Kuga–Satake abelian varieties from Theorem 1.2 will automatically be projective over the disc, that is $\mathcal{A} \subset \mathbb{P}^n \times \Delta$ for some n , see the discussion in Section 2.3. Under this assumption, it is possible to replace the analytic family α by an algebraic one over the formal disc $\text{Spec}R$, where $R = \mathbb{C}[[t]]$ denotes the ring of formal power series. Indeed, $\mathcal{A} \subset \mathbb{P}^n \times \Delta$ is cut out by finitely many polynomials whose coefficients are complex analytic functions on the disc and so we may regard them as elements of R , cf. Section 2.3 below. This defines a projective scheme $\mathcal{A}_R \subset \mathbb{P}_R^n$, flat over the formal disc $\text{Spec}R$. The base change of \mathcal{A}_R to the fraction field $K = \mathbb{C}((t))$ will be denoted by \mathcal{A}_K ; it is an abelian variety over K . Note that the special fibre $\mathcal{A}_R \times_R \mathbb{C}$ of $\mathcal{A}_R \rightarrow \text{Spec}R$ coincides with the central fibre \mathcal{A}_0 of $\mathcal{A} \rightarrow \Delta$.

Blowing-up a smooth subvariety in the smooth locus of \mathcal{A}_0 shows that the semi-stable model $\mathcal{A} \rightarrow \Delta$ is not unique, in fact, the isomorphism type of each component of the special fibre as well as the number of such components depends on the particular choice of a semi-stable model. Instead of semi-stable models, it is thus more convenient to work with the Néron model, which is canonical. The Néron model $\mathcal{A}^{\text{Né}} \rightarrow \text{Spec}R$ is a quasi-projective commutative group scheme over R with generic fibre $\mathcal{A}_K^{\text{Né}}$ isomorphic to \mathcal{A}_K and such that for any smooth separated R -scheme X with generic fibre $X_K := X \times_R K$, any morphism $X_K \rightarrow \mathcal{A}_K^{\text{Né}}$ extends to a unique R -morphism $X \rightarrow \mathcal{A}^{\text{Né}}$. For abelian varieties over K , Néron models exist and are unique up to unique isomorphism, cf. [BLR].

As pointed out to us by Johannes Nicaise, Halle and Nicaise [HN2] showed that the special fibre $\mathcal{A}_0^{\text{Né}} := \mathcal{A}^{\text{Né}} \times_R \mathbb{C}$ of the Néron model can be described in terms of the limit mixed Hodge structure of the family $\mathcal{A} \rightarrow \Delta$. Moreover, according to [HN1] and [HN2], the limit mixed Hodge structure essentially determines the motivic zeta function $Z_{\mathcal{A}_K} \in (K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]][[T]]$ of the abelian variety \mathcal{A}_K . This zeta function is a formal power series with coefficients in the Grothendieck ring of varieties localized at the class of the affine line; for a precise definition of $Z_{\mathcal{A}_K}$, see [HN1, Section 2]. For K3 surfaces over K with semi-stable model over R , this zeta function has already been computed by Stewart and Vologodsky [SV].

Corollary 1.4. *Let $\alpha: \mathcal{A} \rightarrow \Delta$ be the family of Kuga–Satake abelian varieties as in the Theorem 1.2. Then the special fibre $\mathcal{A}_0^{\text{Né}}$ of the Néron model is a disjoint union of isomorphic components $\mathcal{A}_0^{\text{Né}} = \bigsqcup_i A$, where A is a semi-abelian variety given by an extension*

$$0 \longrightarrow (\mathbb{C}^*)^w \longrightarrow A \longrightarrow B \longrightarrow 0,$$

where B is an abelian variety with rational weight one Hodge structure isomorphic to $gr_1^W(\text{KS}^{\text{lim}}(\bar{\mathcal{V}}))$ and w is the dimension of $gr_0^W(\text{KS}^{\text{lim}}(\bar{\mathcal{V}}))$. In particular:

- (1) If $\bar{\mathcal{V}}$ is of type I, then $w = 0$ and A is an abelian variety of dimension 2^{r-1} ;
- (2) If $\bar{\mathcal{V}}$ is of type II, then $w = 2^{r-2}$ and B is of dimension 2^{r-2} ;
- (3) If $\bar{\mathcal{V}}$ is of type III, then $w = 2^{r-1}$, B is trivial and A is an algebraic torus.

The motivic zeta-function (see [HN1]) of the abelian variety \mathcal{A}_K is given by

$$Z_{\mathcal{A}_K}(T) = N[B](\mathbb{L} - 1)^w \sum_{d \geq 1} d^w T^d,$$

where N is the number of connected components of the special fibre $\mathcal{A}_0^{\text{Né}}$ and $[B]$ denotes the class of B in $K_0(\text{Var}_{\mathbb{C}})$.

By [JM, Theorem 1.4] (see also [BLR]), there is a close relationship between the semi-stable model \mathcal{A} and the Néron model $\mathcal{A}^{\text{Né}}$, which we describe next. To this end, note that the canonical bundle $K_{\mathcal{A}}$ is trivial away from the central fibre \mathcal{A}_0 and so we can write

$$K_{\mathcal{A}} \sim \sum_i a_i \mathcal{A}_{0i},$$

where \mathcal{A}_{0i} denote the components of \mathcal{A}_0 and we may assume that $a_i \geq 0$ for all i and $a_i = 0$ for at least one i . We then define the support of $K_{\mathcal{A}}$ as

$$\text{supp}(K_{\mathcal{A}}) := \bigcup_{i: a_i \neq 0} \mathcal{A}_{0i}.$$

Note that $\text{supp}(K_{\mathcal{A}})$ is empty if $K_{\mathcal{A}}$ is trivial; such models are called good models in [JM].

We further consider

$$\mathcal{A}^{\text{sm}} := \mathcal{A}_R \setminus \mathcal{A}_0^{\text{sing}}$$

and

$$\mathcal{A}^{\text{mo}} := \mathcal{A}^{\text{sm}} \setminus \text{supp}(K_{\mathcal{A}}).$$

By [JM, Theorem 1.4] we have an open immersion $\mathcal{A}^{\text{mo}} \hookrightarrow \mathcal{A}^{\text{Né}}$ that gives a one-to-one correspondence between components of the special fibres. When \mathcal{A} is a good model, we have $\mathcal{A}^{\text{Né}} \simeq \mathcal{A}^{\text{sm}} \simeq \mathcal{A}^{\text{mo}}$. Using this description, Corollary 1.4 implies the following.

Corollary 1.5. *In the notation of Theorem 1.2, any component \mathcal{A}_{0i} of the central fibre \mathcal{A}_0 that is not contained in the support of $K_{\mathcal{A}}$ is up to birational equivalence given as follows.*

- (1) *If $\bar{\mathcal{V}}$ is of type I, then \mathcal{A}_{0i} is birational to an abelian variety of dimension 2^{r-1} , which up to isogeny is uniquely determined by the pure weight one Hodge structure $gr_1^W(\text{KS}^{\text{lim}}(\bar{\mathcal{V}}))$.*
- (2) *If $\bar{\mathcal{V}}$ is of type II, then \mathcal{A}_{0i} is birational to a $\mathbb{P}^{2^{r-2}}$ -bundle over an abelian variety of dimension 2^{r-2} , which up to isogeny is uniquely determined by the pure weight one Hodge structure $gr_1^W(\text{KS}^{\text{lim}}(\bar{\mathcal{V}}))$.*
- (3) *If $\bar{\mathcal{V}}$ is of type III, then \mathcal{A}_{0i} is rational.*

2. PRELIMINARIES

2.1. The four categories. We denote by Δ the unit disc, $\Delta^* = \Delta \setminus \{0\}$ the punctured unit disc and $\tau: \mathbb{H} \rightarrow \Delta^*$ denotes the universal covering, where \mathbb{H} is the upper half-plane. We further fix a base point $t \in \Delta^*$ and consider the following categories.

2.1.1. The category $(\text{PVHS}_{\text{K3}})$ of polarized variations of Hodge structures of K3 type over Δ^* : its objects are triples $(\mathcal{V}, q, \mathcal{F}^\bullet)$, where \mathcal{V} is a local system of \mathbb{Q} -vector spaces over Δ^* , $q \in \Gamma(\Delta^*, S^2\mathcal{V}^*)$ is a polarization, and \mathcal{F}^\bullet is a decreasing filtration of $\mathcal{V} \otimes \mathcal{O}_{\Delta^*}$ by holomorphic subbundles. These structures must satisfy the following conditions: the monodromy transformation of \mathcal{V} is unipotent, q has signature $(2, r-2)$, the filtration is of the form $0 = \mathcal{F}^3\mathcal{V} \subset \mathcal{F}^2\mathcal{V} \subset \mathcal{F}^1\mathcal{V} \subset \mathcal{F}^0\mathcal{V} = \mathcal{V} \otimes \mathcal{O}_{\Delta^*}$, where $\mathcal{F}^2\mathcal{V}$ is of rank one, $\mathcal{F}^1\mathcal{V} = (\mathcal{F}^2\mathcal{V})^\perp$, and for any non-vanishing local section σ of $\mathcal{F}^2\mathcal{V}$, we have $q(\sigma, \sigma) = 0$ and $q(\sigma, \bar{\sigma}) > 0$. The morphisms in $(\text{PVHS}_{\text{K3}})$ are morphisms of local systems that preserve polarizations and filtrations. Since they have to preserve the polarizations, all the morphisms are embeddings of Hodge structures.

2.1.2. The category (VHS_{Ab}) of polarizable variations of Hodge structures of abelian type over Δ^* : the objects are pairs $(\mathcal{H}, \mathcal{F}^\bullet)$, where \mathcal{H} is a local system of \mathbb{Q} -vector spaces over Δ^* with unipotent monodromy and \mathcal{F}^\bullet is a filtration of the bundle $\mathcal{H} \otimes \mathcal{O}_{\Delta^*}$. The filtration has to be of the form $0 = \mathcal{F}^2\mathcal{H} \subset \mathcal{F}^1\mathcal{H} \subset \mathcal{F}^0\mathcal{H} = \mathcal{H} \otimes \mathcal{O}_{\Delta^*}$ and has to admit a polarization, that is, an element $\omega \in \Gamma(\Delta^*, \Lambda^2\mathcal{H}^*)$, such that $\mathcal{F}^1\mathcal{H}$ is Lagrangian (in particular its rank is half of the rank of \mathcal{H}) and for any non-vanishing local section v of $\mathcal{F}^1\mathcal{H}$ we have $\sqrt{-1}\omega(v, \bar{v}) > 0$. The morphisms in (VHS_{Ab}) are morphisms of local systems that preserve the filtrations. We do not fix the polarizations and do not require that the morphisms preserve them.

2.1.3. The category $(\text{PMHS}_{\text{K3}})$ of polarized mixed Hodge structures of K3 type (cf. [CKS, Definition 2.26]). Its objects are tuples (V, q, F^\bullet, N) where V is a \mathbb{Q} -vector space, $q \in S^2V^*$ a non-degenerate symmetric bilinear form, $N \in \mathfrak{so}(V, q)$ is a nilpotent operator satisfying $N^3 = 0$, and F^\bullet a filtration on $V_{\mathbb{C}}$ of the form $0 = F^3V \subset F^2V \subset F^1V \subset F^0V = V_{\mathbb{C}}$. Moreover, these structures must satisfy the following conditions. If we denote by W_\bullet the increasing filtration on V defined by N (cf. [Mo, pp. 106]), with the convention that the non-zero graded components have degrees from 0 to 4, then $(V, F^\bullet, W_\bullet)$ is a mixed Hodge structure; the subspace F^2V is one-dimensional and for $0 \neq \sigma \in F^2V$ we have $q(\sigma, \sigma) = 0$, $q(\sigma, \bar{\sigma}) > 0$; the Hodge structures on the primitive parts $P_{2+i} = \ker(N^{i+1}: \text{gr}_{2+i}^W V \rightarrow \text{gr}_{-i}^W V)$ are polarized by $q(-, N^i-)$, $i = 0, 1, 2$. Morphisms in $(\text{PMHS}_{\text{K3}})$ are morphisms of vector spaces preserving polarizations, filtrations, and commuting with the nilpotent operators. In particular, they are morphisms of mixed Hodge structures.

2.1.4. The category (MHS_{Ab}) of polarizable mixed Hodge structures of abelian type: objects are tuples (H, F^\bullet, N) , where H is a \mathbb{Q} -vector space, F^\bullet is a filtration on $H_{\mathbb{C}}$ with $0 = F^2H_{\mathbb{C}} \subset F^1H_{\mathbb{C}} \subset F^0H_{\mathbb{C}} = H_{\mathbb{C}}$ and N is a nilpotent operator on H with $N^2 = 0$ which admits a polarization in the sense of [CKS, Definition 2.26] (analogous to the K3 type case described above). If W_\bullet denotes the weight filtration associated to the operator N (cf. [Mo, pp. 106]), then the tuple $(H, F^\bullet, W_\bullet)$ is a mixed Hodge structure of type $(0, 0) + (1, 0) + (0, 1) + (1, 1)$. Morphisms in (MHS_{Ab}) are morphisms of \mathbb{Q} -vector spaces which preserve F^\bullet and N .

Remark 2.1. It is important for the construction in Theorem 1.1, that the morphisms in the categories $(\text{PVHS}_{\text{K3}})$ and $(\text{PMHS}_{\text{K3}})$ preserve polarizations, while the morphisms in (VHS_{Ab}) and (MHS_{Ab}) do not. This reflects the fact that Kuga-Satake abelian varieties are polarizable, but the polarizations on them are not canonical, in particular they are not compatible with the automorphisms of the initial polarized K3 type Hodge structures.

2.2. The functors computing limit mixed Hodge structures. Let us recall the definition of the functors Lim_{K3} and Lim_{Ab} from the diagram (1.1). For the detailed discussion of limit mixed Hodge structures, we refer to [Sch, §4] and [CKS].

2.2.1. We start from the definition of Lim_{K3} . Consider an object $(\mathcal{V}, q, \mathcal{F}^\bullet) \in (\text{PVHS}_{\text{K3}})$. Let $V = \mathcal{V}_t$ be the fibre of \mathcal{V} above the fixed base point $t \in \Delta^*$. Then q is an element of S^2V^* invariant under the monodromy transformation $T = e^N$ of \mathcal{V} , so that $N \in \mathfrak{so}(V, q)$ and $T \in \text{SO}(V, q)$.

Definition 2.2. *The extended period domain $\hat{\mathcal{D}}_{\text{K3}} \subset \mathbb{P}(V_{\mathbb{C}})$ for (V, q) is the quadric defined by q . The period domain for (V, q) is the open subset $\mathcal{D}_{\text{K3}} = \{[v] \in \hat{\mathcal{D}}_{\text{K3}} \mid q(v, \bar{v}) > 0\}$.*

Our terminology follows Schmid [Sch], where (extended) period domains are defined in terms of flag varieties. We give a slightly different definition, since in our case the period domain \mathcal{D}_{K3} is a Hermitian symmetric domain of noncompact type and $\hat{\mathcal{D}}_{\text{K3}}$ is its compact dual.

Recall that $\tau: \mathbb{H} \rightarrow \Delta^*$ is the universal covering, where \mathbb{H} is the upper half-plane. The local system $\tau^*\mathcal{V}$ is trivial and so $\tau^*\mathcal{F}^2$ defines a morphism $\Phi_{K3}: \mathbb{H} \rightarrow \mathcal{D}_{K3}$ that satisfies the relation $\Phi_{K3}(z+1) = T \cdot \Phi_{K3}(z)$. Define $\Psi_{K3}(z) = e^{-zN} \cdot \Phi_{K3}(z)$, then $\Psi_{K3}(z+1) = \Psi_{K3}(z)$. By the nilpotent orbit theorem [Sch, 4.9], there exists a limit $[v_{lim}] = \lim_{\text{Im}(z) \rightarrow +\infty} \Psi_{K3}(z) \in \hat{\mathcal{D}}_{K3}$. We define the limit Hodge filtration on $V_{\mathbb{C}}$ by setting $F_{lim}^2 V$ to be the subspace spanned by v_{lim} and $F_{lim}^1 V = (F_{lim}^2 V)^\perp$. It follows from the SL_2 -orbit theorem [Sch] that $(V, q, F_{lim}^\bullet, N) \in (\text{PMHS}_{K3})$ and we define $\text{Lim}_{K3}(\mathcal{V}, q, \mathcal{F}^\bullet) = (V, q, F_{lim}^\bullet, N)$. A morphism in (PVHS_{K3}) is an embedding of polarized variations of Hodge structures, and its image under Lim_{K3} is the corresponding embedding of polarized mixed Hodge structures.

2.2.2. The definition of Lim_{Ab} is analogous. Consider $(\mathcal{H}, \mathcal{F}^\bullet) \in (\text{VHS}_{\text{Ab}})$. Let $H = \mathcal{H}_t$ be the fibre of \mathcal{H} above $t \in \Delta^*$ and let $T' = e^{N'}$ be the monodromy transformation. Fix a T' -invariant element $\omega \in \Lambda^2 H^*$ defining a polarization, so that $T' \in \text{Sp}(H, \omega)$.

Definition 2.3. *The extended period domain for (H, ω) is the Grassmannian of Lagrangian subspaces $\hat{\mathcal{D}}_{\text{Ab}} = \text{LGr}(H_{\mathbb{C}}, \omega)$. The period domain for (H, ω) is the open subset*

$$\mathcal{D}_{\text{Ab}} = \{[H^{1,0}] \in \hat{\mathcal{D}}_{\text{Ab}} \mid \sqrt{-1}\omega(v, \bar{v}) > 0, \forall v \in H^{1,0}\}.$$

Analogously to the case of K3 type Hodge structures, $\hat{\mathcal{D}}_{\text{Ab}}$ is the compact dual of \mathcal{D}_{Ab} .

We have a morphism $\Phi_{\text{Ab}}: U \rightarrow \mathcal{D}_{\text{Ab}}$ and define $\Psi_{\text{Ab}}(z) = e^{-zN} \cdot \Phi_{\text{Ab}}(z)$. By the nilpotent orbit theorem [Sch, 4.9] there exists a limit $[H_{lim}^{1,0}] = \lim_{\text{Im}(z) \rightarrow +\infty} \Psi_{\text{Ab}}(z) \in \hat{\mathcal{D}}_{\text{Ab}}$. We define the limit Hodge filtration on $H_{\mathbb{C}}$ by setting $F_{lim}^1 H = H_{lim}^{1,0}$. The weight filtration W_\bullet is determined by the operator N . It follows from the SL_2 -orbit theorem that $(H, F_{lim}^\bullet, W_\bullet) \in (\text{MHS}_{\text{Ab}})$. We note that $H_{lim}^{1,0}$ does not depend on the choice of ω , since the limit can be taken in the Grassmannian of all half-dimensional subspaces in $H_{\mathbb{C}}$. Hence we can define $\text{Lim}_{\text{Ab}}(\mathcal{H}, \mathcal{F}^\bullet) = (H, F_{lim}^\bullet, W_\bullet)$. To define the action of Lim_{Ab} on morphisms, let $\varphi: (\mathcal{H}_1, \mathcal{F}_1^\bullet) \rightarrow (\mathcal{H}_2, \mathcal{F}_2^\bullet)$ be a morphism of VHS on Δ^* . We aim to show that the restriction of φ to the fibre above $t \in \Delta^*$ defines a morphism of mixed Hodge structures. Since the objects in (VHS_{Ab}) are semi-simple, it is enough to consider the case where φ is an automorphism of a simple object. In this case the claim is clear.

2.3. The category (VHS_{Ab}) and families of abelian varieties over the disc. Up to a finite covering of Δ^* , any variation of Hodge structures $(\mathcal{H}, \mathcal{F}^\bullet) \in (\text{VHS}_{\text{Ab}})$ defines a family of abelian varieties $\alpha': \mathcal{A}^* \rightarrow \Delta^*$ (unique up to isogeny). We would like to fill in a central fibre, producing a flat projective family $\alpha: \mathcal{A} \rightarrow \Delta$, smooth over Δ^* . This is always possible by Borel's theorem, as we are going to recall next.

Consider the fibre $H = \mathcal{H}_t$ of the local system \mathcal{H} and fix a polarization $\omega \in \Lambda^2 H^*$. Consider the universal covering $\tau: \mathbb{H} \rightarrow \Delta^*$. The VHS $(\mathcal{H}, \mathcal{F}^\bullet)$ induces a period map $\bar{p}: \mathbb{H} \rightarrow \mathcal{D}_{\text{Ab}}$, where \mathcal{D}_{Ab} is the period domain for (H, ω) defined above (Definition 2.2). Up to a finite covering of Δ^* , we may assume that the monodromy T' is contained in a torsion-free arithmetic subgroup $\Gamma \subset \text{Sp}(H, \omega)$. Then we get a holomorphic map $p: \Delta^* \rightarrow \mathcal{D}_{\text{Ab}}/\Gamma$. By [BB], $\mathcal{D}_{\text{Ab}}/\Gamma$ is quasi-projective, and \mathcal{A}^* is defined as the pull-back of a polarized family of abelian varieties over $\mathcal{D}_{\text{Ab}}/\Gamma$. By [Bor], the map p extends to a map $\bar{p}: \Delta \rightarrow X$, where X is some projective compactification of $\mathcal{D}_{\text{Ab}}/\Gamma$. Then \mathcal{A} can be defined as the pull-back along \bar{p} of a projective family over X . By the semi-stable reduction theorem [KKMSD], we may also assume that α is semi-stable.

Applying this procedure to $\text{KS}(\mathcal{V}, q, \mathcal{F}^\bullet)$ for some $(\mathcal{V}, q, \mathcal{F}^\bullet) \in (\text{PVHS}_{K3})$, we get a degenerating family of abelian varieties over Δ , which we call the Kuga-Satake family attached to $(\mathcal{V}, q, \mathcal{F}^\bullet)$. Note that this

family is not canonically defined, and the central fibre \mathcal{A}_0 is not unique. However, the invariants of \mathcal{A}_0 that we compute do not depend on any particular choices.

By the above construction, the family $\alpha: \mathcal{A} \rightarrow \Delta$ is projective over the disc, meaning that \mathcal{A} is a complex submanifold in $\mathbb{P}^n \times \Delta$ for some n , and α is induced by projection to the second factor. This gives a flat family of subvarieties in the projective space, and this family is the pull-back of the universal family over the Hilbert scheme via a holomorphic map $f: \Delta \rightarrow \text{Hilb}(\mathbb{P}^n)$. Since the analytic Hilbert scheme is given by analytification of the algebraic Hilbert scheme (see e.g. [Si, Proposition 5.3]), we see that \mathcal{A} is defined by finitely many polynomials with complex-analytic coefficients, and we can define the abelian variety \mathcal{A}_K , which is the fibre of \mathcal{A} over the spectrum of $K = \mathbb{C}((t))$. This can be used to apply the construction of [BLR] and obtain the Néron model $\mathcal{A}^{\text{Né}} \rightarrow \text{Spec}R$ of \mathcal{A} , where $R = \mathbb{C}[[t]]$ and $\text{Spec}R$ is the formal disc.

3. THE FUNCTOR KS

In this section we define the functor KS from the diagram (1.1). Consider an object $(\mathcal{V}, q, \mathcal{F}^\bullet) \in (\text{PVHS}_{K3})$. Let $V = \mathcal{V}_t$ be the fibre of \mathcal{V} above the fixed base point $t \in \Delta^*$ and let $T = e^N$ be the monodromy operator. Then $q \in S^2V^*$ is a T -invariant element and we have $N \in \mathfrak{so}(V, q)$ and $T \in \text{SO}(V, q)$. To define $\text{KS}(\mathcal{V}, q, \mathcal{F}^\bullet)$, we need to define a local system \mathcal{H} and a Hodge filtration on $\mathcal{H} \otimes \mathcal{O}_{\Delta^*}$. We will denote by $\mathcal{Cl}(V, q)$ the Clifford algebra of (V, q) , i.e. the quotient of the tensor algebra $T^\bullet(V)$ by the ideal generated by $v \otimes v - q(v, v)$. Similarly, $\mathcal{Cl}^+(V, q)$ denotes the even Clifford algebra of (V, q) , i.e. the sub-algebra of $\mathcal{Cl}(V, q)$, generated by even tensors.

3.1. The local system. The fibre $H = \mathcal{H}_t$ of \mathcal{H} above t is defined to be the \mathbb{Q} -vector space $H := \mathcal{Cl}(V, q)$. To obtain \mathcal{H} , we need to define a monodromy transformation $T' \in \text{GL}(H)$, and we do this by lifting T to the group $\text{Spin}(V, q)$.

Recall that the Clifford group is defined as $G = \{g \in \mathcal{Cl}^\times(V, q) \mid \alpha(g)Vg^{-1} = V\}$, where α is the parity involution. We have the norm homomorphism $\mathcal{N}: G \rightarrow \mathbb{Q}$, $g \mapsto g\bar{g}$, where $g \mapsto \bar{g}$ is the natural anti-involution of the Clifford algebra. By definition, $\text{Spin}(V, q) = \ker(\mathcal{N}) \cap \mathcal{Cl}^+(V, q)$. Recall that we have the embedding

$$(3.1) \quad \eta': \Lambda^2V \hookrightarrow \mathcal{Cl}(V, q), \quad x \wedge y \mapsto \frac{1}{4}(xy - yx).$$

If we use the isomorphism

$$(3.2) \quad \Lambda^2V \xrightarrow{\sim} \mathfrak{so}(V, q), \quad v \wedge w \mapsto q(w, -)v - q(v, -)w$$

to identify Λ^2V with $\mathfrak{so}(V, q)$, then η' induces a homomorphism of Lie algebras

$$(3.3) \quad \eta: \mathfrak{so}(V, q) \longrightarrow \mathcal{Cl}(V, q),$$

which induces an isomorphism of $\mathfrak{so}(V, q)$ with the sub Lie algebra of $\mathcal{Cl}(V, q)$, spanned by the commutators of elements of V .

Define

$$N' := \eta(N) \quad \text{and} \quad T' := e^{N'},$$

and let \mathcal{H} be the local system with fibre $\mathcal{H}_t = H$ and monodromy T' . Define the embedding

$$(3.4) \quad \text{ks}' : V \hookrightarrow \text{End}(H), \quad \text{ks}'(v) = (w \mapsto vw).$$

Lemma 3.1. *The operator T' is the unique unipotent lift of T to $\text{Spin}(V, q)$, and ks' induces an embedding of local systems $\text{ks}' : \mathcal{V} \hookrightarrow \text{End}(\mathcal{H})$.*

Proof. For any $g \in \mathfrak{so}(V, q)$ and any $x \in V \subset Cl(V, q)$, we have $g \cdot x = \text{ad}_{\eta(g)}x = [\eta(g), x] \in V$. For $a = \eta(g)$, $x \in V$ and a formal variable s ,

$$e^{sa}xe^{-sa} = \sum_{i \geq 0} \frac{1}{i!} \text{ad}_a^i(x)s^i = \sum_{i \geq 0} \frac{1}{i!} (g^i \cdot x)s^i.$$

When a is nilpotent, e^{sa} is a polynomial in s , and we see that $e^{sa} \in \text{Spin}(V, q)$. Hence the two possible lifts of T to the Spin-group are $T' = e^{N'}$ and $T'' = -e^{N'}$, and only T' is unipotent.

To prove that ks' defines a map of local systems, we note that the monodromy transformation of $\text{End}(\mathcal{H})$ is the conjugation by T' , and by the formula above, $T \cdot v = T'v(T')^{-1}$ for any $v \in V$. This concludes the lemma. \square

Remark 3.2. Since the monodromy operator T on V respects the bilinear form q , it induces a natural operator on the Clifford algebra $Cl(V, q)$, given by $v_1 \cdots v_k \mapsto T(v_1) \cdots T(v_k)$. However, that operator does not coincide with the monodromy operator $T' = e^{N'}$ defined above. In fact, while T' restricts to the action of T on the image of V inside $\text{End}(H)$, such a compatibility statement fails for the “naive operator” on $H = Cl(V, q)$, considered above. Similarly, while T' is unipotent of index two, the above operator does not have that property.

3.2. The Hodge filtration. In this section we show (see Proposition 3.6) that the Kuga–Satake construction extends to the extended period domain $\hat{\mathcal{D}}_{K3}$. To this end, we introduce a description of the Kuga–Satake construction (see Lemma 3.4), which might be of independent interest.

3.2.1. Let (V, q) be a Hodge structure of K3 type. Let $v \in V^{2,0}$ be a generator with $q(v, \bar{v}) = 2$. Consider $e_1 = \text{Re}(v)$, $e_2 = \text{Im}(v)$ and $I_v = e_1e_2$ (product in the Clifford algebra). We have $I_v^2 = -1$ and the left multiplication by I_v defines a complex structure on the vector space $H_{\mathbb{R}}$. The corresponding weight one Hodge structure on H is called the Kuga–Satake Hodge structure, see [Huy, Chapter 4].

The Kuga–Satake Hodge structures are polarized and the polarization can be defined as follows. Pick a pair of elements $a_1, a_2 \in V$, such that $q(a_1, a_1) > 0$, $q(a_2, a_2) > 0$ and $q(a_1, a_2) = 0$. Let $a = a_1a_2 \in H$. Define the two-form $\omega \in \Lambda^2 H^*$ by $\omega(x, y) = \text{Tr}(xa\bar{y})$, where we use the trace in the Clifford algebra. It is known that either ω or $-\omega$ defines a polarization, see [Huy, Chapter 4].

Lemma 3.3. *The two-form ω defined above is $\text{Spin}(V, q)$ -invariant.*

Proof. Let $g \in \text{Spin}(V, q)$. Then, using $\bar{g}g = 1$, we get: $\omega(gx, gy) = \text{Tr}(gxa\bar{y}\bar{g}) = \text{Tr}(xa\bar{y}\bar{g}g) = \text{Tr}(xa\bar{y}) = \omega(x, y)$. \square

Since the monodromy of the local system constructed above is contained in $\text{Spin}(V, q)$, the above lemma shows that the polarization given by $\pm\omega$ is monodromy-invariant. Hence it will define a polarization of the corresponding VHS.

Recall from Definitions 2.2 and 2.3 the extended period domains $\hat{\mathcal{D}}_{K3}$ and $\hat{\mathcal{D}}_{Ab}$ for (V, q) and (H, ω) . Note that both $\hat{\mathcal{D}}_{K3}$ and $\hat{\mathcal{D}}_{Ab}$ have natural $\text{Spin}(V_{\mathbb{C}}, q)$ -actions: on $\hat{\mathcal{D}}_{K3}$ the action comes from the canonical homomorphism $\text{Spin}(V_{\mathbb{C}}, q) \rightarrow \text{SO}(V_{\mathbb{C}}, q)$, and on $\hat{\mathcal{D}}_{Ab}$ the action is induced from the structure of a left $Cl(V_{\mathbb{C}}, q)$ -module on $H_{\mathbb{C}}$. This action is faithful and preserves the form ω and so it gives rise to an embedding $\text{Spin}(V_{\mathbb{C}}, q) \hookrightarrow \text{Sp}(H_{\mathbb{C}}, \omega)$.

As a point in \mathcal{D}_{Ab} , the Kuga–Satake Hodge structure is given by a subspace $H^{1,0} \subset H_{\mathbb{C}}$ of dimension $\frac{1}{2}d$, where $d := \dim_{\mathbb{Q}}(H)$.

Lemma 3.4. *Let (V, q) be a K3 type Hodge structure. Then the corresponding Kuga–Satake Hodge structure of weight one on $H = Cl(V)$ is given by the half-dimensional subspace $H^{1,0} = V^{2,0} \cdot Cl(V_{\mathbb{C}}, q)$.*

Proof. Let $v \in V^{2,0}$ be a generator with $q(v, \bar{v}) = 2$, and let $e_1 = \operatorname{Re}(v)$ and $e_2 = \operatorname{Im}(v)$. Then, $H^{1,0}$ is the i -eigenspace of the operator $I_v : H_{\mathbb{C}} \rightarrow H_{\mathbb{C}}$, given by left multiplication with $I_v = e_1 e_2$. It is thus clear that $H^{1,0}$ is a right ideal. To see $V^{2,0} \subset H^{1,0}$, note that $q(e_1, e_1) = q(e_2, e_2) = 1$ and $q(e_1, e_2) = 0$ and so we get the following identity in the Clifford algebra $H = Cl(V, q)$:

$$e_1 e_2 \cdot v = -e_2 + i e_1 = i v.$$

Hence, $I_v \cdot v = i v$, which proves $V^{2,0} \subset H^{1,0}$. To conclude the lemma, it now suffices to check

$$\dim(v Cl(V_{\mathbb{C}}, q)) = \frac{1}{2} d.$$

This follows from the next lemma, which concludes the proof. \square

Lemma 3.5. *For any $[v] \in \hat{\mathcal{D}}_{K3}$, the right ideal $v \cdot Cl(V_{\mathbb{C}}, q)$ has dimension $\frac{1}{2} d$.*

Proof. Choose an element $w \in V_{\mathbb{C}}$ with $q(w, w) = 0$, $q(v, w) = 1$ and denote by V' the orthogonal complement to $W = \langle v, w \rangle$. Then $Cl(V_{\mathbb{C}}, q) \simeq Cl(W, q|_W) \otimes Cl(V', q|_{V'})$ and $v Cl(V_{\mathbb{C}}, q) = (v Cl(W, q|_W)) \otimes Cl(V', q|_{V'})$. Since $Cl(W, q|_W) = \langle 1, v, w, vw \rangle$ and $v^2 = 0$, we see that $v Cl(W, q|_W) = \langle v, vw \rangle$ and the claim follows. \square

Lemmas 3.4 and 3.5 show that the Kuga–Satake correspondence extends to $\hat{\mathcal{D}}_{K3}$, an observation that we will need later.

Proposition 3.6. *There exists a $\operatorname{Spin}(V_{\mathbb{C}}, q)$ -equivariant morphism $\kappa : \hat{\mathcal{D}}_{K3} \rightarrow \hat{\mathcal{D}}_{Ab}$, whose restriction to \mathcal{D}_{K3} maps a K3 type Hodge structure to the corresponding Kuga–Satake Hodge structure.*

Proof. Over $\hat{\mathcal{D}}_{K3}$ we have the universal subbundle $\mathcal{O}_{\hat{\mathcal{D}}_{K3}}(-1) \hookrightarrow V \otimes \mathcal{O}_{\hat{\mathcal{D}}_{K3}}$. Using the embedding $V \hookrightarrow Cl(V, q) = H$ we get a subbundle $\mathcal{O}_{\hat{\mathcal{D}}_{K3}}(-1) \hookrightarrow H \otimes \mathcal{O}_{\hat{\mathcal{D}}_{K3}}$. Lemma 3.5 shows that $\mathcal{O}_{\hat{\mathcal{D}}_{K3}}(-1) \cdot (H \otimes \mathcal{O}_{\hat{\mathcal{D}}_{K3}}) \subset H \otimes \mathcal{O}_{\hat{\mathcal{D}}_{K3}}$ is a subbundle of rank $\frac{1}{2} d$. This defines the morphism κ . To check that it is $\operatorname{Spin}(V_{\mathbb{C}}, q)$ -equivariant, let $g \in \operatorname{Spin}(V_{\mathbb{C}}, q)$. We have $g \cdot [v] = [g v g^{-1}]$ where we use multiplication in the Clifford algebra. Then $\kappa(g \cdot [v]) = [g v g^{-1} Cl(V_{\mathbb{C}}, q)] = [g v Cl(V_{\mathbb{C}}, q)] = g \cdot [v Cl(V_{\mathbb{C}}, q)] = g \cdot \kappa([v])$. \square

3.2.2. We go back to the construction of the Hodge filtration on the local system \mathcal{H} . Recall that we have started from an object $(\mathcal{V}, q, \mathcal{F}^{\bullet}) \in (\operatorname{PVHS}_{K3})$ and $V = \mathcal{V}_t$ is the fibre of the local system \mathcal{V} at the base point $t \in \Delta^*$. We consider the universal covering $\tau : \mathbb{H} \rightarrow \Delta^*$ and the subbundle $\tau^* \mathcal{F}^2 \mathcal{V}$ of the trivial bundle $V \otimes \mathcal{O}_U$. This defines a period map $p_{K3} : \mathbb{H} \rightarrow \mathcal{D}_{K3}$. Let $p_{Ab} = \kappa \circ p_{K3}$ and let E be the pull-back of the universal vector bundle over $\hat{\mathcal{D}}_{Ab}$. Since κ is $\operatorname{Spin}(V, q)$ -equivariant by Proposition 3.6 and the monodromy operator T' lies in $\operatorname{Spin}(V, q)$, the bundle E descends to a subbundle $\tilde{\mathcal{F}}^1 \subset \mathcal{H} \otimes \mathcal{O}_{\Delta^*}$. This defines a Hodge filtration $\tilde{\mathcal{F}}^{\bullet}$ on \mathcal{H} . Note that by construction we get a polarizable variation of Hodge structures. We define $\operatorname{KS}(\mathcal{V}, q, \mathcal{F}^{\bullet}) = (\mathcal{H}, \tilde{\mathcal{F}}^{\bullet})$. The action of KS on morphisms is clear from the construction (note that morphisms in $(\operatorname{PVHS}_{K3})$ preserve polarizations, so they induce embeddings of the corresponding Clifford algebras).

Lemma 3.7. *The formula (3.4) defines an embedding of variations of Hodge structures*

$$\operatorname{ks}' : \mathcal{V}(1) \hookrightarrow \operatorname{End}(\mathcal{H}).$$

Proof. We use Lemma 3.1. It remains to check that ks' respects the Hodge filtration, but this is true pointwise (see [Huy, Chapter 4]). \square

4. WEIGHT FILTRATION OF THE KUGA–SATAKE MHS

Let $(\mathcal{V}, q, \mathcal{F}^\bullet) \in (\text{PVHS}_{\text{K3}})$ and $(\mathcal{H}, \tilde{\mathcal{F}}^\bullet) = \text{KS}(\mathcal{V}, q, \mathcal{F}^\bullet)$. Let $V = \mathcal{V}_t$ be the fibre of \mathcal{V} and $H = \text{Cl}(V, q)$ be the fibre of \mathcal{H} at $t \in \Delta^*$ (recall the definition of the functor KS in Section 3). Recall from Section 2 that the weight filtration on $\overline{H} = \text{Lim}_{\text{Ab}}(\mathcal{H}, \tilde{\mathcal{F}}^\bullet)$ is determined by the logarithm N' of the monodromy operator T' . In this section we compute the dimensions of its components.

Recall that the monodromy operator of \mathcal{V} is $T = e^N$, where $N \in \mathfrak{so}(V, q)$ satisfies $N^3 = 0$. The limit weight filtration on V is of the form $0 = W_{-1}V \subset W_0V \subset W_1V \subset W_2V \subset W_3V \subset W_4V = V$, and the components can be described as follows:

$$W_0V = \text{im}(N^2), \quad W_3V = \ker(N^2), \quad W_1V = N(W_3V), \quad W_2V = N^{-1}(W_0V).$$

We recall that $(\mathcal{V}, q, \mathcal{F}^\bullet)$ is of type I if $N = 0$, of type II if $N \neq 0$, $N^2 = 0$ and of type III if $N^2 \neq 0$. The case of type I is trivial and we do not consider it.

The limit weight filtration on H is of the form $0 = W_{-1}H \subset W_0H = \text{im}(N') \subset W_1H = \ker(N') \subset W_2H = H$. We denote by r the rank of \mathcal{V} and by $d = 2^r$ the rank of \mathcal{H} .

Proposition 4.1. *Suppose that $N \neq 0$. Then the image of the operator $N : V \rightarrow V$ is two-dimensional. Under the homomorphism $\eta : \mathfrak{so}(V, q) \rightarrow \text{Cl}(V, q)$ from (3.3), the image $N' = \eta(N)$ is proportional to the bivector corresponding to the image of $N : V \rightarrow V$. Moreover,*

- 1) *in the type II case: $\dim(W_0H) = \frac{1}{4}d$ and $\dim(W_1H) = \frac{3}{4}d$;*
- 2) *in the type III case: $W_0H = W_1H$ and $\dim(W_0H) = \frac{1}{2}d$.*

Proof. In the computations below, we will use the following fact: if $v \in V$ is an isotropic element, then we can find a hyperbolic plane $\langle x, y \rangle \subset V$ with $q(x, x) = 2$, $q(y, y) = -2$, $q(x, y) = 0$ and $2v = x + y$. To prove this fact, note that q is non-degenerate and so we can choose an element $z \in V$ with $q(v, z) = 1$. The element $w := z - \frac{1}{2}q(z, z)v$ has then the property that $q(w, w) = 0$ and $q(v, w) = 1$. Putting $x := v + w$ and $y := v - w$, we obtain a hyperbolic plane $\langle x, y \rangle = \langle v, w \rangle \subset V$ with $q(x, x) = 2$, $q(y, y) = -2$, $q(x, y) = 0$ and $2v = x + y$, as claimed. This proves the above fact.

Type II case. In this case $N \neq 0$, $N^2 = 0$, hence $W_0V = 0$, $W_1V = \text{im}(N)$, $W_2V = \ker(N)$ and $W_3V = V$. Since $\mathcal{F}^2\mathcal{V}$ is of rank one, the Hodge numbers of the limit mixed Hodge structure $\text{Lim}_{\text{K3}}(\mathcal{V}, q, \mathcal{F}^\bullet)$ are $h^{0,0} = h^{2,2} = 0$, $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 1$, $h^{1,1} = r - 4$. It follows that $\dim W_1V = 2$ and so the image of N is two-dimensional. For any $x, y \in V$, we have $q(Nx, Ny) = -q(x, N^2y) = 0$, so W_1V is an isotropic subspace.

Let now $e_1 \in V$ be a non-isotropic element. Since $W_1V = \text{im}(N)$ is a non-trivial isotropic subspace, we can by the above fact assume that e_1 is contained in a hyperbolic plane and $q(e_1, e_1) = 2$. Then Ne_1 is isotropic and $q(Ne_1, e_1) = 0$. By the above fact, applied to the isotropic vector $Ne_1 \in e_1^\perp$, we can find a hyperbolic plane $U \subset e_1^\perp$ with a basis $U = \langle e_2, e_3 \rangle$, $q(e_2, e_2) = 2$, $q(e_2, e_3) = 0$, $q(e_3, e_3) = -2$, such that $2Ne_1 = e_2 + e_3$. Then $q(Ne_2, e_2) = 0$, $q(Ne_2, e_3) = q(Ne_2, 2Ne_1 - e_2) = 0$, so that $Ne_2 = -Ne_3$ is orthogonal to U . We also have $q(2Ne_3, e_1) = -q(e_3, e_2 + e_3) = 2$. Let $e_4 = 2Ne_3 - e_1$. Then $q(e_1, e_4) = 0$, $q(e_4, e_4) = -2$, and $U' = \langle e_1, e_4 \rangle$ is a hyperbolic plane orthogonal to U .

Since $\text{im}(N) \subset U \oplus U'$, the orthogonal complement to $U \oplus U'$ is contained in the kernel of N . The action of N on $U \oplus U'$ is:

$$Ne_1 = -Ne_4 = \frac{1}{2}(e_2 + e_3); \quad Ne_3 = -Ne_2 = \frac{1}{2}(e_1 + e_4).$$

Under the isomorphism $\mathfrak{so}(V, q) \simeq \Lambda^2V$ from (3.2), the operator N is represented by $\frac{1}{4}(e_2 + e_3) \wedge (e_1 + e_4)$; indeed, both sides vanish on the orthogonal complement of $U \oplus U'$ and they agree on the basis e_1, e_2, e_3

and e_4 of $U \oplus U'$. Since U and U' are orthogonal to each other, we get via (3.1):

$$N' = \eta(N) = \frac{1}{8}(e_2 + e_3)(e_1 + e_4).$$

This is the bivector corresponding to $\text{im}(N)$.

It remains to compute the dimension of $W_0H = \text{im}(N')$ and $W_1H = \ker(N')$. Let $f_1 = \frac{1}{2}(e_1 + e_4)$ and $f_2 = \frac{1}{2}(e_2 + e_3)$. Then, $N' = \eta(N) = \frac{1}{2}f_2f_1 \in \mathcal{Cl}(V, q)$. Let V' be the orthogonal complement of $U \oplus U'$. Then $\mathcal{Cl}(V, q) \simeq \mathcal{Cl}(U \oplus U', q|_{U \oplus U'}) \otimes \mathcal{Cl}(V', q|_{V'})$. We have $N' \in \mathcal{Cl}(U \oplus U', q|_{U \oplus U'})$, so it is enough to find the image and the kernel of N' acting by left multiplication on $\mathcal{Cl}(U \oplus U', q|_{U \oplus U'}) \simeq \text{Mat}_4(\mathbb{Q})$. Via this last isomorphism, N' corresponds to a 4×4 matrix with the only non-zero element in the upper right corner. The kernel of this matrix is 3-dimensional and the image is 1-dimensional.

Type III case. In this case $N^2 \neq 0$, $N^3 = 0$. The Hodge numbers of the limit mixed Hodge structure $\text{Lim}_{K3}(\mathcal{V}, q, \mathcal{F}^\bullet)$ are $h^{0,0} = h^{2,2} = 1$, $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0$, $h^{1,1} = r - 2$. It follows that $W_0V = W_1V$, $W_2V = W_3V$ and $\dim(W_0V) = \dim(V/W_3V) = 1$.

Consider the isotropic subspace $V' = \text{im}(N)$, which contains $W_0V = \text{im}(N^2)$. Since $W_1V = N(W_3V)$ is one-dimensional and $W_3V = \ker(N^2)$ is of codimension one in V , we conclude that $\dim V' = 2$. Since q is non-degenerate, we can find $x, y \in V$ such that $q(Nx, Ny) = -q(x, N^2y) \neq 0$. It follows that $q|_{V'}$ is non-zero. Pick an element $v \in V'$ with $q(v, v) \neq 0$. Then, $v = Nv'$ for some $v' \in V$ and $q(v, N^2x) = 0$ for any $x \in V$. Hence, $W_0V \subset v^\perp$ and we get an orthogonal decomposition

$$V' = W_0V \oplus \langle v \rangle.$$

By the above fact, applied to $W_0V \subset v^\perp$, we can find a hyperbolic plane $U \subset v^\perp$, containing W_0V , together with a basis $U = \langle e_1, e_2 \rangle$, such that $q(e_1, e_1) = 2$, $q(e_1, e_2) = 0$, $q(e_2, e_2) = -2$ and $W_0V = \langle e_1 + e_2 \rangle$.

Let $V'' = U \oplus \langle v \rangle$, then $\text{im}(N) = \langle v, e_1 + e_2 \rangle \subset V''$ and so N preserves V'' . Since $e_1 + e_2 \in \ker(N)$, $Nv \in W_0V$ is nonzero. Hence, there is some $\alpha \in \mathbb{Q}$ such that $e_3 = \alpha v$ satisfies $Ne_3 = e_1 + e_2$. Using again $e_1 + e_2 \in \ker(N)$, we get $Ne_1 = -Ne_2$. From the equalities $q(Ne_1, e_1) = 0$ and $q(Ne_2, e_2) = 0$, we conclude that $Ne_1 = -Ne_2 \in U^\perp$ and so $Ne_1 = -Ne_2 = ae_3$ for some $a \in \mathbb{Q}$. To find the coefficient a , note that $aq(e_3, e_3) = q(Ne_1, e_3) = -q(e_1, Ne_3) = -2$, so that $a = -2/q(e_3, e_3)$. We can summarize:

$$Ne_1 = \frac{-2e_3}{q(e_3, e_3)}; \quad Ne_2 = \frac{2e_3}{q(e_3, e_3)}; \quad Ne_3 = e_1 + e_2.$$

Since $U \subset v^\perp$ and $q(v, v) \neq 0$, we have an orthogonal direct sum decomposition $V = V'' \oplus (V'')^\perp$. For any $x \in (V'')^\perp$ and $y \in V$, we have $q(Nx, y) = -q(x, Ny) = 0$, because $Ny \in V''$. Hence, $(V'')^\perp \subset \ker(N)$, and so the image of N under the isomorphism $\mathfrak{so}(V, q) \simeq \Lambda^2V$ from (3.2) is contained in $\Lambda^2(V'')$. Checking the evaluation on the basis e_1, e_2, e_3 of V'' , one sees that N is represented by $1/q(e_3, e_3)(e_1 + e_2) \wedge e_3 \in \Lambda^2V$. Using (3.1), we therefore get

$$N' = \eta(N) = \frac{1}{2q(e_3, e_3)}(e_1 + e_2)e_3,$$

because $e_1 + e_2$ and e_3 anti-commute in H . This shows that N' is proportional to the bi-vector which corresponds to the image of $N: V \rightarrow V$.

It remains to compute the dimension of $W_0H = \text{im}(N')$ and $W_1H = \ker(N')$. We have $N' = \frac{1}{2q(e_3, e_3)}(e_1 + e_2)e_3 \in \mathcal{Cl}(V, q)$. Since the element e_3 is invertible in the Clifford algebra, we have $\text{im}(N') = \ker N' = (e_1 + e_2)\mathcal{Cl}(V, q)$ – the right ideal generated by the isotropic vector $e_1 + e_2$. The dimension of this ideal is $\frac{1}{2}d$. This finishes the proof of the proposition. \square

5. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. 1) We define the functor KS^{lim} . Let $\bar{V} = (V, q, F^\bullet, N) = \text{Lim}_{\text{KS}3}(\mathcal{V}, q, \mathcal{F}^\bullet)$. Let $H = \mathcal{C}l(V, q)$, $N' = \eta(N)$ (cf. Section 3). The subspace F^2V gives a point in the extended period domain $\hat{\mathcal{D}}_{K3}$ (see Definition 2.2). Let $[\tilde{F}^1H] = \kappa([F^2V])$, where κ is defined in Proposition 3.6. We need to check that $(H, \tilde{F}^\bullet, N')$ is a polarizable mixed Hodge structure.

Let $(\mathcal{H}, \tilde{\mathcal{F}}^\bullet) = \text{KS}(\mathcal{V}, q, \mathcal{F}^\bullet)$. As before, $\tau: \mathbb{H} \rightarrow \Delta^*$ is the universal covering. Recall the construction of limit mixed Hodge structures from Section 2. We have $\Phi_{K3}: \mathbb{H} \rightarrow \hat{\mathcal{D}}_{K3}$ and $\Psi_{K3}(z) = e^{zN} \cdot \Phi_{K3}(z)$. Then $\Phi_{Ab} = \kappa \circ \Phi_{K3}$ and since κ is $\text{Spin}(V_{\mathbb{C}}, q)$ -equivariant by Proposition 3.6, we have $\Psi_{Ab} = \kappa \circ \Psi_{K3}$. This implies that

$$\lim_{\text{Im}(z) \rightarrow +\infty} \Psi_{Ab}(z) = \kappa \left(\lim_{\text{Im}(z) \rightarrow +\infty} \Psi_{K3}(z) \right),$$

and thus $(H, \tilde{F}^\bullet, N') = \text{Lim}_{\text{Ab}} \circ \text{KS}(\mathcal{V}, q, \mathcal{F}^\bullet)$ is a polarizable mixed Hodge structure. We define $\text{KS}^{\text{lim}}(\bar{V}) = (H, \tilde{F}^\bullet, N')$ and automatically get $\text{Lim}_{\text{Ab}} \circ \text{KS} = \text{KS}^{\text{lim}} \circ \text{Lim}_{\text{KS}3}$.

2) Given $\bar{V} = (V, q, \mathcal{F}^\bullet, N)$ and $\bar{H} = (H, \tilde{F}^\bullet, N') = \text{KS}^{\text{lim}}(\bar{V})$, we define

$$\text{ks}: \bar{V}(1) \hookrightarrow \text{End}(\bar{H}), \quad v \mapsto f_v,$$

where $f_v(w) = vw$.

We need to check that the embedding ks is compatible with the weight and Hodge filtrations. For the weight filtration, it is enough to check compatibility with the action of the monodromy operators $T = e^N$ and $T' = e^{N'}$. This was done in Lemma 3.1.

For the Hodge filtration, one can use Lemma 3.7 and a straightforward limit argument. For completeness, we check compatibility directly. The Hodge filtrations on H^* and $\text{End}(H)$ have the following components:

$$\begin{aligned} F^{-1}H^* &= H^*, & F^0H^* &= \{\varphi \in H^* \mid \varphi|_{F^1H} = 0\}, \\ F^{-1}\text{End}(H) &= \text{End}(H), & F^0\text{End}(H) &= F^1H \otimes H^* + H \otimes F^0H^*, & F^1\text{End}(H) &= F^1H \otimes F^0H^*. \end{aligned}$$

Fix $v_0 \in V_{\mathbb{C}}$ with $q(v_0) = 0$. We have $F^1V(1) = \langle v_0 \rangle$ and $F^1H = v_0 \mathcal{C}l(V_{\mathbb{C}}, q)$ by Lemma 3.5. The image of f_{v_0} is contained in F^1H , and since v_0 is isotropic, f_{v_0} clearly annihilates F^1H . So we have $f_{v_0} \in F^1\text{End}(H)$.

Next consider $v \in F^0V(1)$, so that $q(v, v_0) = 0$. Choose a subspace $W \subset H$ complementary to F^1H and let $\text{pr}_W, \text{pr}_{F^1H}$ be the projectors. Then, $f_v = \text{pr}_{F^1H} \circ f_v + \text{pr}_W \circ f_v$ and $\text{pr}_{F^1H} \circ f_v \in F^1H \otimes H^*$. Also, $\text{pr}_W \circ f_v(v_0x) = \text{pr}_W(vv_0x) = -\text{pr}_W(v_0vx) = 0$, so $\text{pr}_W \circ f_v \in H \otimes F^0H^*$ and $f_v \in F^0\text{End}(H)$. This completes the proof. \square

Proof of Theorem 1.2. Consider the fixed base point $t \in \Delta^*$. The Clemens–Schmid sequence [Mo] for weight one yields an exact sequence

$$0 \longrightarrow H^1(\mathcal{A}_0, \mathbb{Q}) \longrightarrow H_{\text{lim}}^1(\mathcal{A}_t, \mathbb{Q}) \xrightarrow{N} H_{\text{lim}}^1(\mathcal{A}_t, \mathbb{Q}).$$

Since $\ker(N) = W_1 H_{\text{lim}}^1(\mathcal{A}_t, \mathbb{Q})$, we conclude

$$H^1(\mathcal{A}_0, \mathbb{Q}) \cong W_1 H_{\text{lim}}^1(\mathcal{A}_t, \mathbb{Q}).$$

The Hodge numbers of the mixed Hodge structure $H_{\text{lim}}^1(\mathcal{A}_t, \mathbb{Q})$ are computed via Proposition 4.1, which yields the claimed result. This concludes Theorem 1.2. \square

Proof of Corollary 1.3. By the definition of the weight filtration on the simple normal crossing variety \mathcal{A}_0 , we have $H^k(\Sigma, \mathbb{Q}) \cong W_0 H^k(\mathcal{A}_0, \mathbb{Q})$. The Clemens–Schmid sequence [Mo] further implies $W_0 H_{lim}^k(\mathcal{A}_t, \mathbb{Q}) \cong W_0 H^k(\mathcal{A}_0, \mathbb{Q})$ and so we conclude

$$H^k(\Sigma, \mathbb{Q}) \cong W_0 H_{lim}^k(\mathcal{A}_t, \mathbb{Q}).$$

Moreover, the isomorphism $H^k(\mathcal{A}_t, \mathbb{Q}) \cong \Lambda^k H^1(\mathcal{A}_t, \mathbb{Q})$ induces a canonical isomorphism of limit mixed Hodge structures

$$H_{lim}^k(\mathcal{A}_t, \mathbb{Q}) \cong \Lambda^k H_{lim}^1(\mathcal{A}_t, \mathbb{Q}),$$

see for instance [HN2, Proposition 6.1]. Putting both identities together, we obtain

$$H^k(\Sigma, \mathbb{Q}) \cong \Lambda^k(W_0 H_{lim}^1(\mathcal{A}_t, \mathbb{Q}))$$

By [KLSV, Lemma 6.16], the above isomorphisms are compatible with cup product. Hence,

$$H^*(\Sigma, \mathbb{Q}) \cong \Lambda^*(W_0 H_{lim}^1(\mathcal{A}_t, \mathbb{Q}))$$

is isomorphic to the rational cohomology algebra of a real torus of real dimension $2 \dim_{\mathbb{C}}(W_0 H_{lim}^1(\mathcal{A}_t, \mathbb{Q}))$. The corollary follows therefore from Theorem 1.2. \square

Proof of Corollary 1.4. Since $\mathcal{A}_0^{\text{Né}}$ is a group scheme over \mathbb{C} , all components are isomorphic to the component $A := \mathcal{A}_0^{\text{Né}}$ which contains the identity element. Since \mathcal{A} is a semi-stable model, the monodromy operator is unipotent of index two and so [SGA, IX.3.5] implies that the generic fibre \mathcal{A}_K of the Néron model has semi-abelian reduction; that is, A is a semi-abelian variety over \mathbb{C} . The Chevalley decomposition of A thus reads as follows

$$0 \longrightarrow T \longrightarrow A \longrightarrow B \longrightarrow 0,$$

where $T \cong (\mathbb{C}^*)^w$ is an algebraic torus and B is an abelian variety, cf. [HN2, (3.1)]. In [HN2, Theorem 6.2], Halle and Nicaise describe the dimensions of T and B as well as the rational Hodge structure of B in terms of the limit mixed Hodge structure $H_{lim}^1(\mathcal{A}_t, \mathbb{Q})$. The asserted description of the special fibre $\mathcal{A}_0^{\text{Né}}$ of the Néron model follows therefore from the main results of this paper, see Theorems 1.1 and 1.2 above.

As we will explain next, the expression for the motivic zeta-function can be deduced from [HN1, Proposition 8.3]; we are grateful to Johannes Nicaise for pointing this out. In the notation used in *loc. cit.*, that result reads as follows:

$$Z_{\mathcal{A}_K}(T) = \sum_{d \in \mathbb{N}} \phi_{\mathcal{A}_K}(d) \cdot (\mathbb{L} - 1)^{t_{\mathcal{A}_K}(d)} \cdot \mathbb{L}^{u_{\mathcal{A}_K}(d) + \text{ord}_{\mathcal{A}_K}(d)} \cdot [B_{\mathcal{A}_K}(d)] \cdot T^d.$$

Since \mathcal{A}_K has semi-abelian reduction, the unipotent rank $u_{\mathcal{A}_K}(d)$ vanishes for all d , cf. [HN2, Section 2]. For the same reason, [HN1, Proposition 6.2] implies that the toric rank $t_{\mathcal{A}_K}(d)$ and the abelian quotient $B_{\mathcal{A}_K}(d)$ do not depend on d . Vanishing of the order function $\text{ord}_{\mathcal{A}_K}(d)$ can be deduced e.g. from [HN1, Proposition 7.5 and Corollary 4.20], and for the expression $\phi_{\mathcal{A}_K}(d) = \phi_{\mathcal{A}_K}(1)d^w$, see the proof of Theorem 8.6 in [HN1] and references therein. Finally, $\phi_{\mathcal{A}_K}(1)$ is the number of components of $\mathcal{A}_0^{\text{Né}}$ by [HN1, Definition 3.6]. Altogether, this establishes the formula claimed in Corollary 1.4. \square

Proof of Corollary 1.5. Corollary 1.5 is an immediate consequence of the relation between the Néron model $\mathcal{A}^{\text{Né}}$ and the semi-stable model \mathcal{A} , given by Jordan and Morrison in [JM, Theorem 1.4]. \square

ACKNOWLEDGEMENT

We are grateful to the referee for useful remarks and to Daniel Huybrechts, Radu Laza, Johannes Nicaise and Arne Smeets for discussions. Special thanks to Johannes Nicaise for pointing out the relevance of [HN1] and [HN2].

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