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Geometry & Topology 19 (2015) 295–342

On the construction problem for Hodge numbers

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For any symmetric collection $(h^{p,q})_{p+q=k}$ of natural numbers, we construct a smooth complex projective variety X whose weight-k Hodge structure has Hodge numbers $h^{p,q}(X)=h^{p,q}$; if $k=2m$ is even, then we have to impose that $h^{m,m}$ is bigger than some quadratic bound in m. Combining these results for different weights, we solve the construction problem for the truncated Hodge diamond under two additional assumptions. Our results lead to a complete classification of all nontrivial dominations among Hodge numbers of Kähler manifolds.

32Q15; 14C30, 51M15

1 Introduction

For a Kähler manifold X, Hodge theory yields an isomorphism

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X,\Omega_X^p). \tag{1-1}$$

As a refinement of the Betti numbers of X, one therefore defines the $(p,q)^{\text{th}}$ Hodge number $h^{p,q}(X)$ of X to be the dimension of $H^q(X,\Omega_X^p)$. This way one can associate to each n–dimensional Kähler manifold X its collection of Hodge numbers $h^{p,q}(X)$ with $0\le p,q\le n$. Complex conjugation and Serre duality show that such a collection of Hodge numbers $(h^{p,q})_{p,q}$ in dimension n needs to satisfy the Hodge symmetries

$$h^{p,q}=h^{q,p}=h^{n-p,n-q}. \tag{1-2}$$

Moreover, as a consequence of the hard Lefschetz theorem, the Lefschetz conditions

$$h^{p,q}\ge h^{p-1,q-1} \quad \text{for all } p+q\le n \tag{1-3}$$

hold. Given these classical results, the construction problem for Hodge numbers asks which collections of natural numbers $(h^{p,q})_{p,q}$ satisfying (1-2) and (1-3) actually arise as Hodge numbers of some n–dimensional Kähler manifold. In his survey article on the construction problem in Kähler geometry [20], C Simpson explains our lack of knowledge on this problem. Indeed, even weak versions where instead of all Hodge numbers one only considers small subcollections of them are wide open; for some

Published: 27 February 2015 DOI: 10.2140/gt.2015.19.295

partial results in dimensions two and three we refer to Barth, Hulek, Peters and Van de Ven [3], Chang [5] and Hunt [9].

This paper provides three main results on the above construction problem in the category of smooth complex projective varieties, which is stronger than allowing arbitrary Kähler manifolds. We present them in the following three subsections respectively.

1.1 The construction problem for weight-k Hodge structures

It follows from Griffiths transversality that a general integral weight-k $(k \geq 2)$ Hodge structure (not of K3 type) cannot be realized by a smooth complex projective variety; see Voisin [21, Remark 10.20]. This might lead to the expectation that general weight k Hodge numbers can also not be realized by smooth complex projective varieties. Our first result shows that this expectation is wrong. This answers a question in [20].

Theorem 1 *Fix $k \geq 1$ and let $(h^{p,q})_{p+q=k}$ be a symmetric collection of natural numbers. If $k = 2m$ is even, we assume*

$$h^{m,m} \geq m \cdot \lfloor (m+3)/2 \rfloor + \lfloor m/2 \rfloor^2.$$

Then in each dimension greater than or equal to $k+1$ there exists a smooth complex projective variety whose Hodge structure of weight k realizes the given Hodge numbers.

The examples which realize given weight-k Hodge numbers in the above theorem have dimension greater than or equal to $k+1$. However, if we assume that the outer Hodge number $h^{k,0}$ vanishes and that the remaining Hodge numbers are even, then we can prove a version of Theorem 1 also in dimension k; see Corollary 13 in Section 5.

Since any smooth complex projective variety contains a hyperplane class, it is clear that some kind of bound on $h^{m,m}$ in Theorem 1 is necessary. For $m = 1$, for instance, the bound provided by the above theorem is $h^{1,1} \geq 2$. In Section 7 we will show that in fact the optimal bound $h^{1,1} \geq 1$ can be reached. That is, we will show (Theorem 15) that any natural numbers $h^{2,0}$ and $h^{1,1}$ with $h^{1,1} \geq 1$ can be realised as weight-two Hodge numbers of some smooth complex projective variety. For $m \geq 2$, we do not know whether the bound on $h^{m,m}$ in Theorem 1 is optimal or not.

1.2 The construction problem for the truncated Hodge diamond

Given Theorem 1 one is tempted to ask for solutions to the construction problem for collections of Hodge numbers which do not necessarily correspond to a single cohomology group. In order to explain our result on this problem, we introduce the following notion: An n–dimensional formal Hodge diamond is a table

$$
\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & \iddots & & & \vdots & & & \ddots & \\
h^{n,0} & & & & & & & & h^{0,n} \\
 & \ddots & & & \vdots & & & \iddots & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & h^{0,0} & & & &
\end{array}
\tag{1-4}
$$

of natural numbers $h^{p,q}$ satisfying the Hodge symmetries (1-2), the Lefschetz conditions (1-3) and the connectivity condition $h^{0,0} = h^{n,n} = 1$. The $h^{p,q}$ are referred to as Hodge numbers and the sum over all $h^{p,q}$ with $p+q=k$ as the k^{th} Betti number b_k of this formal diamond; the vector $(b_0, \ldots, b_{2n})$ is called a vector of formal Betti numbers. Finally, for $p+q \leq n$, the primitive $(p,q)^{\text{th}}$ Hodge number of the above diamond is defined as

$$l^{p,q} := h^{p,q} - h^{p-1,q-1}.$$

Definition 2 A truncated n–dimensional formal Hodge diamond is a formal Hodge diamond (1-4) as above where the horizontal middle axis, ie the row of Hodge numbers $h^{p,q}$ with $p+q=n$, is omitted.

We note that for a Kähler manifold X its truncated Hodge diamond together with all holomorphic Euler characteristics $\chi(X, \Omega_X^p)$, where $p = 0, \ldots, \lfloor n/2 \rfloor$, is equivalent to giving the whole Hodge diamond. It is shown by Kotschick and the author in [13] that a linear combination of Hodge numbers can be expressed in terms of Chern numbers if and only if it is a linear combination of these Euler characteristics. Therefore, the Hodge numbers of the truncated Hodge diamond form a complement to the space of Hodge numbers which are determined by Chern numbers; cf [13] where the Hodge numbers in dimension n are regarded as linear forms on the weight-n part of a certain graded ring.

Our second main result solves the construction problem for the truncated Hodge diamond under two additional assumptions:

Theorem 3 *Suppose we are given a truncated n–dimensional formal Hodge diamond whose Hodge numbers $h^{p,q}$ satisfy the following two additional assumptions:*

(1) *For $p < n/2$, the primitive Hodge numbers $l^{p,p}$ satisfy*
$$l^{p,p} \geq p \cdot (n^2 - 2n + 5)/4.$$

(2) *The outer Hodge numbers $h^{k,0}$ vanish either for all $k = 1, \dots, n-3$, or for all $k \neq k_0$ for some $k_0 \in \{1, \dots, n-1\}$.*

Then there exists an n–dimensional smooth complex projective variety whose truncated Hodge diamond coincides with the given one.

Theorem 3 has several important consequences. For instance, for the union of $h^{n-2,0}$ and $h^{n-1,0}$ with the collection of all Hodge numbers which neither lie on the boundary, nor on the horizontal or vertical middle axis of (1-4), the construction problem is solvable without any additional assumptions. That is, the corresponding subcollection of any n–dimensional formal Hodge diamond can be realized by a smooth complex projective variety. The number of Hodge numbers we omit in this statement from the whole diamond (1-4) grows linearly in n, whereas the number of all entries of (1-4) grows quadratically in n. In this sense, Theorem 3 yields very good results on the construction problem in high dimensions.

Theorem 3 deals with Hodge structures of different weights simultaneously. This enables us to extract from it results on the construction problem for Betti numbers. Indeed, the following corollary rephrases Theorem 3 in terms of Betti numbers.

Corollary 4 *Let $(b_0, \dots, b_{2n})$ be a vector of formal Betti numbers with*
$$b_{2k} - b_{2k-2} \geq k \cdot (n^2 - 2n + 5)/8 \quad \text{for all } k < n/2.$$

Then there exists an n–dimensional smooth complex projective variety X such that $b_k(X) = b_k$ for all $k \neq n$.

This corollary says for instance that in even dimensions, the construction problem for the odd Betti numbers is solvable without any additional assumptions.

1.3 Universal inequalities and Kollár–Simpson's domination relation

Following Kollár and Simpson [20, page 9], we say that a Hodge number $h^{r,s}$ dominates $h^{p,q}$ in dimension n if there exist positive constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that for all n–dimensional smooth complex projective varieties X, the following holds:

(1-5)
$$c_1 \cdot h^{r,s}(X) + c_2 \geq h^{p,q}(X).$$

Moreover, such a domination is called nontrivial if $(0,0) \neq (p,q) \neq (n,n)$, and if (1-5) does not follow from the Hodge symmetries (1-2) and the Lefschetz conditions (1-3).

In [20] it is speculated that the middle Hodge numbers should probably dominate the outer ones. In our third main theorem of this paper, we classify all nontrivial dominations among Hodge numbers in any given dimension. As a result we see that the above speculation is accurate precisely in dimension two.

Theorem 5 *The Hodge number $h^{1,1}$ dominates $h^{2,0}$ nontrivially in dimension two and this is the only nontrivial domination in dimension two. Moreover, there are no nontrivial dominations among Hodge numbers in any dimension different from two.*

Firstly, using the classification of surfaces and the Bogomolov–Miyaoka–Yau inequality, we will prove in Section 9 (Proposition 22) that

$$h^{1,1}(X) > h^{2,0}(X)$$

holds for all Kähler surfaces X. That is, the middle degree Hodge number $h^{1,1}$ indeed dominates $h^{2,0}$ nontrivially in dimension two.

Secondly, in addition to Theorem 3, the proof of Theorem 5 will rely on the following result; see Theorem 17 in Section 8. For all $a > b$ with $a + b \leq n$, there are n–dimensional smooth complex projective varieties whose primitive Hodge numbers $l^{p,q}$ satisfy $l^{a,b} \gg 0$ and $l^{p,q} = 0$ for all other $p > q$.

Theorem 5 deals with universal inequalities of the form (1-5). In Section 10 we deduce from the main results of this paper some progress on the analogous problem for inequalities of arbitrary shape (Corollaries 24, 25 and 26). For instance, we will see that any universal inequality among Hodge numbers of smooth complex projective varieties which holds in all sufficiently large dimensions at the same time is a consequence of the Lefschetz conditions.

The problem of determining all universal inequalities among Hodge numbers of smooth complex projective varieties in a fixed dimension remains open. It is however surprisingly easy to solve the analogous problem for inequalities among Betti numbers. Indeed, using products of hypersurfaces of high degree, we will prove (Proposition 27) that in fact any universal inequality among the Betti numbers of n–dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions.

1.4 Some negative results

Theorem 5 shows that at least in dimension two, the constraints which classical Hodge theory puts on the Hodge numbers of Kähler manifolds are not complete. Indeed, given weight-two Hodge numbers can in general not be realized by a surface; by Theorem 1 (resp. Theorem 15) they can however be realized by higher-dimensional varieties. In Sections 11 and 12 of this paper we collect some partial results which demonstrate similar issues in dimensions three and four, respectively. This is one of the reasons which makes the construction problem for Hodge numbers so delicate.

In Section 11 we prove (Proposition 28) that the Hodge numbers $h^{p,q}$ of any smooth complex projective threefold with $h^{1,1}=1$ and $h^{2,0}>0$ satisfy $h^{1,0}=0$, $h^{2,0}<h^{3,0}$ and $h^{2,1}<12^6\cdot h^{3,0}$. Moreover, for $h^{3,0}-h^{2,0}$ bounded from above, only finitely many deformation types of such examples exist. In Section 12 we prove similar results (Proposition 32) for projective fourfolds with $h^{1,1}=1$. (The existence of three- and fourfolds with $h^{1,1}=1$ and $h^{2,0}>0$ is established by Theorem 15 in Section 7.)

Concerning the Betti numbers, we prove the following in Section 12 (Corollary 33). Let X be a Kähler fourfold with $b_2(X)=1$, then $b_3(X)$ can be bounded in terms of $b_4(X)$. Since this phenomenon can neither be explained with the Hodge symmetries, the Lefschetz conditions nor the Hodge–Riemann bilinear relations, we conclude that even for the Betti numbers of Kähler manifolds, the known constraints are not complete.

1.5 Organization of the paper

In Section 2 we outline our construction methods. In Section 3 we consider the hyperelliptic curve C_g given by $y^2=x^{2g+1}+1$ and construct useful subgroups of $\mathrm{Aut}(C_g^k)$. In Section 4 we develop the construction method needed for the proofs of Theorems 1 and 3 in Sections 5 and 6, respectively. In Section 7 we prove Theorem 15, ie we show that for weight-two Hodge structures the bound on $h^{1,1}$ in Theorem 1 can be chosen to be optimal. We produce in Section 8 examples whose primitive Hodge numbers $l^{p,q}$ with $p>q$ are concentrated in a single (p,q)–type, and show in Section 9 how our results lead to a proof of Theorem 5. In Section 10 we apply our results to the problem of finding universal inequalities among Hodge and Betti numbers of smooth complex projective varieties. Finally, we discuss in Sections 11 and 12 the negative results, mentioned in Section 1.4.

1.6 Notation and conventions

The natural numbers $\mathbb{N}:=\mathbb{Z}_{\geq 0}$ include zero. All Kähler manifolds are compact and connected, if not mentioned otherwise. A variety is a separated integral scheme of finite type over $\mathbb{C}$. Using the GAGA principle (see Serre [17]), we usually identify a smooth projective variety with its corresponding analytic space, which is a Kähler manifold. If not mentioned otherwise, cohomology means singular (or de Rham) cohomology with coefficients in $\mathbb{C}$; the cup product on cohomology will be denoted by $\wedge$.

By a group action $G \times Y \to Y$ on a variety Y, we always mean a group action by automorphisms from the left. For any finite subgroup $\Gamma \subseteq G$, the fixed point set of the induced Γ–action on Y will be denoted by

$$\mathrm{Fix}_Y(\Gamma) := \{y \in Y \mid g(y) = y \text{ for all } g \in \Gamma\}. \tag{1-6}$$

This fixed point set has a natural scheme structure. If $\Gamma = \langle \phi \rangle$ is cyclic, then we will frequently write $\mathrm{Fix}_Y(\Gamma) = \mathrm{Fix}_Y(\phi)$.

2 Outline of our construction methods

The starting point of our constructions is the observation that there are finite group actions $G \times T \to T$, where T is a product of hyperelliptic curves, such that the G–invariant cohomology of T is essentially concentrated in a single (p,q)–type; see Section 3.2. In local holomorphic charts, G acts by linear automorphisms. Thus, by the Chevalley–Shephard–Todd theorem, T/G is smooth if and only if G is generated by quasireflections, that is, by elements whose fixed point set is a divisor on T. Unfortunately, it turns out that in our approach this strong condition can rarely be met. We therefore face the problem of a possibly highly singular quotient T/G.

One way to deal with this problem is to pass to a smooth model X of T/G. However, only the outer Hodge numbers $h^{k,0}$ are birational invariants [13]. Therefore, there will be in general only very little relation between the cohomology of X and the G–invariant cohomology of T. Nevertheless, we will find in Section 8 examples T/G which admit smooth models whose cohomology is, apart from (a lot of) additional (p,p)–type classes, indeed given by the G–invariants of T. We will overcome technical difficulties by a general inductive approach which is inspired by work of Cynk and Hulek [7]; see Proposition 19.

In Theorems 1 and 3 we need to construct examples with bounded $h^{p,p}$ and so the above method does not work anymore. Instead, we will use the following lemma, known as the Godeaux–Serre construction; see Atiyah and Hirzebruch [2] and Serre [18].

Lemma 6 *Let G be a finite group whose action on a smooth complex projective variety Y is free outside a subset of codimension greater than n. Then Y/G contains an n–dimensional smooth complex projective subvariety whose cohomology below degree n is given by the G–invariant classes of Y.*

Proof A general n–dimensional G–invariant complete intersection subvariety $Z \subseteq Y$ is smooth by Bertini's theorem. For a general choice of Z, the G–action on Z is free and so Z/G is a smooth subvariety of Y/G which, by the Lefschetz hyperplane theorem applied to $Z \subseteq Y$, has the property we want in the lemma. □

The construction method which we develop in Section 4 (Proposition 12) and which is needed in Theorems 1 and 3 works roughly as follows. Instead of a single group action, we will consider a finite number of finite group actions $G_i \times T_i \to T_i$ indexed by $i \in I$. Blowing up all T_i simultaneously in a large ambient space Y, we are able to construct a smooth complex projective variety $\widetilde{Y}$ which admits an action of the product $G = \prod_{i \in I} G_i$ that is free outside a subset of large codimension and so Lemma 6 applies. Moreover, the G–invariant cohomology of $\widetilde{Y}$ will be given in terms of the G_i–invariant cohomology of the T_i. This is a quite powerful method since it allows us to apply Lemma 6 to a finite number of group actions simultaneously, even without assuming that the group actions we started with are free away from subspaces of large codimension.

3 Hyperelliptic curves and group actions

3.1 Basics on hyperelliptic curves

In this section, following mostly Shafarevich [19, page 214], we recall some basic properties of hyperelliptic curves. In order to unify our discussion, hyperelliptic curves of genus 0 and 1 will be $\mathbb{P}^1$ and elliptic curves, respectively.

For $g \geq 0$, let $f \in \mathbb{C}[x]$ be a degree $2g+1$ polynomial with distinct roots. Then a smooth projective model X of the affine curve Y given by

$$\{y^2 = f(x)\} \subseteq \mathbb{C}^2$$

is a hyperelliptic curve of genus g. Although Y is smooth, its projective closure has a singularity at ∞ for $g > 1$. The hyperelliptic curve X is therefore explicitly given by the normalization of this projective closure. It turns out that X is obtained from Y by adding one additional point at ∞. This additional point is covered by an affine piece, given by

$$\{v^2 = u^{2g+2} \cdot f(u^{-1})\} \quad \text{where } x = u^{-1} \text{ and } y = v \cdot u^{-g-1}.$$

On an appropriate open cover of X, local holomorphic coordinates are given by x, y, u and v respectively. Moreover, the smooth curve X has genus g and a basis of $H^{1,0}(X)$ is given by the differential forms

$$\omega_i := \frac{x^{i-1}}{y} \cdot dx,$$

where $i = 1, \dots, g$.

Let us now specialize to the situation where f equals the polynomial $x^{2g+1}+1$ and denote the corresponding hyperelliptic curve of genus g by C_g. It follows from the explicit description of the two affine pieces of C_g that this curve carries an automorphism ψ_g of order $2g+1$ given by

$$(x,y)\mapsto(\zeta\cdot x,y),\quad (u,v)\mapsto(\zeta^{-1}\cdot u,\zeta^g\cdot v),$$

where ζ denotes a primitive $(2g+1)^{\text{th}}$ root of unity. Similarly,

$$(x,y)\mapsto(x,-y),\quad (u,v)\mapsto(u,-v)$$

defines an involution which we denote by multiplication by -1. Moreover, it follows from the above description of $H^{1,0}(C_g)$ that the ψ_g–action on $H^{1,0}(C_g)$ has eigenvalues $\zeta,\dots,\zeta^g$, whereas the involution acts by multiplication with -1 on $H^{1,0}(C_g)$.

Any smooth curve can be embedded into $\mathbb{P}^3$. For the curve C_g, we fix the explicit embedding given by

$$[1:x:y:x^{g+1}]=[u^{g+1}:u^g:v:1].$$

Obviously, the involution as well as the order $(2g+1)$–automorphism ψ_g of $C_g\subseteq\mathbb{P}^3$ extend to $\mathbb{P}^3$ via

$$[1:1:-1:1]\quad\text{and}\quad[1:\zeta:1:\zeta^{g+1}],$$

respectively.

3.2 Group actions on products of hyperelliptic curves

Let

$$T:=C_g^k$$

be the k–fold product of the hyperelliptic curve C_g with automorphism ψ_g defined in Section 3.1. For $a\geq b$ with $a+b=k$, we define for each $i=1,2,3$ a subgroup $G^i(a,b,g)$ of $\mathrm{Aut}(T)$ whose elements are called automorphisms of the i^{th} kind. The subgroup of automorphisms of the first kind is given by

$$G^1(a,b,g):=\{\psi_g^{j_1}\times\cdots\times\psi_g^{j_{a+b}}\mid j_1+\cdots+j_a-j_{a+1}-\cdots-j_{a+b}\equiv 0 \mod (2g+1)\}.$$

In order to define the automorphisms of the second kind, let us consider the group $\mathrm{Sym}(a)\times\mathrm{Sym}(b)\times\mu_2^{a+b}$, where $\mu_2=\{1,-1\}$ is the multiplicative group on two elements. An element (σ,τ,ϵ), where $\sigma\in\mathrm{Sym}(a)$, $\tau\in\mathrm{Sym}(b)$ and $\epsilon=(\epsilon_1,\dots,\epsilon_{a+b})$ is a vector of signs $\epsilon_i\in\{1,-1\}$, acts on T via

$$(x_1,\dots,x_a,y_1,\dots,y_b)\mapsto(\epsilon_1\cdot x_{\sigma(1)},\dots,\epsilon_a\cdot x_{\sigma(a)},\epsilon_{a+1}\cdot y_{\tau(1)},\dots,\epsilon_{a+b}\cdot y_{\tau(b)}).$$

Here, multiplication with -1 means that we apply the involution $(-1) \in \mathrm{Aut}(C_g)$. We define

$$G^2(a,b,g) \subseteq \mathrm{Sym}(a) \times \mathrm{Sym}(b) \times \mu_2^{a+b}$$

to be the index-four subgroup consisting of those elements (σ, τ, ϵ) which satisfy

$$\mathrm{sign}(\sigma) \cdot \epsilon_1 \cdots \epsilon_a = 1 \quad \text{and} \quad \mathrm{sign}(\tau) \cdot \epsilon_{a+1} \cdots \epsilon_{a+b} = 1,$$

where sign denotes the signum of the corresponding permutation. Via the above action of $\mathrm{Sym}(a) \times \mathrm{Sym}(b) \times \mu_2^{a+b}$ on T, the group $G^2(a,b,g)$ is a finite subgroup of $\mathrm{Aut}(T)$.

Finally, $G^3(a,b,g)$ is trivial if $a \neq b$ and if $a = b$, then it is generated by the automorphism which interchanges the two factors of $T = {C_g}^a \times {C_g}^a$.

Definition 7 The group $G(a,b,g)$ is the subgroup of $\mathrm{Aut}(T)$ which is generated by the union of $G^i(a,b,g)$ for $i = 1,2,3$.

Automorphisms of different kinds do in general not commute with each other. However, it is easy to see that each element in $G(a,b,g)$ can be written as a product $\phi_1 \circ \phi_2 \circ \phi_3$ such that ϕ_i lies in $G^i(a,b,g)$. Therefore, $G(a,b,g)$ is a finite group which naturally acts on the cohomology of T.

Lemma 8 *If $a > b$, then the $G(a,b,g)$–invariant cohomology of T is a direct sum*

$$V^{a,b} \oplus V^{b,a} \oplus \left(\bigoplus_{p=0}^{k} V^{p,p} \right),$$

where $V^{a,b} = \overline{V^{b,a}}$ is a g–dimensional space of (a,b)–classes and $V^{p,p} \cong V^{k-p,k-p}$ is a space of (p,p)–classes of dimension $\min(p+1, b+1)$, where $p \leq k/2$ is assumed.

Proof We denote the fundamental class of the j^{th} factor of T by $\Omega_j \in H^{1,1}(T)$. Moreover, we pick for $j = 1, \dots, k$ a basis $\omega_{j1}, \dots, \omega_{jg}$ of $(1,0)$–classes of the j^{th} factor of T in such a way that

$$\psi_g^* \omega_{jl} = \zeta^l \omega_{jl}$$

for a fixed $(2g+1)^{\text{th}}$ root of unity ζ. Then the cohomology ring of T is generated by the Ω_j, ω_{jl} and their conjugates. Moreover, the involution on the j^{th} curve factor of T acts on ω_{jl} and $\overline{\omega_{jl}}$ by multiplication by -1 and leaves Ω_j invariant.

Suppose that we are given a $G(a,b,g)$–invariant class which contains the monomial

$$\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s} \wedge \omega_{j_1 l_1} \wedge \cdots \wedge \omega_{j_r l_r} \wedge \overline{\omega_{j_{r+1} l_{r+1}}} \wedge \cdots \wedge \overline{\omega_{j_t l_t}} \tag{3-1}$$

nontrivially. Since the product of a $(1,0)$– and a $(0,1)$–class of the i^{th} curve factor is a multiple of Ω_i, and since classes of degree 3 vanish on curves, we may assume the indices $i_1,\dots,i_s,j_1,\dots,j_t$ are pairwise distinct. Therefore, application of a suitable automorphism of the first kind shows $t=0$ if $s\geq 1$ and $t=a+b$ if $s=0$. In the latter case, suppose there are indices i_1 and i_2 with either $i_1,i_2\leq r$ or $i_1,i_2>r$, such that $j_{i_1}\leq a$ and $j_{i_2}>a$. Then, application of a suitable automorphism of the first kind yields $l_{i_1}+l_{i_2}=0$ in $\mathbb{Z}/(2g+1)\mathbb{Z}$, which contradicts $1\leq l_i\leq g$. This shows

$$\{j_1,\dots,j_r\}=\{1,\dots,a\} \quad \text{or} \quad \{j_1,\dots,j_r\}=\{a+1,\dots,a+b\}.$$

By applying suitable automorphisms of the first kind once more, one obtains $l_1=\cdots=l_t$. Thus, we have just shown that a $G(a,b,g)$–invariant class of T is either a polynomial in the Ω_j, or a linear combination of

$$\omega_l := \omega_{1l}\wedge\cdots\wedge\omega_{al}\wedge\overline{\omega_{a+1l}}\wedge\cdots\wedge\overline{\omega_{a+bl}} \tag{3-2}$$

or their conjugates, where $l=1,\dots,g$. Note that ω_l is of (a,b)–type whereas any polynomial in the Ω_j is a sum of (p,p)–type classes. Moreover, by the definition of $G^1(a,b,g)$ and $G^2(a,b,g)$, both groups act trivially on ω_l and $\bar{\omega}_l$. Since $a>b$, the group $G^3(a,b,g)$ is trivial and so it follows that each ω_l and $\bar{\omega}_l$ is $G(a,b,g)$–invariant. Therefore, the span of $\omega_1,\dots,\omega_g$ yields a g–dimensional space $V^{a,b}$ of $G(a,b,g)$–invariant (a,b)–classes. Its conjugate $V^{b,a}:=\overline{V^{a,b}}$ is spanned by the $G(a,b,g)$–invariant (b,a)–classes $\bar{\omega}_1,\dots,\bar{\omega}_g$.

Next, we define $V^{p,p}$ to consist of all $G(a,b,g)$–invariant homogeneous degree-p polynomials in $\Omega_1,\dots,\Omega_{a+b}$. Application of a suitable automorphism of the second kind shows that any element Θ in $V^{p,p}$ is a polynomial in the elementary symmetric polynomials in $\Omega_1,\dots,\Omega_a$ and $\Omega_{a+1},\dots,\Omega_{a+b}$. By standard facts about symmetric polynomials, it follows that Θ can be written as a polynomial in

$$\sum_{j=1}^{a}{\Omega_j}^i \quad\text{and}\quad \sum_{j=a+1}^{a+b}{\Omega_j}^i$$

for $i\geq 0$. Since Ω_j^2 vanishes for all j, we see that a basis of $V^{p,p}$ is given by the elements

$$(\Omega_1+\cdots+\Omega_a)^{p-i}\wedge(\Omega_{a+1}+\cdots+\Omega_{a+b})^i,$$

where $0\leq p-i\leq a$ and $0\leq i\leq b$. Using $a>b$, this concludes the lemma by an easy counting argument. □

Lemma 9 *If $a=b$, then the $G(a,b,g)$–invariant cohomology of T is a direct sum $\bigoplus_{p=0}^k V^{p,p}$, where $V^{p,p}\cong V^{k-p,k-p}$ is a space of (p,p)–classes whose dimension is given by $\lfloor p/2\rfloor+1$ if $p<a$, and by $\lfloor p/2\rfloor+g+1$ if $p=a$.*

Proof We use the same notation as in the proof of Lemma 8 and put $b := a$. Suppose that we are given a $G(a,a,g)$–invariant cohomology class on T which contains the monomial (3-1) nontrivially. This monomial is then necessarily $G^1(a,a,g)$–invariant and the same arguments as in Lemma 8 show that it is either a monomial in the Ω_j, or it coincides with one of the ω_l and their conjugates, defined in (3-2).

For each $l = 1,\dots,g$, the classes ω_l and $\overline{\omega_l}$ are invariant under the action of $G^1(a,a,g)$ and $G^2(a,a,g)$. Moreover, the generator of $G^3(a,a,g)$ interchanges the two factors of $T = {C_g}^a \times {C_g}^a$. Its action on cohomology therefore maps ω_l to $(-1)^a \cdot \overline{\omega_l}$. This shows that a linear combination of the ω_l and their conjugates is $G(a,a,g)$–invariant if and only if it is a linear combination of the classes

$$\omega_l + (-1)^a \cdot \overline{\omega_l}, \tag{3-3}$$

where $l = 1,\dots,g$. This yields a g–dimensional space of $G(a,a,g)$–invariant (a,a)–classes.

It remains to study which homogeneous polynomials in the Ω_j are $G(a,a,g)$–invariant. As in the proof of Lemma 8, one shows that any such polynomial of degree p is necessarily a linear combination of

$$\Omega(p-i,i) := (\Omega_1 + \cdots + \Omega_a)^{p-i} \wedge (\Omega_{a+1} + \cdots + \Omega_{2a})^i,$$

where $0 \le p-i \le a$ and $0 \le i \le a$. The above monomials are clearly invariant under the action of $G^1(a,a,g)$ and $G^2(a,a,g)$. Moreover, the generator of $G^3(a,a,g)$ interchanges the two factors of T and hence its action on cohomology maps $\Omega(p-i,i)$ to $\Omega(i,p-i)$. We are therefore reduced to linear combinations of

$$\Omega(i,p-i) + \Omega(p-i,i),$$

where $0 \le i \le p-i \le a$. Such linear combinations are certainly $G(a,a,g)$–invariant. If $p \le a$, then the condition on the index i means $0 \le i \le p/2$. It follows that for $p \le a$, the space of those $G(a,a,g)$–invariant (p,p)–classes which are given by polynomials in the Ω_j has dimension $\lfloor p/2 \rfloor + 1$. Combining this with our previous observation that the classes in (3-3) span a g–dimensional space of $G(a,a,g)$–invariant (a,a)–classes, this concludes the lemma. □

For later applications, we will also need the following:

Lemma 10 *For all $a \ge b$ there exists some $N > 0$ and an embedding of $G(a,b,g)$ into $\mathrm{GL}(N+1)$ such that a $G(a,b,g)$–equivariant embedding of ${C_g}^{a+b}$ into $\mathbb{P}^N$ exists. Moreover, ${C_g}^{a+b}$ contains a point which is fixed by $G(a,b,g)$.*

Proof For the first statement, we use the embedding of C_g into $\mathbb{P}^3$, constructed in Section 3.1. This yields an embedding of C_g^{a+b} into $(\mathbb{P}^3)^{a+b}$. From the explicit description of that embedding, it follows that the action of $G(a,b,g)$ on C_g^{a+b} extends to an action on $(\mathbb{P}^3)^{a+b}$ which is given by first multiplying homogeneous coordinates with some roots of unity and then permuting these in some way. Using the Segre map, we obtain for some large N an embedding of $G(a,b,g)$ into $\mathrm{GL}(N+1)$ together with a $G(a,b,g)$–equivariant embedding

$$C_g{}^{a+b} \hookrightarrow \mathbb{P}^N.$$

This proves the first statement in the lemma.

For the second statement, note that the point ∞ of C_g is fixed by both ψ_g as well as the involution. Thus ∞ yields a point on the diagonal of C_g^{a+b} which is fixed by $G(a,b,g)$. □

4 Group actions on blown-up spaces

4.1 Cohomology of blow-ups

Let Y be a Kähler manifold, T a submanifold of codimension r and let $\pi\colon \widetilde{Y} \to Y$ be the blow-up of Y along T. Then the exceptional divisor $j\colon E \hookrightarrow \widetilde{Y}$ of this blow-up is a projective bundle of rank $r-1$ over T and we denote the dual of the tautological line bundle on E by $\mathcal{O}_E(1)$. Then the Hodge structure on $\widetilde{Y}$ is given by the following theorem; see [21, page 180].

Theorem 11 *We have an isomorphism of Hodge structures*

$$H^k(Y,\mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(T,\mathbb{Z}) \right) \to H^k(\widetilde{Y},\mathbb{Z}),$$

where on $H^{k-2i-2}(T,\mathbb{Z})$, the natural Hodge structure is shifted by $(i+1,i+1)$. On $H^k(Y,\mathbb{Z})$, the above morphism is given by π^* whereas on $H^{k-2i-2}(T,\mathbb{Z})$ it is given by $j_* \circ h^i \circ \pi|_E^*$, where h denotes the cup product with $c_1(\mathcal{O}_E(1)) \in H^2(E,\mathbb{Z})$ and j_* is the Gysin morphism of the inclusion $j\colon E \hookrightarrow \widetilde{Y}$.

We will need the following property of the ring structure of $H^*(\widetilde{Y},\mathbb{Z})$. Note that the first Chern class of $\mathcal{O}_E(1)$ coincides with the pullback of $-[E] \in H^2(\widetilde{Y},\mathbb{Z})$ to E. For a class $\alpha \in H^{k-2i-2}(T,\mathbb{Z})$, this implies

$$(j_* \circ h^i \circ \pi|_E^*)(\alpha) = j_*\big(j^*(-[E])^i \wedge \pi|_E^*(\alpha)\big) = (-[E])^i \wedge j_*(\pi|_E^*(\alpha)), \tag{4-1}$$

where we used the projection formula.

4.2 Key construction

Let I be a finite nonempty set and let $i_0 \in I$. Suppose that for each $i \in I$, we are given a representation

$$G_i \to \mathrm{GL}(V_i)$$

of a finite group G_i on a finite-dimensional complex vector space V_i. Further, assume that the induced G_i–action on $\mathbb{P}(V_i)$ restricts to an action on a smooth subvariety $T_i \subseteq \mathbb{P}(V_i)$ and that there is a point $p_{i_0} \in T_{i_0}$ which is fixed by G_{i_0}. Then we have the following key result.

Proposition 12 *For any $n > 0$, there exists some complex vector space V and pairwise disjoint embeddings of T_i into $Y := T_{i_0} \times \mathbb{P}(V)$ such that the blow-up $\widetilde{Y}$ of Y along all T_i with $i \neq i_0$ inherits an action of $G := \prod_{i \in I} G_i$ which is free outside a subset of codimension greater than n. Moreover, $\widetilde{Y}/G$ contains an n–dimensional smooth complex projective subvariety X whose primitive Hodge numbers are, for all $p+q<n$, given by*

$$l^{p,q}(X) = \dim\bigl(H^{p,q}(T_{i_0})^{G_{i_0}}\bigr) + \sum_{i \neq i_0} \dim\bigl(H^{p-1,q-1}(T_i)^{G_i}\bigr).$$

Proof The product

$$G := \prod_{i \in I} G_i$$

acts naturally on the direct sum $\bigoplus_{i \in I} V_i$. We pick some $k \gg 0$. Then

$$V := \Bigl(\bigoplus_{i \in I} V_i\Bigr) \oplus \Bigl(\bigoplus_{g \in G} g \cdot \mathbb{C}^k\Bigr)$$

inherits a linear G–action where $h \in G$ acts on the second factor by sending $g \cdot \mathbb{C}^k$ canonically to $(h \cdot g) \cdot \mathbb{C}^k$. Then we obtain G–equivariant inclusions

$$T_i \hookrightarrow \mathbb{P}(V_i) \hookrightarrow \mathbb{P}(V),$$

where for $j \neq i$, the group G_j acts via the identity on T_i and $\mathbb{P}(V_i)$. The product

$$Y := T_{i_0} \times \mathbb{P}(V)$$

inherits a G–action via the diagonal, where for $i \neq i_0$ elements of G_i act trivially on T_{i_0}.

Using the base point $p_{i_0} \in T_{i_0}$, we obtain for all $i \in I$ disjoint inclusions

$$T_i \hookrightarrow Y,$$

and we denote the blow-up of Y along the union of all T_i with $i \neq i_0$ by $\widetilde{Y}$. Since $p_{i_0} \in T_{i_0}$ is fixed by G, the G–action maps each T_i to itself and hence lifts to $\widetilde{Y}$.

We want to prove that the G–action on $\widetilde{Y}$ is free outside a subset of codimension greater than n. For k large enough, the G–action on Y certainly has this property. Hence, it suffices to check that the induced G–action on the exceptional divisor E_j above $T_j \subseteq Y$ is free outside a subset of codimension greater than n.

For $|I| = 1$, this condition is empty. For $|I| \geq 2$, we fix an index $j \in I$ with $j \neq i_0$. Then it suffices to show that for a given nontrivial element $\phi \in G$ the fixed point set $\mathrm{Fix}_{E_j}(\phi)$ has codimension greater than n in E_j. If $t_j \in T_j$ is not fixed by ϕ, then the fiber of $E_j \to T_j$ above t_j is moved by ϕ and hence disjoint from $\mathrm{Fix}_{E_j}(\phi)$. Conversely, if t_j is fixed by ϕ, then ϕ acts on the normal space

$$\mathcal{N}_{T_j,t_j} = T_{Y,t_j} / T_{T_j,t_j}$$

via a linear automorphism and the projectivization of this vector space is the fiber of $E_j \to T_j$ above t_j. The tangent space T_{Y,t_j} equals

$$T_{T_{i_0},p_{i_0}} \oplus (L^* \otimes (V/L)),$$

where L is the line in V which corresponds to the image of t_j under the projection $Y \to \mathbb{P}(V)$. Since $\phi \neq \mathrm{id}$, it follows for large k that the fixed point set of ϕ on the fiber of E_j above t_j has codimension greater than n. Hence, $\mathrm{Fix}_{E_j}(\phi)$ has codimension greater than n in E_j, as we want.

As we have just shown, the G–action on $\widetilde{Y}$ is free outside a subset of codimension greater than n. Hence, by Lemma 6, the quotient $\widetilde{Y}/G$ contains an n–dimensional smooth complex projective subvariety X whose cohomology below the middle degree is given by the G–invariants of $\widetilde{Y}$. In order to calculate the dimension of the latter, we first note that for all $i \in I$, the divisor E_i on $\widetilde{Y}$ is preserved by G. Since $\mathcal{O}_{E_i}(-1)$ is given by the restriction of $\mathcal{O}_{\widetilde{Y}}(E_i)$ to E_i, it follows that $c_1(\mathcal{O}_{E_i}(1))$ is G–invariant. For $p+q<n$, the primitive $(p,q)^{\mathrm{th}}$ Hodge number of X is, by Theorem 11, therefore given by

$$l^{p,q}(X) = \dim\bigl(H^{p,q}(Y)^G\bigr) - \dim\bigl(H^{p-1,q-1}(Y)^G\bigr) + \sum_{i\neq i_0} \dim\bigl(H^{p-1,q-1}(T_i)^{G_i}\bigr),$$

where $H^*(-)^G$ denotes G–invariant cohomology. Since any automorphism of projective space acts trivially on its cohomology, the Künneth formula implies

$$\dim\bigl(H^{p,q}(Y)^G\bigr) - \dim\bigl(H^{p-1,q-1}(Y)^G\bigr) = \dim\bigl(H^{p,q}(T_{i_0})^{G_{i_0}}\bigr).$$

This finishes the proof of Proposition 12. □

5 Proof of Theorem 1

Fix $k \geq 1$ and let $(h^{p,q})_{p+q=k}$ be a symmetric collection of natural numbers. In the case where $k = 2m$ is even, we additionally assume

$$h^{m,m} \geq m \cdot \left(m - \left\lfloor \tfrac{m}{2} \right\rfloor + 1\right) + \left\lfloor \tfrac{m}{2} \right\rfloor^2.$$

Then we want to construct for $n > k$ an n–dimensional smooth complex projective variety X with the above Hodge numbers on $H^k(X, \mathbb{C})$.

Let us consider the index set $I := \{0, \dots, \lfloor (k-1)/2 \rfloor\}$ and put $i_0 := 0$. Then, for all $i \in I$, we consider the $(k-2i)$–fold product

$$T_i := (C_{h^{k-i,i}})^{k-2i},$$

where $C_{h^{k-i,i}}$ denotes the hyperelliptic curve of genus $h^{k-i,i}$, defined in Section 3.1. On T_i we consider the action of

$$G_i := G(k-2i, 0, h^{k-i,i}),$$

defined in Section 3.2.

By Lemma 10, we may apply the construction method of Section 4.2 to the set of data (T_i, G_i, I, i_0). Thus, by Proposition 12, there exists an n–dimensional smooth complex projective variety X whose primitive Hodge numbers are for $p+q < n$ given by

$$l^{p,q}(X) = \dim\left(H^{p,q}(T_{i_0})^{G_{i_0}}\right) + \sum_{i \neq i_0} \dim\left(H^{p-1,q-1}(T_i)^{G_i}\right).$$

Lemma 8 says that for $p > q$, the only G_i–invariant (p,q)–classes on T_i are of type $(k-2i, 0)$. Therefore, $l^{p,q}(X)$ vanishes for $p > q$ and $p+q < n$ in all but the cases

$$l^{k,0}(X) = \dim\left(H^{k,0}(T_{i_0})^{G_{i_0}}\right) = h^{k,0},$$
$$l^{k-2i+1,1}(X) = \dim\left(H^{k-2i,0}(T_i)^{G_i}\right) = h^{k-i,i}$$

for all $1 \leq i < k/2$. Using the formula

$$h^{k-i,i}(X) = \sum_{s=0}^{i} l^{k-i-s,i-s}(X),$$

we deduce, for $0 \leq i < k/2$,

$$h^{k-i,i}(X) = h^{k-i,i}.$$

Thus, if k is odd, then the Hodge symmetries imply that the Hodge structure on $H^k(X,\mathbb{C})$ has Hodge numbers $(h^{k,0},\dots,h^{0,k})$.

We are left with the case where $k=2m$ is even. Since blowing-up a point increases $h^{m,m}$ by one and leaves $h^{p,q}$ with $p\neq q$ unchanged, it suffices to prove that

$$h^{m,m}(X)=m\cdot\big(m-\lfloor\tfrac{m}{2}\rfloor+1\big)+\lfloor\tfrac{m}{2}\rfloor^2.$$

As we have seen,

$$h^{m,m}(X)=\sum_{s=0}^{m} l^{s,s}(X)=\sum_{s=0}^{m}\Big(\dim\big(H^{s,s}(T_0)^{G_0}\big)+\sum_{0<i<k/2}\dim\big(H^{s-1,s-1}(T_i)^{G_i}\big)\Big).$$

By Lemma 8, we have $\dim(H^{s,s}(T_i)^{G_i})=1$ for all $0\le s\le 2\cdot\dim(T_i)$ and so

$$h^{m,m}(X)=m+1+\sum_{s=0}^{m-1}\sum_{0<i<k/2}\dim\big(H^{s,s}(T_i)^{G_i}\big).$$

Since T_i has dimension $2(m-i)$, we see that

$$\sum_{s=0}^{m-1}\dim\big(H^{s,s}(T_i)^{G_i}\big)=\begin{cases}m & \text{if } 2(m-i)>m-1,\\ 2(m-i)+1 & \text{if } 2(m-i)\le m-1.\end{cases}$$

Hence

$$h^{m,m}(X)=m+1+\sum_{i=1}^{\lfloor m/2\rfloor} m+\sum_{i=\lfloor m/2\rfloor+1}^{m-1}(2(m-i)+1),$$

and it is straightforward to check that this simplifies to

$$h^{m,m}(X)=m\cdot\lfloor(m+3)/2\rfloor+\lfloor m/2\rfloor^2.$$

This finishes the proof of Theorem 1.

In Theorem 1 we have only dealt with Hodge structures below the middle degree. Under stronger assumptions, the following corollary of Theorem 1 deals with Hodge structures in the middle degree. We will use this corollary in the proof of Theorem 5 in Section 9.

Corollary 13 *Let $(h^{n,0},\dots,h^{0,n})$ be a symmetric collection of even natural numbers such that $h^{n,0}=0$. If $n=2m$ is even, then we additionally assume that*

$$h^{m,m}\ge 2\cdot(m-1)\cdot\lfloor(m+2)/2\rfloor+2\cdot\lfloor(m-1)/2\rfloor^2.$$

Then there exists an n–dimensional smooth complex projective variety X whose Hodge structure of weight n realizes the given Hodge numbers.

Proof For $n=1$ we may put $X=\mathbb{P}^1$ and for $n=2$ the blow-up of $\mathbb{P}^2$ in $h^{1,1}-1$ points does the job. It remains to deal with $n\geq 3$. Here, by Theorem 1 there exists an $(n-1)$–dimensional smooth complex projective variety Y whose Hodge decomposition on $H^{n-2}(Y,\mathbb{C})$ has Hodge numbers

$$\left(\tfrac{1}{2}\cdot h^{n-1,1},\dots,\tfrac{1}{2}\cdot h^{1,n-1}\right).$$

By the Künneth formula, the product $X:=Y\times\mathbb{P}^1$ has Hodge numbers

$$h^{p,q}(X)=h^{p,q}(Y)+h^{p-1,q-1}(Y).$$

Using the Hodge symmetries on Y, Corollary 13 follows. □

6 Proof of Theorem 3

In this section we prove Theorem 3, stated in Section 1. Our proof will follow the same lines as the proof of Theorem 1 in Section 5.

Given a truncated n–dimensional formal Hodge diamond whose Hodge numbers (resp. primitive Hodge numbers) are denoted by $h^{p,q}$ (resp. $l^{p,q}$). Suppose that one of the following two additional conditions holds:

(1) The number $h^{k,0}$ vanishes for all $k\neq k_0$ for some $k_0\in\{1,\dots,n-1\}$.

(2) The number $h^{k,0}$ vanishes for all $k=1,\dots,n-3$.

We will construct universal constants $C(p,n)$ such that under the additional assumption $l^{p,p}\geq C(p,n)$ for all $1\leq p<n/2$, an n–dimensional smooth complex projective variety X with the given truncated Hodge diamond exists. Then Theorem 3 follows as soon as we have shown $C(p,n)\leq p\cdot(n^2-2n+5)/4$.

Since blowing-up a point on X increases the primitive Hodge number $l^{1,1}(X)$ by one and leaves the remaining primitive Hodge numbers unchanged, it suffices to deal with the case where $l^{1,1}=C(1,n)$ is minimal.

To explain our construction, let us for each $r\geq s>0$ with $2<r+s<n$ consider the $(r+s-2)$–fold product

$$T_{r,s}:=(C_{l^{r,s}})^{r+s-2},$$

where $C_{l^{r,s}}$ is the hyperelliptic curve of genus $l^{r,s}$, constructed in Section 3.1. On $T_{r,s}$ we consider the group action of

$$G_{r,s}:=G(r-1,s-1,l^{r,s}),$$

defined in Section 3.2.

At this point we need to distinguish between the above cases (1) and (2). We begin with (1) and consider the index set

$$I := \{(r,s) \mid r \geq s > 0,\ n > r+s > 2\} \cup \{i_0\},$$

and put

$$T_{i_0} := (C_{l^{k_0,0}})^{k_0} \quad \text{and} \quad G_{i_0} := G(k_0, 0, l^{k_0,0}).$$

By Lemma 10, we may apply the construction method of Section 4.2 to the set of data (T_i, G_i, I, i_0). Thus, Proposition 12 yields an n–dimensional smooth complex projective variety X whose primitive Hodge numbers $l^{p,q}(X)$ with $p+q<n$ are given by

$$l^{p,q}(X) = \dim\bigl(H^{p,q}(T_{i_0})^{G_{i_0}}\bigr) + \sum_{(r,s)\in I\setminus\{i_0\}} \dim\bigl(H^{p-1,q-1}(T_{r,s})^{G_{r,s}}\bigr). \tag{6-1}$$

If $p>q$, then Lemmas 8 and 9 say that

$$\dim\bigl(H^{p-1,q-1}(T_{r,s})^{G_{r,s}}\bigr) = \begin{cases} 0 & \text{if } (r,s) \neq (p,q), \\ l^{p,q} & \text{if } (r,s) = (p,q). \end{cases} \tag{6-2}$$

Moreover,

$$\dim\bigl(H^{p,q}(T_{i_0})^{G_{i_0}}\bigr) = \begin{cases} 0 & \text{if } (k_0,0) \neq (p,q), \\ l^{p,q} & \text{if } (k_0,0) = (p,q). \end{cases} \tag{6-3}$$

In (6-1), the summation condition $(r,s) \in I\setminus\{i_0\}$ means $r \geq s > 0$ and $n > r+s > 2$. It therefore follows from (6-2) and (6-3) that $l^{p,q}(X) = l^{p,q}$ holds for all $p>q$ with $p+q<n$. By the Hodge symmetries on X, $l^{p,q}(X) = l^{p,q}$ then follows for all $p \neq q$ with $p+q<n$.

Next, for $p=q$, one extracts from (6-1) an explicit formula of the form

$$l^{p,p}(X) = l^{p,p} + C_1(p,n),$$

where $C_1(p,n)$ is a constant which only depends on p and n. Replacing $l^{p,p}$ by $l^{p,p} - C_1(p,n)$ in the above argument then shows that in case (1), an n–dimensional smooth complex projective variety with the given truncated Hodge diamond exists as long as

$$l^{p,p} \geq C_1(p,n)$$

holds for all $1 \leq p < n/2$.

In order to find a rough estimate of $C_1(p,n)$, we deduce from Lemmas 8 and 9 the inequalities

$$\dim\bigl(H^{p,p}(T_{i_0})^{G_{i_0}}\bigr) \leq 1 \quad \text{for all } p,$$

and

$$\dim\left(H^{p-1,p-1}(T_{r,s})^{G_{r,s}}\right) \leq \begin{cases} p & \text{if } (r,s) \neq (p,p), \\ p + l^{p,p} & \text{if } (r,s) = (p,p). \end{cases}$$

Using these estimates, (6-1) gives

$$C_1(p,n) \leq 1 + \sum_{\substack{r \geq s > 0 \\ n > r+s > 2}} p, \tag{6-4}$$

where we used that $(r,s) \in I \setminus \{i_0\}$ is equivalent to $r \geq s > 0$ and $n > r+s > 2$. If we write $\lfloor x \rfloor$ for the floor function of x, then (6-4) gives explicitly

$$C_1(p,n) \leq p \cdot n \cdot \left\lfloor \tfrac{n-1}{2} \right\rfloor - p \cdot \left\lfloor \tfrac{n-1}{2} \right\rfloor \cdot \left(\left\lfloor \tfrac{n-1}{2} \right\rfloor + 1 \right).$$

If n is odd, then the above right-hand side equals $p \cdot (n-1)^2/4$ and if n is even, then it is given by $p \cdot n(n-2)/4$. Hence,

$$C_1(p,n) \leq p \cdot (n-1)^2/4.$$

Let us now turn to case (2). Here we consider the same index set I as above, and for all $i \neq i_0$ we also define T_i and G_i as above. However, for $i = i_0$, we put

$$T_{i_0} := (C_{l^{n-1,0}})^{n-1} \times (C_{l^{n-2,0}})^{n-2},$$
$$G_{i_0} := G(n-1, 0, l^{n-1,0}) \times G(n-2, 0, l^{n-2,0}).$$

By Lemma 10, there exist integers N_1 and N_2 such that G_{i_0} admits an embedding into $\mathrm{GL}(N_1+1) \times \mathrm{GL}(N_2+1)$ in such a way that an G_{i_0}–equivariant embedding of T_{i_0} into $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ exists. Using the Segre map, we obtain for some $N > 0$ an embedding of G_{i_0} into $\mathrm{GL}(N+1)$ and an G_{i_0}–equivariant embedding of T_{i_0} into $\mathbb{P}^N$. Moreover, by Lemma 10, T_{i_0} contains a point p_{i_0} which is fixed by G_{i_0}. Hence the construction method of Section 4.2 can be applied to the above set of data. Therefore Proposition 12 yields an n–dimensional smooth complex projective variety X whose primitive Hodge numbers $l^{p,q}(X)$ are given by formula (6-1).

For $p > q$ and $p+q < n$, the G_{i_0}–invariant cohomology of T_{i_0} is trivial whenever (p,q) is different from $(n-2,0)$ and $(n-1,0)$. Moreover, for $(p,q) = (n-1,0)$ it has dimension $l^{n-1,0}$ and for $(p,q) = (n-2,0)$ its dimension equals $l^{n-2,0}$. Thus (6-1) and the Hodge symmetries on X yield $l^{p,q}(X) = l^{p,q}$ for all $p \neq q$ with $p+q < n$. Also, as in case (1), we obtain

$$l^{p,p}(X) = l^{p,p} + C_2(p,n),$$

where $C_2(p,n)$ is a constant in p and n which can be estimated by

$$C_2(p,n) \le p+1+\sum_{\substack{r\ge s>0 \\ n>r+s>2}} p,$$

where we used that $H^{p,p}(T_{i_0})^{G_{i_0}}$ has dimension $p+1$. Our estimation for $C_1(p,n)$ shows

$$C_2(p,n) \le p\cdot(n-1)^2/4+p.$$

Then, for $l^{p,p}\ge C_2(p,n)$, we may replace $l^{p,p}$ by $l^{p,p}-C_2(p,n)$ in the above argument and obtain an n–dimensional smooth complex projective variety with the given truncated Hodge diamond.

Let us now define

$$C(p,n) := \max\bigl(C_1(p,n), C_2(p,n)\bigr). \tag{6-5}$$

Then in both cases (1) and (2), a variety with the desired truncated Hodge diamond exists if $l^{p,p}\ge C(p,n)$. Moreover, $C(p,n)$ can roughly be estimated by

$$C(p,n)\le p\cdot\frac{n^2-2n+5}{4}.$$

This finishes the proof of Theorem 3.

Remark 14 As we have seen in the above proof, we may replace the given lower bound on $l^{p,p}$ in assumption (1) of Theorem 3 by the smaller constant $C(p,n)$, defined in (6-5).

7 Special weight-2 Hodge structures

In this section we show that for weight-two Hodge structures, the lower bound $h^{1,1}\ge 2$ in Theorem 1 can be replaced by the optimal lower bound $h^{1,1}\ge 1$. Our proof uses an ad hoc implementation of the Godeaux–Serre construction. The examples we construct here compare nicely to the results in Sections 11 and 12. However, since the methods of this section are not used elsewhere in the paper, the reader can easily skip this section.

Theorem 15 *Let $h^{2,0}$ and $h^{1,1}$ be natural numbers with $h^{1,1}\ge 1$. Then in each dimension greater than or equal to 3 there exists a smooth complex projective variety X with*

$$h^{2,0}(X)=h^{2,0} \quad \textit{and} \quad h^{1,1}(X)=h^{1,1}.$$

Proof Since blowing-up a point increases $h^{1,1}$ by one and leaves $h^{2,0}$ unchanged, in order to prove Theorem 15, it suffices to construct for given g in each dimension $n > 2$ a smooth complex projective variety X with $h^{2,0}(X) = g$ and $h^{1,1}(X) = 1$.

We fix some large integers N_1 and N_2 and consider $T := {C_g}^2$ together with the subgroups $G^1(2,0,g)$ and $G^2(2,0,g)$ of $\mathrm{Aut}(T)$, defined in Section 3.2. For $j = 1,\dots,N_1$, we denote a copy of T^{N_2} by A_j and we put

$$A := A_1 \times \cdots \times A_{N_1}.$$

That is, A is a $(2 \cdot N_1 \cdot N_2)$–fold product of C_g, but we prefer to think of A to be an N_1–fold product of T^{N_2}, where the j^{th} factor is denoted by A_j.

Next, we explain the construction of a certain subgroup G of automorphisms of A. This group is generated by five finite subgroups $G_1,\dots,G_5$ in $\mathrm{Aut}(A)$. The first subgroup of $\mathrm{Aut}(A)$ is given by

$$G_1 := G^1(2,0,g)^{\times N_1},$$

where $G^1(2,0,g)$ acts on each A_j via the diagonal action. The second one is

$$G_2 := G^1(2,0,g)^{\times N_2},$$

acting on A via the diagonal action. The third one is given by

$$G_3 := G^2(2,0,g),$$

acting on each A_j as well as on A via the diagonal action. The fourth group of automorphisms of A equals

$$G_4 := \mathrm{Sym}(N_1),$$

which acts on A via permutation of the A_j. Finally, we put

$$G_5 := \mathrm{Sym}(N_2),$$

which permutes the T–factors of each A_j and acts on A via the diagonal action.

Suppose we are given some elements $\phi_i \in G_i$. Then, ϕ_3 commutes with ϕ_4 and ϕ_5, and $\phi_3 \circ \phi_1 = \phi_1' \circ \phi_3$, respectively $\phi_1 \circ \phi_3 = \phi_3 \circ \phi_1''$ as well as $\phi_3 \circ \phi_2 = \phi_2' \circ \phi_3$, respectively $\phi_2 \circ \phi_3 = \phi_3 \circ \phi_2''$ holds for some $\phi_i', \phi_i'' \in G_i$, where $i = 1, 2$. Similar relations can be checked for all products $\phi_i \circ \phi_j$ and so we conclude that each element ϕ in the group $G \subseteq \mathrm{Aut}(A)$, which is generated by $G_1,\dots,G_5$, can be written in the form

$$\phi = \phi_1 \circ \phi_2 \circ \phi_3 \circ \phi_4 \circ \phi_5,$$

where ϕ_i lies in G_i.

Suppose that the fixed point set $\mathrm{Fix}_A(\phi)$ contains an irreducible component whose codimension is less than

$$\min(N_1/2, 2N_2).$$

Since ϕ is just some permutation of the $2N_1N_2$ curve factors of A, followed by automorphisms of each factor, we deduce that ϕ needs to fix more than

$$2N_1N_2 - \min(N_1, 4N_2)$$

curve factors. If ϕ_4 were nontrivial, then ϕ would fix at most $2(N_1-2)N_2$ curve factors, and if ϕ_5 were nontrivial, then ϕ would fix at most $2N_1(N_2-2)$ curve factors. Thus $\phi_4 = \phi_5 = \mathrm{id}$. If ϕ_3 were nontrivial, then its action on a single factor $T = {C_g}^2$ cannot permute the two curve factors. Thus ϕ_3 is just multiplication with -1 on each curve factor. This cannot be canceled with automorphisms in $G^1(2,0,g)$, since the latter is a cyclic group of order $2g+1$. Therefore $\phi_3 = \mathrm{id}$ follows as well.

Since ϕ fixes more than $2N_1N_2 - N_1$ curve factors, we see that $\phi = \phi_1 \circ \phi_2$ needs to be the identity on at least one A_{j_0}. Since ϕ_2 acts on each A_j in the same way, it lies in $G_1 \cap G_2$ and so we may assume $\phi_2 = \mathrm{id}$. Finally, any nontrivial automorphism in G_1 has a fixed point set of codimension greater than or equal to $2N_2$. This is a contradiction.

For N_1 and N_2 large enough, it follows that the G–action on A is free outside a subset of codimension greater than n. Then, by Lemma 6, A/G contains a smooth n–dimensional subvariety X whose cohomology below degree n is given by the G–invariants of A.

For the proof of the theorem, it remains to show $h^{2,0}(X) = g$ and $h^{1,1}(X) = 1$. For this purpose, we denote the fundamental class of the j^{th} curve factor of A by

$$\Omega_j \in H^{1,1}(A).$$

Moreover, we pick for $j = 1, \dots, 2N_1N_2$ a basis $\omega_{j1}, \dots, \omega_{jg}$ of $(1,0)$–classes of the j^{th} curve factor of A in such a way that

$$\psi_g^* \omega_{jl} = \zeta^l \omega_{jl},$$

for a fixed $(2g+1)^{\text{th}}$ root of unity ζ. Then the cohomology ring of A is generated by the Ω_j, ω_{jl} and their conjugates.

Suppose that we are given a G–invariant $(1,1)$–class which contains $\omega_{is} \wedge \overline{\omega_{jr}}$ nontrivially. Then application of a suitable automorphism in G_1 shows that after relabeling $A_1, \dots, A_{N_1}$, we may assume $1 \le i, j \le 2N_2$. Moreover, it follows that i and j have the same parity, since otherwise $r+s$ would be zero modulo $2g+1$, contradicting

$1 \le r, s \le g$. Finally, application of a suitable element in G_2 shows that $i = j$. Since $\omega_{is} \wedge \overline{\omega_{ir}}$ is a multiple of Ω_i, it follows that our G–invariant $(1,1)$–class is of the form

$$\lambda_1 \cdot \Omega_1 + \cdots + \lambda_{2N_1N_2} \cdot \Omega_{2N_1N_2}.$$

Since G acts transitively on the curve factors of A, this class is G–invariant if and only if $\lambda_1 = \cdots = \lambda_{2N_1N_2}$. This proves $h^{1,1}(X) = 1$.

It remains to show $h^{2,0}(X) = g$. We define for $l = 1, \dots, g$ the $(2,0)$–class

$$\omega_l := \sum_{i=1}^{N_1N_2} \omega_{2i-1l} \wedge \omega_{2il}$$

and claim that these form a basis of the G–invariant $(2,0)$–classes of A. Clearly, they are linearly independent and it is easy to see that they are G–invariant.

Conversely, suppose that a G–invariant class contains $\omega_{il_1} \wedge \omega_{jl_2}$ nontrivially. Then, application of a suitable element in G_1 shows that $l_1 \pm l_2$ is zero modulo $2g+1$. This implies $l_1 = l_2$. Therefore, our G–invariant $(2,0)$–class is of the form

$$\sum_{ijl} \lambda_{ijl} \cdot \omega_{il} \wedge \omega_{jl}.$$

For fixed $l = 1, \dots, g$, we write $\lambda_{ij} = \lambda_{ijl}$ and note that

$$\sum_{ij} \lambda_{ij} \cdot \omega_{il} \wedge \omega_{jl}$$

is also G–invariant. We want to show that this class is a multiple of ω_l. Applying suitable elements of G_1 shows that the above $(2,0)$–class is a sum of $(2,0)$–classes of the factors $A_1, \dots, A_{N_1}$. Since this sum is invariant under the permutation of the factors $A_1, \dots, A_{N_1}$, it suffices to consider the class

$$\sum_{i,j=1}^{2N_2} \lambda_{ij} \cdot \omega_{il} \wedge \omega_{jl}$$

on A_1, which is invariant under the induced G_2– and G_5–action on A_1. In this sum we may assume $\lambda_{ij} = 0$ for all $i \ge j$ and application of a suitable element in G_2 shows that the above class is given by

$$\sum_{i=1}^{N_2} \lambda_{2i-12i} \cdot \omega_{2i-1l} \wedge \omega_{2il}.$$

Finally, application of elements of G_5 proves that our class is a multiple of

$$\sum_{i=1}^{N_2} \omega_{2i-1l} \wedge \omega_{2il}.$$

This finishes the proof of $h^{2,0}(X) = g$ and thereby establishes Theorem 15. □

Remark 16 The above construction does not generalize to higher degrees, at least not in the obvious way.

8 Primitive Hodge numbers away from the vertical middle axis

In this section we produce examples whose primitive Hodge numbers away from the vertical middle axis of the Hodge diamond (1-4) are concentrated in a single (p,q)–type. These examples will then be used in the proof of Theorem 5 in Section 9. Our precise result is as follows:

Theorem 17 *For $a > b \geq 0$, $n \geq a+b$ and $c \geq 1$, there exists an n–dimensional smooth complex projective variety whose primitive (p,q)–type cohomology has dimension $(3^c-1)/2$ if $p = a$ and $q = b$, and vanishes for all other $p > q$.*

In comparison with Theorem 3, the advantage of Theorem 17 is that it also controls the Hodge numbers $h^{p,q}$ with $p \neq q$ and $p+q = n$. These numbers lie in the horizontal middle row of the Hodge diamond (1-4) and so they were excluded in the statement of Theorem 3.

Using an iterated resolution of $(\mathbb{Z}/3\mathbb{Z})$–quotient singularities whose local description is given in Section 8.1, we explain an inductive construction method in Section 8.2. Using this construction, Theorem 17 will easily follow in Section 8.3. Our approach is inspired by Cynk–Hulek's construction of rigid Calabi–Yau manifolds [7].

8.1 Local resolution of $\mathbb{Z}/3\mathbb{Z}$–quotient singularities

Fix a primitive third root of unity ξ and choose affine coordinates $(x_1, \dots, x_n)$ on $\mathbb{C}^n$. For an open ball $Y \subseteq \mathbb{C}^n$ centered at 0 and for some $r \geq 0$, we consider the automorphism $\phi\colon Y \to Y$ given by

$$(x_1, \dots, x_n) \mapsto (\xi \cdot x_1, \dots, \xi \cdot x_r, \xi^2 \cdot x_{r+1}, \dots, \xi^2 \cdot x_n).$$

Let Y' be the blow-up of Y in the origin with exceptional divisor $E' \subseteq Y'$. Then ϕ lifts to an automorphism $\phi' \in \mathrm{Aut}(Y')$ and we define Y'' to be the blow-up of Y' along $\mathrm{Fix}_{Y'}(\phi')$. The exceptional divisor of this blow-up is denoted by $E'' \subseteq Y''$ and ϕ' lifts to an automorphism $\phi'' \in \mathrm{Aut}(Y'')$. In this situation, we have the following lemma.

Lemma 18 *The fixed point set of ϕ'' on Y'' equals E''. Moreover:*

(1) *If $r = 0$ or $r = n$, then $E'' \cong E' \cong \mathbb{P}^{n-1}$. Otherwise, $E' \cong \mathbb{P}^{n-1}$ and E'' is a disjoint union of $\mathbb{P}^{r-1} \times \mathbb{P}^{n-r}$ and $\mathbb{P}^r \times \mathbb{P}^{n-r-1}$.*

(2) *The quotient Y''/ϕ'' is smooth and it admits local holomorphic coordinates $(z_1, \dots, z_n)$, where each z_j comes from a ϕ–invariant meromorphic function on Y, explicitly given by a quotient of two monomials in $x_1, \dots, x_n$.*

Proof This lemma is proved by a calculation similar to Kollár [11, pages 84–87], where the case $n = 2$ is carried out.

The automorphism ϕ' acts on the exceptional divisor $E' \cong \mathbb{P}^{n-1}$ of $Y' \to Y$ by

$$[x_1 : \dots : x_n] \mapsto [\xi \cdot x_1 : \dots : \xi \cdot x_r : \xi^2 \cdot x_{r+1} : \dots : \xi^2 \cdot x_n].$$

Hence, if $r = 0$ or $r = n$, then $\mathrm{Fix}_{Y'}(\phi')$ equals E'. Since this is a smooth divisor on Y', the blow-up $Y'' \to Y'$ is an isomorphism and the quotient Y''/ϕ'' is smooth. Moreover, $E' \cong E''$ is covered by n charts $U_1, \dots, U_n$ such that on U_i, coordinates are given by

$$\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right). \tag{8-1}$$

The quotient Y''/ϕ'' is then covered by $U_1/\phi'', \dots, U_n/\phi''$. Coordinate functions on U_i/ϕ'' are given by the following ϕ–invariant rational functions on Y:

$$\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i^3, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

This proves the lemma for $r = 0$ or $r = n$.

If $0 < r < n$, then $\mathrm{Fix}_{Y'}(\phi')$ equals the disjoint union of $E'_1 \cong \mathbb{P}^{r-1}$ and $E'_2 \cong \mathbb{P}^{n-r-1}$, sitting inside E'. The exceptional divisor E' is still covered by the n–charts $U_1, \dots, U_n$, defined above. Moreover, the charts $U_1, \dots, U_r$ cover E'_1 and $U_{r+1}, \dots, U_n$ cover E'_2. Fix a chart U_i with coordinate functions $(z_1, \dots, z_n)$. If $i \le r$, then ϕ' acts on $r-1$ of these coordinates by the identity and on the remaining coordinates by multiplication with ξ. Conversely, if $i > r$, then ϕ' acts on $n-r-1$ coordinates by the identity and on the remaining coordinates by multiplication with ξ^2. We are therefore in the situation discussed in the previous paragraph and the lemma follows by an application of that result in dimension $n-r+1$ and $r+1$ respectively. □

8.2 Inductive approach

In this section we explain a general construction method which will allow us to prove Theorem 17 in Section 8.3 by induction on the dimension.

For natural numbers $a \neq b$ and $c \geq 0$, let $\mathcal{S}_c^{a,b}$ denote the family of pairs (X, ϕ), consisting of a smooth complex projective variety X of dimension $a+b$ and an automorphism $\phi \in \mathrm{Aut}(X)$ of order 3^c, such that properties (1)–(5) below hold. Here ζ denotes a fixed primitive $(3^c)^{\text{th}}$ root of unity and $g := (3^c - 1)/2$.

(1) The Hodge numbers $h^{p,q}$ of X are given by $h^{a,b} = h^{b,a} = g$ and $h^{p,q} = 0$ for all other $p \neq q$.

(2) The action of ϕ on $H^{a,b}(X)$ has eigenvalues $\zeta, \dots, \zeta^g$.

(3) The group $H^{p,p}(X)$ is for all $p \geq 0$ generated by algebraic classes which are fixed by the action of ϕ.

(4) The set $\mathrm{Fix}_X(\phi^{3^{c-1}})$ can be covered by local holomorphic charts such that ϕ acts on each coordinate function by multiplication with some power of ζ.

(5) For $0 \leq l \leq c-1$, the cohomology of $\mathrm{Fix}_X(\phi^{3^l})$ is generated by algebraic classes which are fixed by the action of ϕ.

For $0 \leq l \leq c-1$, we have obvious inclusions

$$\mathrm{Fix}_X\big(\phi^{3^l}\big) \subseteq \mathrm{Fix}_X\big(\phi^{3^{c-1}}\big).$$

It follows from (4) that $\mathrm{Fix}_X(\phi^{3^l})$ can be covered by local holomorphic coordinates on which ϕ^{3^l} acts by multiplication with some power of ζ^{3^l}. In particular, $\mathrm{Fix}_X(\phi^{3^l})$ is smooth for all $0 \leq l \leq c-1$; its cohomology is of (p,p)–type, since it is generated by algebraic classes by (5). We also remark that condition (3) implies that each variety in $\mathcal{S}_c^{a,b}$ satisfies the Hodge conjecture. Finally, note that $(X, \phi) \in \mathcal{S}_c^{a,b}$ is equivalent to $(X, \phi^{-1}) \in \mathcal{S}_c^{b,a}$.

The inductive approach to Theorem 17 is now given by the following.

Proposition 19 *Let $(X_1, \phi_1^{-1}) \in \mathcal{S}_c^{a_1,b_1}$ and $(X_2, \phi_2) \in \mathcal{S}_c^{a_2,b_2}$. Then*

$$(X_1 \times X_2)/\langle \phi_1 \times \phi_2 \rangle$$

admits a smooth model X such that the automorphism $\mathrm{id} \times \phi_2$ on $X_1 \times X_2$ induces an automorphism $\phi \in \mathrm{Aut}(X)$ with $(X, \phi) \in \mathcal{S}_c^{a,b}$, where $a = a_1 + a_2$ and $b = b_1 + b_2$.

Proof We define the subgroup

$$G := \langle \phi_1 \times \mathrm{id}, \mathrm{id} \times \phi_2 \rangle$$

of $\mathrm{Aut}(X_1 \times X_2)$. For $i = 1, \dots, c$ we consider the element

$$\eta_i := (\phi_1 \times \phi_2)^{3^{c-i}}$$

of order 3^i in G. This element generates a cyclic subgroup

$$G_i := \langle \eta_i \rangle \subseteq G,$$

and we obtain a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_c = \langle \phi_1 \times \phi_2 \rangle,$$

such that each quotient G_i / G_{i-1} is cyclic of order three, generated by the image of η_i.

By definition, G acts on

$$Y_0 := X_1 \times X_2.$$

Using the assumptions that (X_1, ϕ_1^{-1}) and (X_2, ϕ_2) satisfy (1)–(3), it is easily seen (and we will give the details later in this proof) that the $\langle \phi_1 \times \phi_2 \rangle$–invariant cohomology of Y_0 has Hodge numbers $h^{a,b} = h^{b,a} = g$ and $h^{p,q} \neq 0$ for all other $p \neq q$. The strategy of the proof of Proposition 19 is now as follows.

We will construct inductively for $i = 1, \dots, c$ smooth models Y_i of Y_0 / G_i, fitting into the following diagram:

(8-2)
$$\begin{tikzcd}
 & Y''_{c-1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Y''_1 \arrow[dl] \arrow[dr] & & Y''_0 \arrow[dl] \arrow[dr] & \\
Y_c & & Y_{c-1} & & Y_2 & & Y_1 & & Y_0
\end{tikzcd}$$

Here, $Y''_{i-1} \to Y_i$ will be a $3:1$ cover, branched along a smooth divisor, and $Y''_i \to Y_i$ will be the composition $Y''_i \to Y'_i \to Y_i$ of two blow-down maps. This way we obtain a smooth model

$$X := Y_c$$

of $Y_0 / \langle \phi_1 \times \phi_2 \rangle$. At each stage of our construction, the group G will act (in general noneffectively) and we will show that each blow-up and each triple quotient changes the $\langle \phi_1 \times \phi_2 \rangle$–invariant cohomology only by algebraic classes which are fixed by the G–action. Since $\langle \phi_1 \times \phi_2 \rangle$ acts trivially on X, it follows that $H^*(X, \mathbb{C})$ is generated by $\langle \phi_1 \times \phi_2 \rangle$–invariant classes on Y_0 together with algebraic classes which are fixed by the action of G. Hence X satisfies (1). We then define $\phi \in \mathrm{Aut}(X)$ via the action of $\mathrm{id} \times \phi_2 \in G$ on Y_c and show carefully that the technical conditions (2)–(5) are met by (X, ϕ).

In the following, we give the details of the approach outlined above.

We begin with the explicit construction of diagram (8-2). Firstly, let Y_0' be the blow-up of Y_0 along $\mathrm{Fix}_{Y_0}(\eta_1)$. Since G is an abelian group, its action on Y_0 restricts to an action on $\mathrm{Fix}_{Y_0}(\eta_1)$ and so it lifts to an action on the blow-up Y_0'. This allows us to define Y_0'' via the blow-up of Y_0' along $\mathrm{Fix}_{Y_0'}(\eta_1)$. Again, G lifts to Y_0'' since it is abelian. Using this action, we define

$$Y_1 := Y_0''/\langle \eta_1 \rangle,$$

where by abuse of notation, $\langle \eta_1 \rangle$ denotes the subgroup of $\mathrm{Aut}(Y_0'')$ which is generated by the action of $\eta_1 \in G$.

We claim that Y_1 is a smooth model of $Y_0/\langle \eta_1 \rangle$. To see this, we define

$$U_0 := Y_0 \setminus \mathrm{Fix}_{Y_0}(\eta_1)$$

and note that the preimage of this set under the blow-down maps

$$Y_0'' \longrightarrow Y_0' \longrightarrow Y_0$$

gives Zariski open subsets

$$U_0' \subseteq Y_0' \quad \text{and} \quad U_0'' \subseteq Y_0'',$$

both isomorphic to U_0. The group G acts on these subsets and so

$$U_1 := U_0''/\langle \eta_1 \rangle$$

is a Zariski open subset in Y_1 which is isomorphic to the Zariski open subset

$$U_0/\langle \eta_1 \rangle \subseteq Y_0/\langle \eta_1 \rangle.$$

The latter is smooth since η_1 acts freely on U_0 and so it remains to see that Y_1 is smooth at points of the complement of $U_1 \subseteq Y_1$. To see this, note that by (4),

$$\mathrm{Fix}_{Y_0}(\eta_1) = \mathrm{Fix}_{X_1}\big(\phi_1^{3^{c-1}}\big) \times \mathrm{Fix}_{X_2}\big(\phi_2^{3^{c-1}}\big)$$

inside Y_0 can be covered by local holomorphic coordinates on which $\phi_1 \times \phi_2$ acts by multiplication with some powers of ζ. On these coordinates, η_1 acts by multiplication with some powers of a third root of unity. The local considerations of Lemma 18 therefore apply and we deduce that Y_1 is indeed a smooth model of Y_0/G_1.

Since G is abelian, the G–action on Y_0'' descends to a G–action on Y_1. The subgroup $G_1 \subseteq G$ acts trivially on Y_1 and the induced G/G_1–action on Y_1 is effective. Also note that G_i acts freely on $U_0 \subseteq Y_0$ and so G_i/G_1 acts, for $2 \le i \le c$, freely on the Zariski open subset $U_1 \subseteq Y_1$. By (4), the complement of U_0 in Y_0 can be covered by local holomorphic coordinates on which G acts by multiplication with some roots of unity on each coordinate. It therefore follows from the second statement in Lemma 18 that the complement of U_1 in Y_1 can also be covered by local holomorphic coordinates in which G acts by multiplication with some roots of unity on each coordinate. This shows that we can repeat the above construction inductively.

We obtain for $i \in \{1, \dots, c\}$ smooth models

$$Y_i := Y''_{i-1}/\langle \eta_i \rangle$$

of Y_0/G_i on which G acts (noneffectively). The smooth model Y_i contains a Zariski open subset

$$U_i \cong U_0/\langle \eta_i \rangle$$

on which G_l/G_i acts freely for all $i+1 \le l \le c$; explicitly, $U_i := U''_{i-1}/\langle \eta_i \rangle$, where $U''_{i-1} \subseteq Y''_{i-1}$ is isomorphic to U_{i-1}. The complement of U_i is covered by local holomorphic coordinates on which G acts by multiplication with some roots of unity on each coordinate.

Y''_i is then defined via the two-fold blow-up

$$Y''_i \longrightarrow Y'_i \longrightarrow Y_i, \tag{8-3}$$

where one blows up the fixed point set of the action of η_{i+1} on Y_i and Y'_i respectively. The preimage of U_i in Y'_i and Y''_i gives Zariski open subsets

$$U'_i \subseteq Y'_i \quad \text{and} \quad U''_i \subseteq Y''_i,$$

which are both isomorphic to U_i. Since G is abelian, the G–action on Y_i induces actions on Y'_i and Y''_i and these actions restrict to actions on $U_i \cong U'_i \cong U''_i$. The complement of U'_i in Y'_i (resp. U''_i in Y''_i) is by Lemma 18 covered by local holomorphic coordinates on which G acts by multiplication with some roots of unity on each coordinate. Using the local considerations in Lemma 18, it follows that $Y_{i+1} = Y''_i/\langle \eta_{i+1} \rangle$ is a smooth model of Y_0/G_{i+1} which has the above stated properties. This finishes the inductive construction of diagram (8-2).

Our next aim is to compute the cohomology of Y_c. Since G_c acts trivially on Y_c, we may as well compute the G_c–invariant cohomology of Y_c. This point of view has the advantage that it allows an inductive approach, since for $i = 0, \dots, c-1$, the G_c–invariant cohomology of Y_i is easier to compute than its ordinary cohomology.

Before we can carry out these calculations, we have to study the action of arbitrary subgroups $\Gamma \subseteq G$ on Y_i, Y'_i and Y''_i. Since G is an abelian group, it follows that it acts on the fixed point sets $\mathrm{Fix}_{Y_i}(\Gamma)$, $\mathrm{Fix}_{Y_i'}(\Gamma)$ and $\mathrm{Fix}_{Y_i''}(\Gamma)$, defined in (1-6). These actions have the following important properties, where as usual, cohomology means singular cohomology with coefficients in $\mathbb{C}$ (see our conventions in Section 1.6).

Lemma 20 *Let $\Gamma \subseteq G$ be a subgroup which is not contained in G_i. Then $\mathrm{Fix}_{Y_i}(\Gamma)$, $\mathrm{Fix}_{Y_i'}(\Gamma)$ and $\mathrm{Fix}_{Y_i''}(\Gamma)$ are smooth, their G–actions restrict to actions on each irreducible component and their G_c–invariant cohomology is generated by G–invariant algebraic classes.*

Note that the assumption $\Gamma \not\subseteq G_i$ is equivalent to saying that the action of Γ is nontrivial on each of the spaces Y_i, Y_i' and Y_i''.

Proof of Lemma 20 To begin with, we want to verify the lemma for $\mathrm{Fix}_{Y_0}(\Gamma)$, where $\Gamma \subseteq G$ is nontrivial. Recall that $Y_0 = X_1 \times X_2$ and that each element in Γ is of the form $\phi_1^j \times \phi_2^k$. The fixed point set of such an element is then given by

$$\mathrm{Fix}_{Y_0}\big(\phi_1^j \times \phi_2^k\big) = \mathrm{Fix}_{X_1}\big(\phi_1^j\big) \times \mathrm{Fix}_{X_2}\big(\phi_2^k\big).$$

The intersection of sets of the above form is still of the above form and so

$$\mathrm{Fix}_{Y_0}(\Gamma) = \mathrm{Fix}_{X_1}\big(\phi_1^j\big) \times \mathrm{Fix}_{X_2}\big(\phi_2^k\big),$$

for some natural numbers j and k. Since (X_1, ϕ_1^{-1}) and (X_2, ϕ_2) satisfy (4), it follows that $\mathrm{Fix}_{Y_0}(\Gamma)$ is smooth. Also, G acts trivially on $H^0(\mathrm{Fix}_{Y_0}(\Gamma), \mathbb{C})$ by (5) and so the G–action restricts to an action on each irreducible component of $\mathrm{Fix}_{Y_0}(\Gamma)$.

Since Γ is not the trivial group, we now assume without loss of generality that j is not divisible by 3^c. Since (X, ϕ_1^{-1}) satisfies (5), the cohomology of $\mathrm{Fix}_{X_1}(\phi_1^j)$ is then generated by $\langle\phi_1\rangle$–invariant algebraic classes. The G_c–invariant cohomology of $\mathrm{Fix}_{Y_0}(\Gamma)$ is therefore generated by products of these algebraic classes with $\langle\phi_2\rangle$–invariant classes on $\mathrm{Fix}_{X_2}(\phi_2^k)$. Since (X_2, ϕ_2) satisfies (1)–(3) and (5), the latter are, regardless whether k is divisible by 3^c or not, given by $\langle\phi_2\rangle$–invariant algebraic classes. This shows that the G_c–invariant cohomology of $\mathrm{Fix}_{Y_0}(\Gamma)$ is generated by G–invariant algebraic classes, as we want.

Using induction, let us now assume that the lemma is true for $\mathrm{Fix}_{Y_i}(\Gamma)$ for some $i \geq 0$ and for all $\Gamma \not\subseteq G_i$. Blowing-up $\mathrm{Fix}_{Y_i}(\eta_{i+1})$ on Y_i, we obtain the diagram

$$\begin{tikzcd} \mathrm{Fix}_{Y_i'}(\Gamma) \arrow[r, hook] \arrow[d] & Y_i' \arrow[d] \\ \mathrm{Fix}_{Y_i}(\Gamma) \arrow[r, hook] & Y_i \end{tikzcd}$$

and we denote the exceptional divisor of the blow-up $Y_i' \to Y_i$ by $E_i' \subseteq Y_i'$.

Let us first prove that $\mathrm{Fix}_{Y_i'}(\Gamma)$ is smooth and that G acts on its irreducible components. To see this, note that away from E_i', the blow-down map $Y_i' \to Y_i$ is an isomorphism onto its image. Since $\mathrm{Fix}_{Y_i}(\Gamma)$ is smooth, it is then clear that the intersection of $\mathrm{Fix}_{Y_i'}(\Gamma)$ with $Y_i' \setminus E_i'$ is smooth. Also, G acts on the irreducible components of $\mathrm{Fix}_{Y_i'}(\Gamma)$ which are not contained in E_i', since the analogous statement holds for the components of $\mathrm{Fix}_{Y_i}(\Gamma)$. On the other hand, E_i' can be covered by local holomorphic coordinates on which G acts by multiplication with roots of unity. In each of these charts, $\mathrm{Fix}_{Y_i'}(\Gamma)$ corresponds to a linear subspace on which G acts. We conclude that $\mathrm{Fix}_{Y_i'}(\Gamma)$ is smooth and that G acts on each of its irreducible components.

Next, let P be an irreducible component of $\mathrm{Fix}_{Y_i'}(\Gamma)$. We have to prove the following.

Claim *The G_c–invariant cohomology of P is generated by G–invariant algebraic classes.*

Proof Let us denote the image of P in Y_i by Z. Then Z is contained in $\mathrm{Fix}_{Y_i}(\Gamma)$ and the proof of the claim is divided into two cases.

In the first case, we suppose that Z is not contained in the intersection

$$\mathrm{Fix}_{Y_i}(\langle \Gamma, \eta_{i+1}\rangle) = \mathrm{Fix}_{Y_i}(\Gamma) \cap \mathrm{Fix}_{Y_i}(\eta_{i+1}). \tag{8-4}$$

In this case, P is the strict transform of Z in Y_i'. Conversely, if $\tilde{Z} \subseteq \mathrm{Fix}_{Y_i}(\Gamma)$ is any irreducible component, not contained in (8-4), then its strict transform in Y_i' is contained in $\mathrm{Fix}_{Y_i'}(\Gamma)$. Hence Z is in fact an irreducible component of $\mathrm{Fix}_{Y_i}(\Gamma)$. This implies that $\mathrm{Fix}_Z(\eta_{i+1})$ consists of irreducible components of (8-4) and so $\mathrm{Fix}_Z(\eta_{i+1})$ is smooth by induction. Moreover, the strict transform P of Z in Y_i' can be identified with the blow-up of Z along $\mathrm{Fix}_Z(\eta_{i+1})$. We denote the exceptional divisor of this blow-up by D and obtain natural maps

$$f\colon D \hookrightarrow P \quad \text{and} \quad g\colon D \to \mathrm{Fix}_Z(\eta_{i+1}),$$

where f denotes the inclusion and g the projection map respectively. Using Theorem 11 and (4-1), we see that the cohomology of P is generated (as a $\mathbb{C}$–module) by pullback classes of Z together with products

$$[D']^j \wedge f_*(g^*(\alpha)),$$

where D' is an irreducible component of D, j is some natural number and α is a cohomology class on $\mathrm{Fix}_Z(\eta_{i+1})$.

The image $g(D')$ is an irreducible component of $\mathrm{Fix}_Z(\eta_{i+1})$. By induction, G acts on $g(D')$ and hence also on D', the projectivization of the normal bundle of $g(D')$

in Z. This implies that $[D'] \in H^*(P,\mathbb{C})$ is a G–invariant algebraic class. Moreover, the G_c–invariant cohomology of Z as well as the G_c–invariant cohomology of $\mathrm{Fix}_Z(\eta_{i+1})$ is generated by G–invariant algebraic classes by induction. It therefore follows from the above description of $H^*(P,\mathbb{C})$ that the G_c–invariant cohomology of P is indeed generated by G–invariant algebraic classes.

It remains to deal with the case where the image Z of P in Y_i is contained in (8-4). In this case, around each point of Z there are local holomorphic coordinates $(z_1,\dots,z_n)$ on which G acts by multiplication with some roots of unity. In these local coordinates, the fixed point set of η_{i+1} corresponds to the vanishing set of certain coordinate functions. After relabeling these coordinate functions if necessary, we may therefore assume that locally, $\mathrm{Fix}_{Y_i}(\eta_{i+1})$ corresponds to $\{z_m=\dots=z_n=0\}$ for some $m\le n$. This yields local homogeneous coordinates

$$(z_1,\dots,z_{m-1},[z_m:\dots:z_n]) \tag{8-5}$$

along the exceptional divisor E_i' of $Y_i'\to Y_i$. After relabeling of the first $m-1$ coordinates if necessary, we may assume that Γ acts trivially on $z_1,\dots,z_{k-1}$ and nontrivially on $z_k,\dots,z_{m-1}$ for some $1\le k\le m-1$. After relabeling $z_m,\dots,z_n$ if necessary, we may then assume that in the homogeneous coordinates (8-5), P corresponds to $\{z_k=\dots=z_h=0\}$ for some $m\le h\le n$. Here, each element $\gamma\in\Gamma$ acts trivially on $[z_{h+1}:\dots:z_n]$, that is, γ acts by multiplication with the same root of unity on $z_{h+1},\dots,z_n$.

The above local description shows that $P\to Z$ is a PGL–subbundle of the PGL–bundle $E_i'|_Z\to Z$; explicit bundle charts for P are given by $(z_1,\dots,z_{k-1},[z_{h+1}:\dots:z_n])$ as above. The exceptional divisor E_i' carries the line bundle $\mathcal{O}_{E_i'}(1)$ and we denote its restriction to P by $\mathcal{O}_P(1)$. The cohomology of P is then generated (as a $\mathbb{C}$–module) by products of pullback classes on the base Z with powers of $c_1(\mathcal{O}_P(1))$. The line bundle $\mathcal{O}_{E_i'}(1)$ on the exceptional divisor E_i' is isomorphic to the restriction of the line bundle $\mathcal{O}_{Y_i'}(-E_i')$ on Y_i'. The first Chern class of the latter line bundle is G–invariant since G acts on E_i'. It follows that $c_1(\mathcal{O}_P(1))$ is a G–invariant algebraic cohomology class on P.

In the above local coordinates $(z_1,\dots,z_n)$ on Y_i, Z is given by $\{z_k=\dots=z_n=0\}$. The latter set is in fact the fixed point set of $\langle\Gamma,\eta_{i+1}\rangle$ in this local chart and so it follows that Z is an irreducible component of (8-4). By induction, the G_c–invariant cohomology of Z is therefore generated by G–invariant algebraic classes. By the above description of $H^*(P,\mathbb{C})$, we conclude that the G_c–invariant cohomology of P is generated by G–invariant algebraic classes, as we want. This finishes the proof of our claim. □

Altogether, we see that the lemma holds for $\mathrm{Fix}_{Y_i'}(\Gamma)$. Repeating the above argument, we then deduce the same assertion for $\mathrm{Fix}_{Y_i''}(\Gamma)$.

Next, let Γ be a subgroup of G, not contained in G_{i+1}. We denote by

$$p_i\colon Y_i'' \longrightarrow Y_{i+1}$$

the quotient map. Then

$$p_i^{-1}(\mathrm{Fix}_{Y_{i+1}}(\Gamma)) = \bigl\{y \in Y_i'' \bigm| g(y) \in \{y, \eta_{i+1}(y), \eta_{i+1}^2(y)\} \text{ for all } g \in \Gamma\bigr\}.$$

If this set is contained in $\mathrm{Fix}_{Y_i''}(\eta_{i+1})$, then it is given by $\mathrm{Fix}_{Y_i''}(\langle \Gamma, \eta_{i+1}\rangle)$. The restriction of p_i to $\mathrm{Fix}_{Y_i''}(\eta_{i+1})$ is an isomorphism onto its image and so we deduce that in this case, $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$ satisfies the lemma.

Conversely, if $p_i^{-1}(\mathrm{Fix}_{Y_{i+1}}(\Gamma))$ is not contained in $\mathrm{Fix}_{Y_i''}(\eta_{i+1})$, then we pick some

$$y \in p_i^{-1}(\mathrm{Fix}_{Y_{i+1}}(\Gamma)) \quad \text{with } y \notin \mathrm{Fix}_{Y_i''}(\eta_{i+1}).$$

Since η_{i+1} acts trivially on Y_{i+1} and since we are interested in $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$, we assume without loss of generality that η_{i+1} is contained in Γ. Then, Γ acts transitively on $\{y, \eta_{i+1}(y), \eta_{i+1}^2(y)\}$. This gives rise to a short exact sequence

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 1,$$

where $H \subseteq \Gamma$ acts trivially on y and where $g \in \Gamma$ is mapped to $j + 3\mathbb{Z}$ if and only if $g(y) = \eta_{i+1}^j(y)$. Recall that $G \cong \mathbb{Z}/3^c\mathbb{Z} \times \mathbb{Z}/3^c\mathbb{Z}$, and so $\Gamma \cong \mathbb{Z}/3^k\mathbb{Z} \times \mathbb{Z}/3^m\mathbb{Z}$ for some $k, m \geq 0$. In the above short exact sequence, η_{i+1} is mapped to a generator in $\mathbb{Z}/3\mathbb{Z}$ and so η_{i+1} cannot be a multiple of 3 in Γ. That is,

$$\Gamma \cong \langle \eta_{i+1}\rangle \times \langle \gamma\rangle$$

for some $\gamma \in \Gamma$. Since η_{i+1} acts trivially on Y_{i+1}, one easily deduces that

$$\mathrm{Fix}_{Y_{i+1}}(\Gamma) = \mathrm{Fix}_{Y_{i+1}}(\gamma) = \bigcup_{j=0}^{2} p_i\bigl(\mathrm{Fix}_{Y_i''}\bigl(\gamma \circ \eta_{i+1}^j\bigr)\bigr). \tag{8-6}$$

The irreducible components of $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$ are therefore of the form $p_i(Z)$, where Z is an irreducible component of

$$\bigcup_{j=0}^{2} \mathrm{Fix}_{Y_i''}\bigl(\gamma \circ \eta_{i+1}^j\bigr).$$

As we have already proven the lemma on Y_i'', we know that the G–action on Y_i'' restricts to an action on Z. In particular,

$$p_i(Z) = Z/\langle \eta_{i+1}\rangle.$$

Since the abelian group G acts on Z, it also acts on the above quotient.

For the moment we assume that $p_i(Z)$ is smooth. Its cohomology is then given by the η_{i+1}–invariant classes on Z. Since η_{i+1} is contained in G_c, it follows that the G_c–invariant cohomology of $p_i(Z)$ is given by the G_c–invariant cohomology of Z. Since we know the lemma on Y_i'', the latter is generated by G–invariant algebraic classes, as we want.

It remains to see that $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$ is smooth. In the local holomorphic charts which cover the complement of U_{i+1} in Y_{i+1}, this fixed point set is given by linear subspaces which are clearly smooth. It therefore suffices to prove that the fixed point set of Γ on U_{i+1} is smooth. By (8-6), the latter is given by

$$\mathrm{Fix}_{U_{i+1}}(\Gamma) = \Bigl(\bigcup_{j=0}^{2} \mathrm{Fix}_{U_i''}\bigl(\gamma \circ \eta_{i+1}^{j}\bigr)\Bigr) \Big/ \langle \eta_{i+1} \rangle.$$

Since we know the lemma already on Y_i'', the set $\mathrm{Fix}_{U_i''}(\gamma \circ \eta_{i+1}^{j})$ is smooth and η_{i+1} acts on it. This action is free of order three since G_{i+1}/G_i acts freely on U_i''. Therefore,

$$\mathrm{Fix}_{U_i''}\bigl(\gamma \circ \eta_{i+1}^{j}\bigr)/\langle \eta_{i+1} \rangle$$

is smooth for all j. The smoothness of $\mathrm{Fix}_{U_{i+1}}(\Gamma)$ follows since

$$\mathrm{Fix}_{U_i''}\bigl(\gamma \circ \eta_{i+1}^{j_1}\bigr) \cap \mathrm{Fix}_{U_i''}\bigl(\gamma \circ \eta_{i+1}^{j_2}\bigr) = \varnothing$$

holds for $j_1 \not\equiv j_2 \pmod 3$. This concludes Lemma 20 by induction on i. □

Via diagram (8-2), we have constructed a smooth model

$$X := Y_c$$

of $Y_0/\langle \phi_1 \times \phi_2 \rangle$. The group G acts on X and the automorphism $\phi \in \mathrm{Aut}(X)$ which we have to construct in Proposition 19 is simply given by the action of $\mathrm{id} \times \phi_2 \in G$ on X. This automorphism has order 3^c since this is true on the Zariski open subset $U_c \subseteq X$. By Lemma 20, the pair (X, ϕ) satisfies (5); it remains to show that (X, ϕ) satisfies (1)–(4).

The cohomology of X Using Lemma 20, we are now able to read off the cohomology of X from diagram (8-2). Indeed, the cohomology of Y_i'' is given by the cohomology of Y_i (via pullbacks) plus some classes which are introduced by blowing up $\mathrm{Fix}_{Y_i}(\eta_{i+1})$ on Y_i and $\mathrm{Fix}_{Y_i'}(\eta_{i+1})$ on Y_i' respectively. By Lemma 20, these blown-up loci are smooth and their G_c–invariant cohomology is generated by G–invariant algebraic classes. Moreover, G acts on each irreducible component of the blown-up locus and so G acts on each irreducible component of the exceptional divisors of the blow-ups. In particular, the corresponding divisor classes in cohomology are G–invariant. It follows that the G_c–invariant cohomology of Y_i'' is given by the G_c–invariant cohomology of Y_i plus some G–invariant algebraic classes. Also, since η_{i+1} is contained in G_c, the quotient map $Y_i'' \to Y_{i+1}$ induces an isomorphism on G_c–invariant cohomology. It follows inductively that the G_c–invariant cohomology of X — which coincides with the whole cohomology of X — is given by the G_c–invariant cohomology of Y_0 plus G–invariant algebraic classes.

Let us now calculate the G_c–invariant cohomology of Y_0. For $i = 1, 2$, there is by assumption on (X_i, ϕ_i) a basis $\omega_{i1}, \dots, \omega_{ig}$ of $H^{a_i,b_i}(X_i)$ with

$$\phi_1^*(\omega_{1j}) = \zeta^{-j}\omega_{1j} \quad \text{and} \quad \phi_2^*(\omega_{2j}) = \zeta^{j}\omega_{2j}. \tag{8-7}$$

This shows that for $j = 1, \dots, g$, the following linearly independent (a,b)–classes on Y_0 are G_c–invariant:

$$\omega_j := \omega_{1j} \wedge \omega_{2j}.$$

Since (X_1, ϕ_1^{-1}) and (X_2, ϕ_2) satisfy (1), (2) and (3), it follows that apart from the above (a,b)–classes (and their complex conjugates), all G_c–invariant classes on Y_c are generated by products of algebraic classes on X_1 and X_2. These products are G–invariant by (3). Finally, ϕ acts on ω_j by multiplication with ζ^j. Altogether, we have just shown that (X, ϕ) satisfies (1), (2) and (3).

Charts around $\mathrm{Fix}_X(\phi^{3^{c-1}})$ By our construction, there are holomorphic charts which cover the complement of U_c in Y_c, such that ϕ acts on each coordinate function by multiplication with some power of ζ. Therefore, in order to show that (X, ϕ) satisfies (4), it remains to see that around points of

$$W_c := \mathrm{Fix}_{Y_c}(\phi^{3^{c-1}}) \cap U_c,$$

the same holds true.

Let us first prove that the preimage of W_c under the $3^c : 1$ étale covering $\pi\colon U_0 \to U_c$ coincides with the set

$$W_0 := \big(\big(\mathrm{Fix}_{X_1}(\phi_1^{3^{c-1}}) \times X_2\big) \cup \big(X_1 \times \mathrm{Fix}_{X_2}(\phi_2^{3^{c-1}})\big)\big) \cap U_0.$$

Clearly $W_0 \subseteq \pi^{-1}(W_c)$. Conversely, suppose that $(x_1, x_2) \in \pi^{-1}(W_c)$. Then there exists a natural number $1 \le k \le 3^c$ with

$$x_1 = \phi_1^k(x_1) \quad \text{and} \quad \phi_2^{3^{c-1}}(x_2) = \phi_2^k(x_2).$$

If x_1 is not fixed by $\phi_1^{3^{c-1}}$, then 3^{c-1} does not lie in the mod 3^c orbit of k. That is, k is divisible by 3^c and we deduce that x_2 is fixed by $\phi_2^{3^{c-1}}$. This shows that $(x_1, x_2) \in W_0$ as we want.

Since $\pi\colon U_0 \to U_c$ is an étale covering, local holomorphic charts on U_0 give local holomorphic charts on U_c. Around each point

$$x \in \big(\mathrm{Fix}_{X_1}(\phi_1^{3^{c-1}}) \times X_2\big) \cap U_0$$

we may choose local holomorphic coordinates $(z_1, \dots, z_n)$, such that $\phi_1^{-1} \times \mathrm{id}$ acts on each z_j by multiplication with some power of ζ, by assumptions on (X_1, ϕ_1^{-1}). Moreover, the images of $\phi_1^{-1} \times \mathrm{id}$ and $\mathrm{id} \times \phi_2$ in the quotient G/G_c coincide and so the action of $\phi_1^{-1} \times \mathrm{id}$ on X actually coincides with the automorphism ϕ. This shows that $(z_1, \dots, z_n)$ give local holomorphic coordinates around $\pi(x)$ on which ϕ acts by multiplication with some powers of ζ.

The case

$$x \in \big(X_1 \times \mathrm{Fix}_{X_2}(\phi_2^{3^{c-1}})\big) \cap U_0$$

is done similarly and so we conclude that (4) holds for (X, ϕ). This finishes the proof of Proposition 19. □

8.3 Proof of Theorem 17

For $a > b \geq 0$, $n \geq a + b$ and $c \geq 1$, we need to construct an n–dimensional smooth complex projective variety $Z_c^{a,b,n}$ whose primitive (p,q)–type cohomology has dimension $(3^c - 1)/2$ if $p = a$ and $q = b$, and vanishes for all other $p > q$. Suppose that we have already settled the case when $n = a + b$. Then, for $n > a + b$, the product

$$Z_c^{a,b,n} := Z_c^{a,b,a+b} \times \mathbb{P}^{n-a-b}$$

has the desired properties. To prove Theorem 17, it therefore suffices to show that the set $\mathcal{S}_c^{a,b}$, defined in Section 8.2, is nonempty for all $a > b \geq 0$ and $c \geq 1$. We will prove the latter by induction on $a + b$.

We put $g = (3^c - 1)/2$ and consider the hyperelliptic curve C_g with automorphism ψ_g from Section 3.1. It is then straightforward to check that

$$(C_g, \psi_g) \in \mathcal{S}_c^{1,0}. \tag{8-8}$$

Indeed, it is clear that (C_g, ψ_g) satisfies (1)–(3) in the definition of $\mathcal{S}_c^{1,0}$. Moreover, the complement of the point $\infty \in C_g$ is given by the affine curve $y^2 = x^{2g+1} + 1$ and ψ_g acts by multiplication with a primitive $(3^c)^{\text{th}}$ root of unity ζ on x. For all $0 \leq l \leq c-1$, the fixed point set $\mathrm{Fix}_{C_g}(\psi_g^{3^l})$ is therefore given by the points $(x,y)=(0,\pm 1)$ and ∞. These points are ψ_g–invariant and so their cohomology is generated by ψ_g–invariant algebraic classes, which shows that (5) holds. It remains to establish (4). That is, we need to find suitable holomorphic coordinates around the three fixed points of $\psi_g^{3^{c-1}}$. Differentiating the affine equation $y^2=x^{2g+1}+1$ gives $2y\cdot dy=(2g+1)x^{2g}\cdot dx$. This shows that dx spans the cotangent space at $(0,\pm 1)$ and so x is a local coordinate function near $(0,\pm 1)$. The automorphism ψ_g acts on this function by multiplication with ζ, as we want in (4). In order to find a suitable coordinate function around ∞, we use the coordinates (u,v), introduced in Section 3.1. In these coordinates, the curve C_g is given by the equation $v^2=u+u^{2g+2}$ and ∞ corresponds to the point $(u,v)=(0,0)$. Around this point, the function v yields a coordinate function on which ψ_g acts via multiplication with ζ^g; see Section 3.1. This establishes (8-8) and hence settles the case $a+b=1$.

Let now $a>b$ with $a+b>1$. If $b=0$, then by induction, the sets $\mathcal{S}_c^{1,0}$ and $\mathcal{S}_c^{a-1,0}$ are nonempty and so Proposition 19 yields an element in $\mathcal{S}_c^{a,0}$, as desired. If $b\geq 1$, then $\mathcal{S}_c^{a,b-1}$ is nonempty by induction. Also, $\mathcal{S}_c^{0,1}$ is nonempty since it contains (C_g,ψ_g^{-1}) by (8-8). Application of Proposition 19 then yields an element in $\mathcal{S}_c^{a,b}$, as we want. This concludes Theorem 17.

Remark 21 The variety in $\mathcal{S}_c^{a,b}$ which the above proof produces inductively is easily seen to be a smooth model of the quotient of C_g^{a+b} by the group action of $G^1(a,b,g)$, defined in Section 3.2.

9 Proof of Theorem 5

In this section we give a proof of Theorem 5, stated in Section 1. To begin with, we prove that $h^{1,1}$ dominates $h^{2,0}$ nontrivially in dimension two.

Proposition 22 *For a Kähler surface X, the following inequality holds:*

$$h^{1,1}(X)>h^{2,0}(X)$$

Proof First observe that for the product of $\mathbb{P}^1$ with another smooth curve, $h^{2,0}$ vanishes and so the inequality trivially holds because $h^{1,1}>0$ is true for any Kähler manifold. Since any Kähler surface of Kodaira dimension $-\infty$ is birationally equivalent to such a product [3], and since $h^{2,0}$ is a birational invariant, we deduce that the asserted inequality is true in the case of Kodaira dimension $-\infty$. Since blowing-up a point increases $h^{1,1}$ by one and leaves $h^{2,0}$ unchanged, we conclude that it suffices to prove

$h^{1,1}(X) > h^{2,0}(X)$ for all minimal surfaces X of nonnegative Kodaira dimension. For such a surface X, the Bogomolov–Miyaoka–Yau inequality

$$c_1^2(X) \leq 3c_2(X)$$

holds. For Kodaira dimensions 0 and 1, this can be seen in [3, Table 10, page 244] where all possible Chern numbers for minimal surfaces with these Kodaira dimensions are listed. If the Kodaira dimension of X is equal to 2, that is, if X is a minimal surface of general type, then the above inequality is due to Bogomolov–Miyaoka–Yau; see [3, page 275].

In order to translate the above inequality into an inequality between the Hodge numbers of X, we need the following identities which hold for all Kähler surfaces:

$$c_2(X) = 2 - 2b_1(X) + b_2(X),$$
$$c_1^2(X) = 10 - 4b_1(X) + 10h^{2,0}(X) - h^{1,1}(X).$$

Using these, the Bogomolov–Miyaoka–Yau inequality turns out to be equivalent to

$$1 + h^{1,0}(X) + h^{2,0}(X) \leq h^{1,1}(X). \tag{9-1}$$

This clearly implies $h^{1,1}(X) > h^{2,0}(X)$, which finishes the proof of Proposition 22. □

Conversely, let us suppose that the Hodge number $h^{r,s}$ dominates $h^{p,q}$ nontrivially in dimension n. That is, there are positive constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that for all n–dimensional smooth complex projective varieties X, the following holds:

$$c_1 \cdot h^{r,s}(X) + c_2 \geq h^{p,q}(X) \tag{9-2}$$

By the Hodge symmetries (1-2), we may assume $r \geq s$, $p \geq q$, $r + s \leq n$ and $1 \leq p + q \leq n$. The nontriviality of the above domination then means that (9-2) does not follow from the Lefschetz conditions (1-3). In order to prove Theorem 5, it now remains to show $n = 2$, $r = s = 1$ and $p = 2$.

Suppose that $r + s < n$. Since (9-2) does not follow from the Lefschetz conditions (1-3), Theorem 3 (or Corollary 24 below) shows $p + q = n$. Using the Lefschetz hyperplane theorem and the Hirzebruch–Riemann–Roch formula, we see however that a smooth hypersurface $V_d \subseteq \mathbb{P}^{n+1}$ of degree d satisfies $h^{r,s}(V_d) \leq 1$, whereas $h^{p,q}(V_d)$ tends to infinity if d does. This is a contradiction and so $r + s = n$ holds.

Suppose that $r \neq s$. Then, considering a blow-up of $\mathbb{P}^n$ in sufficiently many distinct points proves $p \neq q$. Since $p \neq q$ and $r \neq s$, we may then use certain examples from Theorem 17 to deduce that (9-2) follows from the Lefschetz conditions (1-3). This contradicts the nontriviality of our given domination. Hence $r=s$ and in particular $n=2r$ is even.

Suppose that $p=q$. Considering again a blow-up of $\mathbb{P}^n$ in sufficiently many distinct points then proves $c_1 \geq 1$ and so (9-2) follows from the Lefschetz conditions. This contradicts the nontriviality of (9-2) and so it proves $p \neq q$.

Suppose that $p+q<n$. Using high-degree hyperplane sections of n–dimensional examples from Theorem 3, one proves that there is a sequence of $(n-1)$–dimensional smooth complex projective varieties $(Y_j)_{j\geq 1}$ such that $h^{r-1,r-1}(Y_j)$ is bounded whereas $h^{p,q}(Y_j)$ tends to infinity if j does. (Note that we used $p\neq q$ here.) Since $n=2r$, we have $h^{r-1,r-1}(Y_j)=h^{r,r}(Y_j)$ by the Hodge symmetries. Therefore, the sequence of n–dimensional smooth complex projective varieties

$$(Y_j \times \mathbb{P}^1)_{j\geq 1}$$

has bounded $h^{r,r}$ but unbounded $h^{p,q}$. This is a contradiction and hence shows $p+q=n$.

Next, using Corollary 13 from Section 5, it follows that $p=2r$ and $q=0$ holds. By what we have shown so far we are thus left with the case where $n=2r=2s$, $p=2r$ and $q=0$. In order to finish the proof of Theorem 5, it therefore suffices to show $r=1$. For a contradiction, we assume that $r\geq 2$. By Theorem 17 there exists a $(2r-1)$–dimensional smooth complex projective variety Y with $h^{2r-1,0}(Y)=h^{0,2r-1}(Y)=1$ and $h^{p,q}(Y)=0$ for all other $p\neq q$. Since $r\geq 2$, this implies for a smooth curve C_g of genus g that

$$h^{2r,0}(Y\times C_g)=g \quad \text{and} \quad h^{r,r}(Y\times C_g)=2\cdot h^{r-1,r-1}(Y).$$

Hence $(Y\times C_g)_{g\geq 1}$ is a sequence of $2r$–dimensional smooth complex projective varieties such that $h^{r,r}$ is constant but $h^{2r,0}$ tends to infinity if g does. This is the desired contradiction and shows $r=1$. This finishes the proof of Theorem 5. □

Remark 23 One could of course strengthen Simpson's domination relation between Hodge numbers by requiring that (1-5) holds for all n–dimensional Kähler manifolds X. However, since Proposition 22 holds for all Kähler surfaces, it is immediate that Theorem 5 remains true for this stronger domination relation.

10 Inequalities among Hodge and Betti numbers

It is a very difficult and wide open problem to determine all universal inequalities among Hodge numbers in a fixed dimension; see [20]. In Theorem 5 we basically solved this problem for inequalities of the form (1-5).[1] In this section we deduce from the main results of this paper some further progress on this problem. We formulate our results in the category of smooth complex projective varieties, which is stronger than allowing arbitrary Kähler manifolds.

Our first result is a consequence of Theorem 3:

Corollary 24 *Any universal inequality among the Hodge numbers below the horizontal middle axis in (1-4) of n–dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions (1-3).*

Proof Assume that we are given a universal inequality between the Hodge numbers of the truncated Hodge diamond of smooth complex projective n–folds. In terms of the primitive Hodge numbers $l^{p,q}$, this means that for all natural numbers p and q with $0 < p+q < n$ there are real numbers $\lambda_{p,q}$ and a constant $C \in \mathbb{R}$ such that

$$\sum_{0<p+q<n} \lambda_{p,q} \cdot l^{p,q}(X) \geq C \tag{10-1}$$

holds for all smooth n–folds X. Using the Hodge symmetries (1-2), we may further assume that $\lambda_{p,q} = \lambda_{q,p}$ holds for all p and q. If we put $X = \mathbb{P}^n$, then we see $C \leq 0$. Moreover, for any natural numbers p and q with $0 < p+q < n$, there exists by Theorem 3 a smooth complex projective variety X with $l^{p,q}(X) \gg 0$, whereas (modulo the Hodge symmetries) all remaining primitive Hodge numbers of its truncated Hodge diamond are bounded from above, by n^3 say. This proves $\lambda_{p,q} \geq 0$. That is, the universal inequality (10-1) is a consequence of the Lefschetz conditions (1-3), as we want. □

As an immediate consequence of the above corollary, we note the following:

Corollary 25 *Any universal inequality among the Hodge numbers of smooth complex projective varieties which holds in all sufficiently large dimensions at the same time is a consequence of the Lefschetz conditions.*

In the same way we deduced Corollary 24 from Theorem 3, one deduces the following from Theorem 17.

Corollary 26 *Any universal inequality among the Hodge numbers away from the vertical middle axis in (1-4) of n–dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions (1-3).*

[1] "Basically" means that we did not determine the optimal coefficients in the universal inequality we found in dimension two.

Corollary 24 implies that in dimension n, the Betti numbers b_k with $k \neq n$ do not satisfy any universal inequalities other than the Lefschetz conditions

$$b_k \geq b_{k-2} \quad \text{for all } k \leq n. \tag{10-2}$$

Using a simple construction, we improve this result now. Indeed, the following proposition determines all universal inequalities among the Betti numbers of smooth complex projective varieties in any given dimension.

Proposition 27 *Any universal inequality among the Betti numbers b_k of smooth complex projective n–folds is a consequence of the Lefschetz conditions (10-2).*

Proof By the same argument as in the proof of Corollary 24, it clearly suffices to prove the following claim.

Claim *Let X be the product of $\mathbb{P}^{n-k}$ with some smooth hypersurface $V_d \subseteq \mathbb{P}^{k+1}$ of degree d. Then, for $0 \leq j \leq n$ with $j \neq k$, the j^{th} primitive cohomology $P^j(X)$ of X has dimension less than or equal to 1, whereas $\dim(P^k(X))$ tends to infinity if d does.*

It remains to prove the claim. For $j < k$, the Lefschetz hyperplane theorem yields

$$b_j(V_d) = b_{2k-j}(V_d) = b_j(\mathbb{P}^{k+1}).$$

Moreover, from the adjunction formula we deduce that the topological Euler number $c_k(V_d)$ tends to $\pm\infty$ if $d \to \infty$. This proves that $b_k(V_d)$ tends to infinity if d does.

Using these Betti numbers of V_d, it is straightforward to check that

$$X := V_d \times \mathbb{P}^{n-k}$$

has the primitive cohomology we want. This proves the above claim and thus finishes the proof of Proposition 27. □

11 Threefolds with $h^{1,1} = 1$

Here we show that in dimension three, the constraints which classical Hodge theory puts on the Hodge numbers of smooth complex projective varieties are not complete. Our result generalizes a result of Amorós and Biswas [1, Proposition 4.3], asserting that there is no simply connected Kähler threefold with $h^{2,0} = h^{1,1} = 1$ and $b_3 = 0$.

Proposition 28 *Let X be a smooth complex projective threefold with Hodge numbers $h^{p,q} := h^{p,q}(X)$. Suppose that $h^{1,1} = 1$. Then the following holds:*

- *The outer Hodge numbers satisfy $h^{1,0} = 0$ and $h^{2,0} < \max(h^{3,0}, 1)$.*
- *The canonical bundle of X is antiample if $h^{3,0} = 0$, numerically trivial if $h^{3,0} = 1$ and ample if $h^{3,0} > 1$.*

Moreover, if $h^{3,0} > 1$, then $h^{2,1} < 12^6 \cdot h^{3,0}$ holds and for $h^{3,0} - h^{2,0}$ bounded from above, only finitely many deformation types of such examples exist.

Proposition 28 nicely compares to the examples in Theorem 15, where we have constructed threefolds X with $h^{1,1}(X) = 1$ such that $h^{2,0}(X)$ is equal to any given natural number.

Before we can prove Proposition 28, let us show the following general result.

Lemma 29 *Let X be a Kähler manifold of dimension n and let k be an odd natural number with $2k \leq n$ such that $h^{k,k}(X) = 1$. Then, $b_j(X) = 0$ for all odd $j \leq k$.*

Proof Let ω denote the Kähler class of X. For a contradiction, suppose that the assumptions of the proposition hold and that additionally $b_j(X) \neq 0$ for some odd $j \leq k$. We may assume that j is minimal with this property. Then all j^{th} cohomology is primitive and we pick some nonzero primitive (p,q)–cohomology class α with $p + q = j$. Since $h^{k,k}(X) = 1$ and since $2k \leq n$, the Lefschetz conditions (1-3) imply that $H^{j,j}(X)$ is spanned by ω^j. Thus, by the Hodge–Riemann bilinear relations, we have

$$\alpha \wedge \overline{\alpha} = \lambda \cdot \omega^j$$

for some $\lambda \in \mathbb{C} - \{0\}$. Since $2j \leq 2k \leq n$, we have $\omega^{2j} \neq 0$. As α is of odd degree, this is a contradiction to the above equation and hence establishes the lemma. □

Proof of Proposition 28 Let X be a smooth complex projective threefold with

$$h^{1,1}(X) = 1.$$

The Riemann–Roch formula in dimension three says that

$$c_1(X)c_2(X) = 24\chi(X, \mathcal{O}_X). \tag{11-1}$$

By Lemma 29 we have $h^{1,0}(X) = h^{0,1}(X) = 0$. From the exponential sequence, it therefore follows that X has Picard number one and hence the canonical class K_X of X is either ample, antiample or numerically trivial.

If $-K_X$ is ample, then $h^{2,0}$ and $h^{3,0}$ vanish.

If K_X is numerically trivial, then (11-1) shows that $1+h^{2,0}=h^{3,0}$. Since numerically trivial line bundles have at most one nontrivial section, we deduce $h^{2,0}=0$ and $h^{3,0}=1$.

If K_X is ample, then Yau's inequality holds [22]:

$$c_1(X)c_2(X) \le \tfrac{3}{8}c_1^3(X). \tag{11-2}$$

Together with (11-1), this implies

$$\chi(X,\mathcal{O}_X) \le \tfrac{1}{64}c_1^3(X) < 0. \tag{11-3}$$

Thus $-c_1^3(X)$ can be bounded from above in terms of $h^{3,0}-h^{2,0}$ and hence Kollár–Matsusaka's theorem (see Lazarsfeld [14, page 239]) yields that only finitely many deformation types of threefolds with $h^{1,1}=1$, $h^{3,0}>1$ and $h^{3,0}-h^{2,0}$ bounded from above exist. Furthermore, (11-3) shows that $1+h^{2,0}<h^{3,0}$ holds for any such threefold.

Altogether, this proves firstly $h^{2,0}<\max(h^{3,0},1)$, and secondly that K_X is antiample if $h^{3,0}=0$, it is numerically trivial if $h^{3,0}=1$ and it is ample if $h^{3,0}>1$.

Finally, let us assume that $h^{3,0}>1$ or $h^{2,0}>0$. Then K_X is ample so Fujita's conjecture predicts that $6\cdot K_X$ is very ample; see [14, page 252]. Although this conjecture is still open, Lee proves in [15] that $10\cdot K_X$ is very ample. Thus the following argument due to Catanese and Schneider [4] applies. Firstly, the linear series $|10\cdot K_X|$ embeds X into some $\mathbb{P}^N$ and hence $\Omega_X(20\cdot K_X)$ is a quotient of $\Omega_{\mathbb{P}^N}(2)$ restricted to X. Since the latter is globally generated, it is nef and hence $\Omega_X(20\cdot K_X)$ is nef. Secondly, by Demailly, Peternell and Schneider [8, Corollary 2.6], any Chern number of a nef bundle F on an n–dimensional smooth complex projective variety X is bounded from above by $c_1^n(F)$. In our situation, this yields

$$c_3(\Omega_X^1(20\cdot K_X)) \le c_1^3(\Omega_X^1(20\cdot K_X)). \tag{11-4}$$

A standard computation gives

$$c_3(\Omega_X^1(20\cdot K_X)) = -8400\cdot c_1^3(X) - 20\cdot c_1(X)c_2(X) - c_3(X),$$
$$c_1^3(\Omega_X^1(20\cdot K_X)) = -61^3\cdot c_1^3(X).$$

Together with Yau's inequality (11-2), this yields in (11-4) that

$$1748588\cdot c_1(X)c_2(X) \le 3\cdot c_3(X). \tag{11-5}$$

By the Riemann–Roch formula, this inequality is in fact one between the Hodge numbers of threefolds with ample canonical bundle. In our case, $h^{1,1}=1$ and $h^{1,0}=0$

yield

$$6994346 + 6994346 \cdot h^{2,0} + 3 \cdot h^{2,1} \leq 6994349 \cdot h^{3,0}.$$

Thus, a rough estimation yields

$$h^{2,1} < 12^6 \cdot h^{3,0}.$$

This concludes the proof of the proposition. □

Remark 30 Instead of using [8], but still relying on [15], Chang and Lopez prove in [6] that there is a computable constant $C > 0$ such that $C \cdot c_1(X) c_2(X) \leq c_3(X)$ holds for all threefolds X with ample canonical bundle. Computing C explicitly shows that it is about four times smaller than the analogous constant which appears in (11-5). However, since the explicit extraction of C is slightly tedious and since this constant is still far from being realistic, we did not try to carry this out here.

Using Proposition 28 together with the classification of Fano threefolds in Iskovkikh and Prokhorov [10, page 215], we obtain the following classification of Hodge diamonds of threefolds with $h^{1,1} = 1$ and $h^{3,0} = 0$.

Corollary 31 *Let $h^{p,q}$ be the Hodge numbers of a smooth complex projective threefold with $h^{1,1} = 1$ and $h^{3,0} = 0$. Then $h^{1,0}$ and $h^{2,0}$ vanish, and for $h^{2,1}$ precisely one of the following values occurs:*

$$h^{2,1} \in \{0, 2, 3, 5, 7, 10, 14, 20, 21, 30, 52\}.$$

12 Fourfolds with $h^{1,1} = 1$

Here we show that in dimension four, the constraints which classical Hodge theory puts on the Hodge numbers of smooth complex projective varieties are not complete.

Proposition 32 *Let X be a smooth complex projective fourfold with Hodge numbers $h^{p,q} := h^{p,q}(X)$. If $h^{1,1} = 1$, then $h^{1,0} = 0$ and for bounded $h^{2,0}$, $h^{4,0}$ and $h^{2,2}$, only finitely many values for $h^{3,0}$, $h^{2,1}$ and $h^{3,1}$ occur.*

Since Kähler manifolds with $b_2 = 1$ are projective, Proposition 32 implies immediately that even for the Betti numbers of Kähler manifolds, the known constraints are not complete.

Corollary 33 *Let X be a Kähler fourfold with $b_2(X) = 1$. Then $b_3(X)$ is bounded in terms of $b_4(X)$.*

Proof of Proposition 32 Let X be a smooth complex projective fourfold with Hodge numbers $h^{p,q}$ and Chern numbers $c_1^4, c_1^2c_2, \dots, c_4$. Suppose that $h^{1,1}=1$ and that $h^{2,0}$, $h^{4,0}$ and $h^{2,2}$ are bounded. Then Lemma 29 shows $h^{1,0}=0$. Moreover:

Lemma 34 *The following inequality holds:*

$$224+228h^{2,0}-224h^{3,0}+h^{2,2}-2h^{3,1}+226h^{4,0} \geq \tfrac{1}{3}\cdot(4c_1^2c_2-c_1^4).$$

Proof Since $h^{1,1}=1$, we see that $c_2(X)=\lambda\cdot\omega^2+\alpha$, where α is a primitive $(2,2)$–class and ω the Kähler class on X. Since ω and $c_2(X)$ are real cohomology classes, we obtain

$$\alpha-\overline{\alpha}=-(\lambda-\overline{\lambda})\cdot\omega^2.$$

In this equation, the left-hand side is primitive. However, no nonzero multiple of ω^2 is primitive and we conclude $\alpha=\overline{\alpha}$ and $\lambda\in\mathbb{R}$. Thus, by the Hodge–Riemann bilinear relations, we have

$$\int_X \alpha\wedge\overline{\alpha}=\int_X \alpha^2\geq 0.$$

This implies, since $\alpha\wedge\omega=0$ and $\lambda\in\mathbb{R}$, that

$$\int_X c_2(X)^2=\int_X(\lambda^2\omega^4+2\lambda\cdot\omega^2\wedge\alpha+\alpha^2)\geq 0. \tag{12-1}$$

Let us now use the formula, due to Libgober and Wood [16],

$$c_1c_3=12\chi^2-36\chi^3+72\chi^4-14c_4, \tag{12-2}$$

where $\chi^p=\sum_q(-1)^q h^{p,q}$. By the Riemann–Roch theorem and (12-1), we also have

$$\chi^4=\tfrac{1}{720}(-c_4+c_1c_3+3c_2^2+4c_1^2c_2-c_1^4)\geq\tfrac{1}{720}(-c_4+c_1c_3+4c_1^2c_2-c_1^4).$$

Using Libgober and Wood's expression for c_1c_3, this reads

$$\chi^4\geq\tfrac{1}{720}(-15c_4+12\chi^2-36\chi^3+72\chi^4+4c_1^2c_2-c_1^4).$$

Finally, expressing the topological Euler characteristic c_4 as well as all the χ^p in terms of Hodge numbers, one obtains the inequality, claimed in the lemma. □

Since $h^{1,1}=1$ and $h^{1,0}=0$, we see that X has Picard number one. Thus the canonical class K_X is either antiample, numerically trivial or ample.

In fixed dimension, there are only finitely many deformation types of smooth complex projective varieties with antiample canonical class; see Kollár, Miyaoka and Mori [12]. Since deformation equivalent varieties have the same Hodge numbers [21, page 235], the proposition is true in this case.

If K_X is numerically trivial, then Lemma 34 implies that $h^{3,0}$ and $h^{3,1}$ are bounded.

Moreover, Libgober and Wood's formula (12-2) shows that

$$52 + 40h^{2,0} - 4h^{2,1} - 2h^{2,2} - 52h^{3,0} + 8h^{3,1} + 44h^{4,0} = 0.$$

Since we already know that apart from $h^{2,1}$ all Hodge numbers in the above identity are bounded, it follows that $h^{2,1}$ is bounded as well.

It remains to deal with the case where K_X is ample. Here, Yau's inequality [22] holds:

$$c_1^2 c_2 \geq \tfrac{2}{5} c_1^4.$$

Using this, we obtain from Lemma 34 that

$$224 + 228h^{2,0} - 224h^{3,0} + h^{2,2} - 2h^{1,3} + 226h^{4,0} \geq \tfrac{1}{5} c_1^4.$$

Since $h^{2,0}$, $h^{4,0}$ and $h^{2,2}$ are bounded, we deduce that c_1^4 is bounded from above. Thus Kollár and Matsusaka's theorem [14, page 239] implies only finitely many deformation types of such fourfolds exist. As in the case of antiample canonical class, $h^{3,0}$, $h^{2,1}$ and $h^{3,1}$ are bounded. This concludes the proof of Proposition 32. □

Acknowledgments Thanks to Burt Totaro for drawing my attention to the construction problem and for informing me about the results in [4; 6]. I am grateful to my advisor Daniel Huybrechts for stimulating discussions and to Ciaran Meachan, Dieter Kotschick and Burt Totaro for useful comments. I would also like to warmly thank two anonymous referees; this paper benefited significantly from their careful reading and suggestions. The author is supported by an IMPRS Scholarship of the Max Planck Society.

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Proposed: Lothar Göttsche — Received: 7 August 2013
Seconded: Richard Thomas, Ciprian Manolescu — Revised: 22 February 2014