Algebraic Geometry 1

Prof. Stefan Schreieder

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Introduction

This course is an introduction to algebraic geometry. That is, we study the geometry of zero sets of polynomials in several variables over some field k. For $k = \mathbb{R}$, the following examples are familiar from high school:

$$\{x^2 + y^2 - 1 = 0\} \quad \text{or} \quad \{y - x^2 = 0\}.$$

Generalizing the first example, we get

$$\{x^n + y^n - 1 = 0\}.$$

The question whether for $k = \mathbb{Q}$ and $n \geq 3$, this last example has any solution besides the trivial ones (i.e. $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$ if n is odd and $(x, y) = (\pm 1, 0)$ and $(x, y) = (0, \pm 1)$ if n is even) is known as Fermat's Last Theorem. For a very long time, this was one of the most famous open problems in number theory, which has only been solved in the 1990s by Andrew Wiles.

The above simple example shows that for arbitrary fields k, understanding the zero sets of polynomial equations has immediately an arithmetic flavour. To turn the problem into a geometric one, we keep our favourite polynomial equations (e.g. those chosen above), but we replace the arbitrary field k by its algebraic closure $\overline{k}$. Once we understand the geometry and shape of the solutions over $\overline{k}$, one can recover the solutions over k as the $Gal(\overline{k}/k)$-invariants.

So for the first part of the course, which will be devoted to varieties over algebraically closed fields, we will always assume that k is algebraically closed. Our aim is then to study the zero set of polynomial equations over k from a geometric point of view. The most important technical tool will be commutative algebra. The reader is assumed to have basic knowledge of commutative algebra (e.g. to the extent of [1]); we will however try to clearly state without proofs the facts from commutative algebra that we use.

Disclaimer. These are rough lecture notes that I have written for the preparation of the Algebraic Geometry I class that I have taught at LMU München during the WS 2018/19. The notes may contain many typos and actual mistakes. If you find any, please send me an email: schreieder@math.lmu.de

1 Algebraic sets and affine varieties

Let k be an algebraically closed field.

Definition 1.1. *Let $n \in \mathbb{N}$. The affine n-space over k is the set*

$$\mathbb{A}^n := \mathbb{A}^n_k := \{(a_1, \dots, a_n) \in k^n \mid a_i \in k\}.$$

Polynomials $f \in k[x_1, \dots, x_n]$ can be regarded as functions $f : \mathbb{A}^n \to k$.

Remark 1.2. *If $f, g \in k[x_1, \dots, x_n]$ induce the same function on $\mathbb{A}^n$, then $f - g$ is a polynomial that vanishes identically on $\mathbb{A}^n$. Since k is algebraically closed, one can easily show that this implies $f = g$ as polynomials.*

Definition 1.3. *Let $I \subset k[x_1, \dots, x_n]$ be any subset. The associated (affine) algebraic set is*

$$V(I) := \{a \in \mathbb{A}^n \mid f(a) = 0 \ \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n$ is called algebraic if $X = V(I)$ for some subset $I \subset k[x_1, \dots, x_n]$.

Remark 1.4. *It follows immediately that $V(I)$ coincides with the affine algebraic set of the ideal that is generated by I. In particular, we do not loose anything if we consider only affine algebraic sets $V(I)$ associated to ideals $I \subset k[x_1, \ldots, x_n]$. Moreover, since $k[x_1, \ldots, x_n]$ is noetherian by Hilbert's basis theorem, $I = (f_1, \ldots, f_r)$ is generated by finitely many polynomials and so*

$$V(I) = \{f_1 = \cdots = f_r = 0\}$$

is the zero set of finitely many polynomials.

Example 1.5. *The following sets $X \subset \mathbb{A}^n$ are algebraic:*

$$X = \emptyset, \quad X = \mathbb{A}^n \quad \textit{and} \quad X = \{a\},$$

where $a \in \mathbb{A}^n$ is a point.

The next lemma shows that intersections and unions of two algebraic sets are again algebraic.

Lemma 1.6. *Let $I, J \subset k[x_1, \ldots, x_n]$ be ideals. Then,*

(a) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$;

(b) $V(I \cup J) = V(I + J) = V(I) \cap V(J)$.

Proof. This follows easily from the definitions. To prove (a), let $x \in V(IJ)$. Then for any $f \in I \cap J$ we have $f^2 \in IJ$ and so $f^2(x) = 0$. Hence, $f(x) = 0$ and so $x \in V(I \cap J)$. This shows $V(IJ) \subset V(I \cap J)$.

Next, let $x \notin V(I) \cup V(J)$. Then there are polynomials $f \in I$ and $g \in J$ with $f(x) \neq 0$ and $g(x) \neq 0$. Hence, $fg(x) \neq 0$ and so $x \notin V(I \cap J)$, because $fg \in I \cap J$. This proves $V(I \cap J) \subset V(I) \cup V(J)$.

Finally, $V(I) \cup V(J) \subset V(IJ)$ is clear, which finishes the proof of (a).

To prove (b), note first that $V(I \cup J) = V(I + J)$ because $I + J$ is the ideal generated by I and J. Next, $V(I + J) \subset V(I) \cap V(J)$ is clear. For the converse inclusion, let $x \notin V(I + J)$. Then there are some elements $f \in I$ and $g \in J$ with $(f + g)(x) \neq 0$. Hence, $f(x) \neq 0$ or $g(x) \neq 0$. This shows $V(I) \cap V(J) \subset V(I + J)$, as we want. □

Corollary 1.7. *Finite unions and arbitrary intersections of algebraic subsets of $\mathbb{A}^n$ are algebraic.*

Proof. For finite intersections and finite unions, this follows from the previous lemma. Similarly, part (b) follows easily from the Hilbert basis theorem (i.e. the fact that any ideal in $k[x_1, \ldots, x_n]$ is finitely generated) and part (b) of the previous lemma. □

By the corollary, the algebraic subsets of $\mathbb{A}^n$ are the closed sets of some topology on $\mathbb{A}^n$, which we call Zariski topology.

Definition 1.8. *The Zariski topology on $\mathbb{A}^n$ is the topology whose closed subsets are exactly the algebraic subsets. That is, $U \subset \mathbb{A}^n$ is open if its complement is algebraic.*

Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set $X \subset \mathbb{A}^n$; this topology is also called Zariski topology.

Recall from general topology that a topological space X is irreducible if it cannot be written as union of two proper closed subsets: $X \neq X_1 \cup X_2$ with $X_i \subsetneq X$ closed.

Definition 1.9. *An affine algebraic variety is an irreducible closed subset $X \subset \mathbb{A}^n$. That is, X is algebraic and cannot be written as the union of two proper non-empty algebraic subsets.*

Definition 1.10. *Let $X \subset \mathbb{A}^n$ be an arbitrary subset. We define the ideal*

$$I(X) := \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \quad \text{for all } x \in X\}.$$

Lemma 1.11. *Let $X \subset \mathbb{A}^n$ and $S \subset k[x_1, \ldots, x_n]$ be arbitrary subsets. Then,*

(a) $X \subset V(I(X))$ and $S \subset I(V(S))$;

(b) $V(I(X)) = \overline{X}$ is the closure of X, i.e. the smallest closed subset containing X.

Proof. To prove (a), note that for each $f \in I(X)$, $f(x) = 0$ for all $x \in X$ and so $X \subset V(I(X))$. Moreover, for any $f \in S$ we have $f(x) = 0$ for all $x \in V(S)$ and so $S \subset I(V(S))$. This proves (a).

To prove (b), note that $V(I(X))$ is closed and contains X by part (a). Hence, $\overline{X} \subset V(I(X))$. Conversely, let $\overline{X} = V(S)$ for some subset $S \subset k[x_1, \ldots, x_n]$. Then

$$V(I(X)) \subset V(I(V(S))) \subset V(S) = \overline{X},$$

by part (b). Altogether, $V(I(X)) = \overline{X}$, as we want. □

Proposition 1.12. *An affine algebraic set X is a variety if and only if $I(X)$ is a prime ideal in $k[x_1, \ldots, x_n]$.*

Proof. "$\Rightarrow$": Let X be a variety and let $f \cdot g \in I(X)$. Then $X \subset V(fg) = V(f) \cup V(g)$ by the lemma. Hence,

$$X = (X \cap V(f)) \cup (X \cap V(g))$$

is the union of two closed subsets. Since X is irreducible, we may wlog assume that $X \cap V(f) = X$ and so f vanishes on X. That is, $f \in I(X)$ and so $I(X)$ is prime.

"$\Leftarrow$": Suppose $X = A \cup B$ with non-empty proper closed subsets $A, B \subsetneq X$. We can pick points $a \in A$, $b \in B$ with $a \notin B$, $b \notin A$.

For any subset $S \subset \mathbb{A}^n$, $V(I(S)) = \overline{S}$ by Lemma 1.11. Since A and B are closed by assumptions, $V(I(A)) = A$ and $V(I(B)) = B$. Since $a \notin B$ and $b \notin A$, we find an element $f \in I(B)$ with $f(a) \neq 0$ and $g \in I(A)$ with $g(b) \neq 0$. But then $fg \in I(X)$ with $f \notin I(X)$ and $g \notin I(X)$, because fg vanishes on X, while neither f nor g vanishes separately on X. Hence, $I(X)$ is not a prime ideal, as we want. □

Recall the following result from commutative algebra (see e.g. [1]), where we emphasize that k is assumed to be an algebraically closed field.

Theorem 1.13 (Hilbert Nullstellensatz)**.** *Let $J \subset k[x_1, \ldots, x_n]$ be an ideal. Then,*

(a) $V(J) = \emptyset \Leftrightarrow J = k[x_1, \ldots, x_n]$;

(b) $I(V(J)) = \sqrt{J}$;

(c) if $J \subset k[x_1, \ldots, x_n]$ is a maximal ideal, then $J = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$, i.e. J is the ideal of functions that vanish at a single point of $\mathbb{A}^n$.

Corollary 1.14. *There are bijections*

$$\{\text{affine algebraic subsets } X \subset \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals } J \subset k[x_1, \ldots, x_n]\},$$

$$\{\text{affine algebraic varieties } X \subset \mathbb{A}^n\} \longleftrightarrow \{\text{prime ideals } J \subset k[x_1, \ldots, x_n]\},$$

$$\{\text{points } p \in \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals } J \subset k[x_1, \ldots, x_n]\}$$

induced by $X \mapsto I(X)$ with inverse $J \mapsto V(J)$.

Proof. The first bijection follows from Lemma 1.6 and the Hilbert Nullstellensatz. The second bijection follows from this and Proposition 1.12. For the last bijection, note that by the Nullstellensatz, $V(J)$ is a point if J is a maximal ideal. Conversely, let $p = (p_1, \ldots, p_n)$ be a point, then

$$k[x_1, \ldots, x_n] \to k, \quad f \mapsto f(p)$$

is a surjective homomorphism of rings with kernel $I(p)$. Hence, $k[x_1, \ldots, x_n]/I(p) \cong k$ is a field and so $I(p)$ is maximal. This concludes the proof. □

We aim to prove the following, which justifies to restrict our attention to varieties.

Theorem 1.15. *Any affine algebraic set is a union of finitely many algebraic varieties.*

The above theorem can be seen as consequence of the primary decomposition theorem for ideals in noetherian rings and Proposition 1.12 above. A more elementary proof uses the following definition.

Definition 1.16. *A topological space X is noetherian if any chain of closed subsets*

$$\cdots \subset X_n \subset X_{n-1} \subset \cdots \subset X_2 \subset X_1 \subset X$$

becomes stationary, i.e. $X_n = X_{n+1}$ for all n sufficiently large.

Lemma 1.17. *Affine space $\mathbb{A}^n$ is noetherian.*

Proof. Let

$$\cdots \subset X_n \subset X_{n-1} \subset \cdots \subset X_2 \subset X_1 \subset \mathbb{A}^n$$

be a chain of closed subsets. Then

$$(0) \subset I(X_1) \subset I(X_2) \subset \cdots \subset I(X_{n-1}) \subset I(X_n) \subset \cdots \subset k[x_1, \ldots, x_n]$$

is a chain of ideals. This chain becomes stationary, because $k[x_1, \ldots, x_n]$ is noetherian by Hilbert's basis theorem. Hence, $I(X_n) = I(X_{n+1})$ for sufficiently large n and so $X_n = X_{n+1}$ for sufficiently large n, because $V(I(X)) = \overline{X}$ by Lemma 1.11. □

Since subspaces of noetherian spaces are obviously noetherian, we get the following.

Corollary 1.18. *Any affine algebraic set $X \subset \mathbb{A}^n$ together with the Zariski topology is a noetherian topological space.*

The key property of noetherian topological spaces that we will use is the following simple lemma.

Lemma 1.19. *Let X be a noetherian topological space. Then X can be written as the finite union of irreducible topological spaces.*

Proof. For a contradiction, we assume that the statement fails for X. Then X is not irreducible and so $X = X_1 \cup X_2$ for non-empty closed subsets $X_i \subsetneq X$. Moreover, either X_1 or X_2 is not irreducible and so it admits a similar decomposition. Repeating this argument, we find an infinite chain of closed subsets of X that does not become stationary, which contradicts the fact that X is noetherian. □

Proof of Theorem 1.15. By Corollary 1.18, any algebraic subset $X \subset \mathbb{A}^n$ is noetherian. The result follows therefore from the lemma above. □

2 Regular maps of affine varieties

Recall that the natural functions on $\mathbb{A}^n$ are those that are given by polynomials. Similarly, if $X \subset \mathbb{A}^n$ is algebraic, the natural functions on X are given by restrictions of polynomials.

Definition 2.1. *Let $X \subset \mathbb{A}^n$ be an algebraic set. A function $f : X \to k$ is regular if there is a polynomial $F \in k[x_1, \dots, x_n]$ with $f(x) = F(x)$ for all $x \in X$. The set of all regular functions on X is denoted by $k[X]$.*

Note that $k[X]$ is in a natural way a ring, where addition and multiplication are defined pointwise. Moreover, there is a natural surjective ring homomorphism

$$k[x_1, \dots, x_n] \to k[X], \quad F \mapsto F|_X$$

The krnel of the above surjection is given by all polynomials F with $F|_X = 0$. By definition, this is exactly $I(X)$ and so we get an isomorphism

$$k[x_1, \dots, x_n]/I(X) \cong k[X].$$

Remark 2.2. *By Corollary 1.14, we see that $k[X]$ is always reduced. Moreover, $k[X]$ is an integral domain (resp. a field) if and only if X is a variety (resp. a point).*

Definition 2.3. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine algebraic sets. A map $\phi : X \to Y$ is called regular map if $\phi = (f_1, \dots, f_m)$ for regular functions $f_i \in k[X]$. A regular map ϕ is an isomorphism if it has an inverse which is also regular.*

We note that a regular map $\phi : X \to \mathbb{A}^1$ is nothing but a regular function.

Lemma 2.4. *Let $\phi : X \to Y$ be a regular map. Then*

$$\phi^* : k[Y] \to k[X], \quad f \mapsto f \circ \phi$$

is a ring homomorphism.

Proof. The main point to prove here is that ϕ^* is well-defined. For this we need to show that $f \circ \phi$ is regular for all $f \in k[Y]$. Let $\phi = (f_1, \dots, f_n)$, then $f \circ \phi = f(f_1, \dots, f_n)$ is regular, because f and f_i are regular and so they are given by restriction of polynomial functions. This shows that ϕ^* is well-defined and one easily checks that it is a ring homomorphism. □

The following result describes some of the most basic properties of regular maps.

Theorem 2.5. *Let $\phi : X \to Y$ be a regular map between affine algebraic sets. Then*

(i) ϕ is a continuous map;

(ii) ϕ^* is injective if and only if $\overline{\phi(X)} = Y$;

(iii) ϕ^* is surjective if and only if $\phi : X \xrightarrow{\sim} \phi(X)$ and $\phi(X) \subset Y$ is closed;

(iv) ϕ is an isomorphism if and only if $\phi^* : k[Y] \to k[X]$ is an isomorphism.

In order to prove this theorem, we will use the following notation, which generalizes $V(I)$ from above.

Definition 2.6. *Let X be an affine algebraic set and let $I \subset k[X]$ be a subset. Then we put*

$$V_X(I) = \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}.$$

As before, it suffices to consider the above definition in the case where I is an ideal. If J is the preimage of I via the surjection $k[x_1, \ldots, x_n] \to k[X]$, then we have $V_X(I) = V(J)$. In particular, the closed subsets of X are precisely those that are of the form $V_X(I)$ for some (reduced) ideal $I \subset k[x_1, \ldots, x_n]$ with $I(X) \subset I$.

Proof of Theorem 2.5. Item (i) follows from the equality

$$\phi^{-1}(V_Y(I)) = V_X(\phi^*(I) \cdot k[X])$$

for $I \subset k[Y]$. (For this note that if $I = (f_1, \ldots, f_r)$, then the ideal $\phi^*(I) \cdot k[X]$ is generated by $(\phi^* f_1, \ldots, \phi^* f_r)$.

Item (ii):

"$\Rightarrow$:" Suppose that $\phi(X) \subset V_Y(f)$ for some $f \in k[Y]$. Then $\phi^* f = 0$ and so $f = 0$ because ϕ^* is injective by assumptions.

"$\Leftarrow$:" Suppose $\phi^* f = 0$ for some $f \in k[Y]$. Then $\phi(X) \subset V_Y(f) \subset Y$. Since $\overline{\phi(X)} = Y$, we get $V_Y(f) = Y$ and so $f = 0$, as we want.

Item (iv):

"$\Rightarrow$": If $\phi : X \to Y$ is an isomorphism, then ϕ^{-1} is regular and so ϕ^* has as inverse $(\phi^{-1})^*$, which shows that ϕ^* is an isomorphism.

"$\Leftarrow$": Suppose that $\phi^* : k[Y] \to k[X]$ is an isomorphism with inverse $(\phi^*)^{-1} : k[X] \to k[Y]$. Since ϕ^* is a k-algebra homomorphism, so is $(\phi^*)-1$; that is, $(\phi^*)^{-1}(\lambda \cdot f) = \lambda(\phi^*)^{-1}(f)$ for all $\lambda \in k$ and $f \in k[X]$. Suppose that $X \subset \mathbb{A}^n$ with affine coordinates $t_1, \ldots, t_n$ on $\mathbb{A}^n$. We may then consider the map

$$\psi : Y \to \mathbb{A}^n, \quad y \mapsto (f_1(y), \ldots, f_n(y))$$

where $f_i := (\phi^*)^{-1}(t_i)$. One easily checks that $\psi(Y) \subset X$ and that ψ is in fact an inverse of ϕ. This proves (iv).

Item (iii):

"$\Rightarrow$": Let $Z := V_Y(\ker(\phi^* : k[Y] \to k[X]))$. Then, $\phi(X) \subset Z$ by definition and so $\phi : X \to Z$ is a morphism. Moreover, $k[Z] = k[Y]/\ker(\phi^*)$ and so $\phi^* : k[Z] \xrightarrow{\sim} k[X]$ since ϕ^* is surjective. This shows $\phi : X \xrightarrow{\sim} Z$ by (iv), as we want.

"$\Leftarrow$": Suppose that $Z \subset Y$ is a closed subset such that ϕ factors as $X \xrightarrow{\sim} Z \subset Y$. Then $\phi^* : k[Y] \to k[X] \cong k[Z] \cong k[Y]/I(Z)$, and so ϕ^* is onto. This concludes the proof of the theorem. □

3 Rational maps of affine varieties

Let $X \subset \mathbb{A}^n$ be an affine algebraic variety. Then the ideal $I(X)$ is prime and so $k[X] \cong k[x_1, \ldots, x_n]/I(X)$ is an integral domain. In particular, we may form the fraction field $\mathrm{Frac}(k[X])$, which is a field that contains $k[X]$.

Definition 3.1. *Let X be an affine algebraic variety. Then $k(X) := \mathrm{Frac}(k[X])$ is the field of rational functions on X.*

We say that an element $\varphi \in k(X)$ is regular at $x \in X$ if there are regular functions $f, g \in k[X]$ with $\varphi = \frac{f}{g}$ such that $g(x) \neq 0$.

Lemma 3.2. *Let X be an affine variety. A rational function $\varphi \in k(X)$ is regular (i.e. $\varphi \in k[X]$) if and only if it is regular at any point of X.*

Proof. If φ is regular, then it is clearly regular at any point. Conversely, assume that $\varphi \in k(X)$ is regular at any point. Consider $I := \{f \in k[X] \mid f \cdot \varphi \in k[X]\}$. This is clearly an ideal and we need to show that $1 \in I$, i.e. $I = k[X]$. By the Nullstellensatz, this is equivalent to showing that $V_X(I) = \emptyset$. So for a contradiction, assume that there is some $x \in X$ with $f(x) = 0$ for all $f \in I$. Since φ is regular at x, we have $\varphi = \frac{f}{g}$ for some regular functions f and g with $g(x) \neq 0$. But then $g \in I$ and so $x \notin V_X(I)$, a contradiction. □

Definition 3.3. *Let X be an affine algebraic variety, $\varphi \in k(X)$ a rational function. Then the domain* $\mathrm{dom}(\varphi)$ *of φ is the set of points where φ is regular.*

Lemma 3.4. *Let X be an affine algebraic variety. Then* $\mathrm{dom}(\varphi) \subset X$ *is open and non-empty (hence dense) for all $\varphi \in k(X)$.*

Proof. To see that $\mathrm{dom}(\varphi)$ is open, we simply note that

$$\mathrm{dom}(\varphi) = X \setminus V_X(I),$$

where $I := \{f \in k[x_1, \dots, x_n] \mid f\varphi \in k[X]\}$.

To see that it is non-empty, note that $\varphi = \frac{f}{g}$ for some $f, g \in k[X]$ with $g \neq 0$. Since g is not identically zero, there is a point $x \in X$ where g does not vanish and so $x \in \mathrm{dom}(\varphi)$. This concludes the proof of the lemma. □

Definition 3.5. *Let $X \subset \mathbb{A}^n$ be an affine algebraic variety and let $Y \subset \mathbb{A}^m$ be an affine algebraic set. A rational map*

$$\varphi : X \dashrightarrow Y$$

is given by $\varphi = (\varphi_1, \dots, \varphi_m)$ for some rational functions $\varphi_i \in k(X)$, such that $\varphi(x) \in Y$ for all

$$x \in \mathrm{dom}(\varphi) := \bigcap_{i=1}^{m} \mathrm{dom}(\varphi_i).$$

A rational map φ is called dominant if $\varphi(\mathrm{dom}(\varphi))$ *is dense in Y.*

Remark 3.6. *By the above lemma,* $\mathrm{dom}(\varphi_i)$ *is nonempty open for all i. It follows that* $\mathrm{dom}(\varphi)$ *is non-empty open as well (because any two non-empty open subsets of an irreducible topological space have nonzero intersection, see Exercise 3 on sheet 1).*

Definition 3.7. *Let X and Y be affine varieties, $\varphi : X \dashrightarrow Y$ a rational map. Then φ is birational (or a birational isomorphism) if there is a rational map $\psi : Y \dashrightarrow X$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are the identity maps wherever they are defined.*

Let $\varphi : X \to Y$ be a dominant rational map. Then we can define a pullback map on rational functions:

$$\varphi^* : k(Y) \to k(X), \quad \pi \mapsto \pi \circ \varphi.$$

Note that this is a homomorphism of fields and so it is always injective (because fields have non nontrivial ideals). Note also that it is essential that φ is dominant for this definition, as otherwise, there is a regular function $g \in k[Y]$ with $g \circ \varphi = 0$ and so the pullback of $\frac{1}{g}$ cannot be defined.

Lemma 3.8. *Let $\varphi : X \dashrightarrow Y$ be a dominant rational map between affine algebraic varieties. Then the following are equivalent:*

(i) φ is birational;

(ii) $\varphi^* : k(Y) \to k(X)$ *is an isomorphism of fields.*

Proof. This is similar to Theorem 2.5, see Exercise sheet 3. □

While rational functions and rational maps among affine varieties are very natural, their definition takes us automatically outside of the world of affine varieties. Indeed, rational functions and rational maps are only defined on non-empty open subsets of affine varieties and such sets might in general not be affine varieties themselves. This leads to the following definition.

Definition 3.9. *(1) A quasi-affine algebraic set* $X \subset \mathbb{A}^n$ *is an open subset of an affine algebraic set. A quasi-affine variety is a quasi-affine algebraic set that is irreducible.*

(2) A regular function $\phi : X \to k$ *is a function which locally around each point* x *can be written as* $\varphi = \frac{F}{G}$ *for some polynomials* $F, G \in k[t_1, \dots, t_n]$ *with* $G(x) \neq 0$. *The set of regular functions is denoted by* $k[X]$

(3) A regular map between quasi-affine algebraic sets $X \subset \mathbb{A}^n$ *and* $Y \subset \mathbb{A}^m$ *is a map* $\phi : X \to Y$ *given by* $\phi = (\phi_1, \dots, \phi_m)$ *for some regular functions* $\phi_i \in k[X]$. *The map* ϕ *is an isomorphism if it has an inverse that is also regular.*

Example 3.10. *Let* $X := \mathbb{A}^2 \setminus \{0\}$. *Then the inclusion* $i : X \hookrightarrow \mathbb{A}^2$ *induces an isomorphism* $k[\mathbb{A}^2] \cong k[X]$, *see Exercise sheet 3. Since* i *is not an isomorphism, it follows from Theorem 2.5 that* X *is not isomorphic to an affine variety.*

The above example shows that not every quasi-affine set is affine. A key property of quasi-affine varieties is the fact that they look at least locally like affine varieties.

Lemma 3.11. *Let* X *be a quasi-affine algebraic set and let* $x \in X$ *be a point. Then there is an open subset* $U \subset X$ *with* $x \in U$ *which is isomorphic to an affine algebraic set.*

Proof. Note that $X \subset \overline{X}$ is an open subset of some affine algebraic set $\overline{X} \subset \mathbb{A}^n$. Let $Z := \overline{X} \setminus X$. Since $x \notin Z$, we may find a function $f \in I(Z) \subset k[\overline{X}]$ with $f(x) \neq 0$. Hence, $U := \overline{X} \setminus V(f)$ is an open neighbourhood of x, which is isomorphic to an affine variety by Exercise sheet 3. □

Proposition 3.12. *Let* X *and* Y *be affine algebraic varieties. Then the following are equivalent:*

(i) X *and* Y *are birational;*

(ii) there are non-empty open subsets $U \subset X$ *and* $V \subset Y$ *that are isomorphic.*

Proof. (i) ⇒ (ii): Let $\varphi : X \dashrightarrow Y$ be a rational map with inverse $\psi : Y \dashrightarrow X$. Then, $U = \mathrm{dom}(\varphi) \subset X$ and $V := \mathrm{dom}(\psi) \subset Y$ are open and we have morphisms $\varphi|_U : U \to Y$ and $\psi|_V : V \to X$. Let $U' := \varphi|_U^{-1}(V)$, then the composition of $\varphi|_{U'} : U' \to V \subset Y$ with $\psi|_V$ is well-defined and so it must be the identity, because ψ is an inverse of φ. Let $V' = \psi|_V^{-1}(U')$, then the above observation shows that $\varphi(U') \subset V'$ and so we get morphisms

$$\varphi|_{U'} : U' \to V' \quad \text{and} \quad \psi|_{V'} : V' \to U'.$$

These morphisms must be inverses of each other, because $\psi = \varphi^{-1}$ as rational maps (and so $\psi \circ \varphi$ and $\varphi \circ \psi$ are the identity wherever they are defined). This proves (ii).

(ii) ⇒ (i): Let $U \subset X$ and $V \subset Y$ be open subsets with an isomorphism

$$\phi : U \xrightarrow{\sim} V.$$

Let $Y \subset \mathbb{A}^m$ and let $\phi = (\phi_1, \ldots, \phi_m)$ for some $\phi_i \in k[U]$. By definition, locally around a given point of U we have $\phi_i = \frac{f_i}{g_i}$ for some polynomials f_i, g_i. We thus get a rational map

$$\left(\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n}\right) : U \dashrightarrow V$$

which coincides with ϕ on a non-empty open subset (which is automatically dense). The same is true for the inverse of ϕ and so we find that X and Y are birational, as we want. □

Example 3.13. *Let $f \in k[x_1, x_2]$ be an irreducible quadratic polynomial. Then $X := V(f) \subset \mathbb{A}^2$ is birational to $\mathbb{A}^1$.*

Proof. Wlog, we may assume that $0 \in X$. For $s \in k$, let $L_s := V(sx_1 - x_2) \subset \mathbb{A}^2$ be the line through the origin of slope $\frac{1}{s}$. Let $(x_1, x_2) \in X \cap L_s$, then $x_2 = sx_1$ and $f(x_1, sx_1) = 0$. For a Zariski open subset of $s \in \mathbb{A}^1$, we have that $f(x_1, sx_1)$ is a quadratic polynomial in x_1. Since $(0,0) \in X$, we find

$$f(x_1, sx_1) = x_1 \cdot (a(s)x_1 - b(s))$$

for some polynomials $a(s)$ and $b(s)$. But then we get rational maps

$$\varphi : X \dashrightarrow \mathbb{A}^1, \quad (x_1, x_2) \mapsto \frac{x_2}{x_1}$$

and

$$\psi : \mathbb{A}^1 \dashrightarrow X, \quad s \mapsto \left(\frac{b(s)}{a(s)}, s\frac{b(s)}{a(s)}\right)$$

that are inverses of each other. □

4 Quasi-projective varieties

One essential draw back of affine or quasi-affine varieties is that they are not proper; a term that we will define only next semester, but which you should think of as an algebraic analogue of compactness. To see why affine varieties should not be thought of as compact objects, note that they have a lot of non-constant global regular functions, while compact objects should have only constant global functions (compare this with the fact that a connected compact complex manifold has only constant global holomorphic functions). More concretely, if we have $k = \mathbb{C}$ and $X \subset \mathbb{A}^n$ is an affine algebraic variety over $\mathbb{C}$ which is not a point, then in the usual Euclidean topology, X is not compact, because it is unbounded.

The lack of compactness is reflected in the fact that intersection theory does not work well on affine or quasi-affine varieties. For instance, two distinct lines in $\mathbb{A}^2$ meet in a single point, unless they are parallel and so do not meet at all. This implies that by moving two lines in the plane in a 'continuous' way, their intersection may jump from one point to the emtpy set, which is a very unpleasant behaviour. Intuitively the point is that two parallel lines still meet, but they meet at some point at infinity and one should add these points to obtain a compact space.

The above discussion motivates us to introduce projective n-space $\mathbb{P}^n$, which you should think of as a natural "compactification" of $\mathbb{A}^n$ in which intersection products work much better. For instance, we will see that any two distinct lines in $\mathbb{P}^2$ meet in exactly one point.

Definition 4.1. *Let k be a field. Projective n-space over k is given by*

$$\mathbb{P}^n_k := \left(\mathbb{A}^{n+1} \setminus \{0\}\right) / \sim$$

where $x \sim y$ if there is some $\lambda \in k^*$ with $x = \lambda y$. We denote the equivalence class of $x = (x_0, \dots, x_n)$ by $[x_0 : \dots : x_n]$.

If no confusion is likely, we usually suppress the ground field in our notation and write $\mathbb{P}^n$ instead of $\mathbb{P}^n_k$.
Note that a point $[x_0 : \dots : x_n] \in \mathbb{P}^n$ has at least one non-zero entry, i.e. $X_i \neq 0$ for at least one i. Moreover, for any $\lambda \in k^*$,

$$[x_0 : x_1 : \dots : x_n] = [\lambda \cdot x_0 : \lambda \cdot x_1 : \dots : \lambda \cdot x_n].$$

Let $x_0, \dots, x_n$ be coordinates on $\mathbb{A}^{n+1}$. Then any point on $\mathbb{P}^n$ is of the form $[x_0 : \dots : x_n]$ with $x_i \neq 0$ for some i and we call the x_i homogeneous coordinates on $\mathbb{P}^n$.
Consider the inclusion $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ given by

$$(x_1, \dots, x_n) \mapsto [1 : x_1, \dots : x_n].$$

The complement is naturally isomorphic to $\mathbb{P}^{n-1}$ and so we find inductively a decomposition

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

If $F \in k[x_0, \dots, x_n]$ is a non-zero polynomial, then $F(x)$ and $F(\lambda x)$ are in general different and so F does not yield a function on $\mathbb{P}^n$. However, if F is homogeneous of degree d, then $F(\lambda x) = \lambda^d F(x)$ and so at least the condition $F(x) = 0$ is well-defined for a point $x \in \mathbb{P}^n$. This gives rise to the following.

Definition 4.2. *Let $I \subset k[x_0, x_1, \dots, x_n]$ be a homogeneous ideal (i.e. an ideal generated by homogeneous elements). Then*

$$V(I) := \{x \in \mathbb{P}^n \mid F(x) = 0 \text{ for all } F \in I \subset k[x_0, \dots, x_n] \text{ homogeneous}\}$$

is a projective algebraic set.

Definition 4.3. *Conversely, if $X \subset \mathbb{P}^n$ is a projective algebraic set, then*

$$I(X) := \{F \in k[x_0, \dots, x_n] \mid F \text{ homogeneous with } F(x) = 0 \text{ for all } x \in X\}$$

is the (homogeneous) ideal of X. The homogeneous coordinate ring of X is given by

$$S[X] := k[x_0, \dots, x_n]/I(X).$$

Note that the homogeneous coordinate ring $S[X]$ of a projective algebraic set behaves quite different from the ring of regular functions $k[X]$ of an affine algebraic set X, because elements of $S[X]$ cannot be regarded as functions on X. We have nonetheless the following.

Proposition 4.4. *Consider the graded ring $S := k[x_0, \dots, x_n]$, where $|x_i| = 1$ for all i. Let $S_+ := \bigoplus_{d \geq 1} S_d$ and consider a homogeneous ideal $I \subset S$. Then,*

(i) $V(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$;

(ii) if $X := V(I)$ is non-empty, then $I(X) = \sqrt{I}$.

Proof. The ideal I defines an affine algebraic subset $Y := V_{\mathbb{A}^{n+1}}(I) \subset \mathbb{A}^{n+1}$. Considering the projection $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, we see that $V(I) \subset \mathbb{P}^n$ is empty if and only if $Y \subset \{0\}$. By the Nullstellensatz, the latter is equivalent to $S_+ \subset \sqrt{I}$, which proves (i).

To prove (ii), note that $\pi : Y \setminus \{0\} \to X$ is surjective (with fibre $\mathbb{A}^1$ above each point of X) and that $I(Y) = \sqrt{I}$ by the Nullstellensatz. Since any homogeneous polynomial that vanishes on X also vanishes on Y, we find $I(X) \subset I(Y)$. Conversely, any homogeneous $f \in I(Y)$ vanishes on X. Finally, $I(Y)$ is homogeneous (e.g. because I is homogeneous and $I(Y) = \sqrt{I}$, or because Y is homogeneous in the sense that if $y \in Y$ then $\lambda y \in Y$ for all $\lambda \in k$) and so $I(X) = I(Y) = \sqrt{I}$. This concludes the proposition. □

Definition 4.5. *(1) A quasi-projective algebraic set is an open subset of a projective algebraic set.*

(2) A projective variety is an irreducible projective algebraic set.

(3) A quasi-projective variety is an open subset of a projective variety.

Definition 4.6. *Let X be a quasi-projective algebraic set.*

(1) A function $\phi : X \to k$ is regular if locally around every point $x \in X$ it is given by $\frac{F}{G}$ for homogeneous polynomials F and G of the same degree. The set of regular functions, denoted by $k[X]$, is a ring.

(2) Suppose that X is irreducible, i.e. a quasi-projective variety. A rational function $\phi : X \dashrightarrow k$ is the equivalence class of a pair (U, ϕ), where $U \subset X$ is non-empty and open, and $\phi \in k[U]$ is a regular function. Two pairs (U, ϕ) and (V, ψ) are equivalent if there is some non-empty open subset $W \subset U \cap V$ with $\phi|_W = \psi|_W$. The set of rational functions, denoted by $k(X)$, is a field.

Remark 4.7. *Let X be a quasi-projective variety. Then any rational function $\phi \in k(X)$ can be written as fraction $\phi = \frac{F}{G}$ where F and G are homogeneous polynomials of the same degree with $G \neq 0$. Note however that this description is not unique, i.e. a single rational function might have several descriptions as such a fraction.*

Definition 4.8. *Let X and Y be quasi-projective algebraic sets.*

(1) A regular map $\phi : X \to Y$ is a continuous map such that $\phi^*(f) \in k[\phi^{-1}(U)]$ for all $U \subset Y$ open and $f \in k[U]$. A regular map $\phi : X \to Y$ is an isomorphism if it admits an inverse that is also regular.

(2) Suppose that X is irreducible. A rational map $\phi : X \dashrightarrow Y$ is the equivalence class of a regular map $\phi_U : U \to Y$ for some non-empty open subset $U \subset X$. A rational map $\phi : X \dashrightarrow Y$ is birational if Y is irreducible and if ϕ admits a rational inverse.

Proposition 4.9. *Let $X \subset \mathbb{P}^n_k$ and $Y \subset \mathbb{P}^m_k$ be quasi-projective algebraic sets and let $\phi : X \to Y$ be a map. Then the following are equivalent:*

(a) ϕ is regular;

(b) locally around each $x \in X$ we have $\phi|_U = [F_0 : \cdots : F_m]|_U$ for some homogeneous polynomials $F_0, \ldots, F_m$ of the same degree.

Proof. "(a) ⇒ (b)":

For $x \in X$, consider $y := \phi(x) \in Y$. Wlog $y = [y_0 : \cdots : y_m]$ with $y_0 \neq 0$, i.e. y is contained in the open subset $V = Y \setminus V_Y(t_0)$. Hence, $\phi^*(\frac{t_i}{t_0})$ is regular and so it coincides with $\frac{F_i}{G_i}$ near

x, where F_i, G_i are polynomials of the same degree with $G_i(x) \neq 0$. Hence, locally at x we have

$$\phi = [1 : \frac{F_1}{G_1} : \cdots : \frac{F_m}{G_m}] = [G : \frac{F_1 G}{G_1} : \cdots : \frac{F_m G}{G_m}]$$

where $G = G_1 \cdots G_m$.

"(b) $\Rightarrow$ (a)":

To see that ϕ is continuous, it suffices to check this locally on X and so it suffices to see that $\phi|_U = [F_0 : \cdots : F_m]|_U : U \to Y$ is continuous, where F_i are polynomials of the same degree without common zero on U. The latter is clear because

$$\phi|_U^{-1}(V_Y(F)) = V_U(F(F_0, \ldots, F_m)).$$

Next, let $V \subset Y$ be any open subset and let $f \in k[V]$. Then we need to see that $\phi^* f$ is regular on $\phi^{-1}(V)$. That's a local condition on $\phi^{-1}(V)$. For $x \in \phi^{-1}(V)$, we have $\phi|_U = [F_0 : \cdots : F_m]|_U$ locally on some $x \in U \subset \phi^{-1}(V)$, where F_i are homogeneous of the same degree. Since f is regular at $\phi(x)$, it is locally at $\phi(x)$ given by $f = \frac{H}{G}$ for homogeneous polynomials G, H of the same degree. Then,

$$\phi^* f = \frac{H(F_0, \ldots, F_m)}{G(F_0, \ldots, F_m)}$$

is regular near x, as we want. This concludes the proof of the proposition. □

Corollary 4.10. *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety and let $Y \subset \mathbb{P}^m$ be a quasi-projective algebraic set. Then a rational map $\phi : X \dashrightarrow Y$ is uniquely determined by $[F_0 : \cdots : F_m]$ on some non-empty open subset $U \subset X$ where the homogeneous polynomials F_i have no common zero.*

Proposition 4.11. *(a) Quasi-affine sets are quasi-projective.*

(b) Quasi-projective algebraic sets are covered by affine algebraic sets.

Proof. Let $t_0, \ldots, t_n$ be homogeneous coordinates on $\mathbb{P}^n$ and let $U_i := \mathbb{P}^n \setminus V(t_i)$. Then $\mathbb{P}^n = \bigcup U_i$ is an open covering and

$$\phi_i : U_i \to \mathbb{A}^n, \quad [x_0 : \cdots : x_n] \mapsto (\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i})$$

is an isomorphism, with inverse

$$\psi_i : \mathbb{A}^n \to U_i, \quad (t_0, \ldots, \hat{t}_i, \ldots, t_n) \mapsto (t_0 : \cdots : 1 : \cdots : t_n).$$

This proves (a). Item (b) follows then from item (a) and Lemma 3.11. This concludes the proof of the proposition. □

Example 4.12. *We have $k[\mathbb{P}^n] = k$. In particular, any regular map $\mathbb{P}^n \to \mathbb{A}^m$ is constant.*

Proof. Let $X := \mathbb{P}^n$ and let $\phi \in k[X]$ be a regular function on X. For each $x \in X$, we then have $\phi = \frac{F_x}{G_x}$ locally at x for homogeneous polynomials $F_x, G_x \in k[x_0, \ldots, x_n]$ of the same degree with $G_x(x) \neq 0$. Since $k[x_0, \ldots, x_n]$ is a UFD, we may further assume that F_x and G_x are coprime.

Fix some $x \in X$. If G_x has degree zero, then we are done. Otherwise, there will be a point $y \in X$ with $G_x(y) = 0$. On some non-empty open subset of X, $\frac{F_y}{G_y}$ and $\frac{F_x}{G_x}$ coincide and so

$$F_y G_x = F_x G_y \in k[x_0, \ldots, x_n]$$

Since $gcd(F_x, G_x) = 1$ and $gcd(F_y, G_y) = 1$, we find that $G_x \mid G_y$ and so $G_y(y) = 0$, which is a contradiction. This proves our claim. □

Example 4.13. *The conic curve* $X := V(t_1t_2 - t_0^2) \subset \mathbb{P}^2$ *is isomorphic to* $\mathbb{P}^1$.

Proof. The isomorphism is given by projection to the line $V(x_2)$:

$$\phi : X \to \mathbb{P}^1, \quad [x_0 : x_1 : x_2] \mapsto [x_0 : x_1].$$

Note that this is indeed a regular map, because

$$[x_0 : x_1] = [x_0^2 : x_0x_1] = [x_1x_2 : x_0x_1] = [x_2 : x_0].$$

The inverse is given by

$$\psi : \mathbb{P}^1 \to X, \quad [y_0 : y_1] \mapsto [y_0 : y_1 : \frac{y_0^2}{y_1}] = [y_0y_1 : y_1^2 : y_0^2].$$

□

Example 4.14 (Cremona transformation)**.** *The map*

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0 : x_1 : x_2] \mapsto [\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}] = [x_1x_2 : x_0x_2 : x_0x_1]$$

is a birational automorphism of $\mathbb{P}^2$. *This map is called Cremona transformation.*

Example 4.15 (Projection from a point)**.** *Let* $t_0, \dots, t_n$ *be homogeneous coordinates on* $\mathbb{P}^n$. *Identify* $V(t_n) \subset \mathbb{P}^n$ *with* $\mathbb{P}^{n-1}$ *and let* $y \in \mathbb{P}^n$ *be a point that does not lie on* $\mathbb{P}^{n-1}$. *Then the morphism*

$$\phi : \mathbb{P}^n \setminus \{y\} \to \mathbb{P}^{n-1}, \quad x \mapsto (\textit{line through } x \textit{ and } y) \cap \mathbb{P}^{n-1},$$

induces a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$.

Proof. Up to applying a linear transformation, we may assume that $y = [0 : \cdots : 0 : 1]$. We then have

$$\phi[x_0 : \cdots : x_n] = [x_0 : \cdots : x_{n-1}],$$

which is clearly a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ which restricts to a morphism on $\mathbb{P}^n \setminus \{y\}$, as we want. □

Example 4.16 (Blow-up of $\mathbb{A}^n$ in the origin)**.** *Let* $\phi : \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$ *be given by*

$$(x_1, \dots, x_n) \mapsto [x_1 : \cdots : x_n].$$

Let $\Gamma_\phi \subset \mathbb{A}^n \times \mathbb{P}^n$ *be the closure of the graph of* ϕ *(i.e. the closure of the graph of the morphism that is given by* ϕ *restricted to* $\mathbb{A}^n \setminus \{0\}$*). Then* $Bl_0\mathbb{A}^n := \Gamma_\phi$ *is called the blow-up of* $\mathbb{A}^n$ *in* 0 *and* $\pi : Bl_0\mathbb{A}^n \to \mathbb{A}^n$ *is called the blow-down map. This map has the following properties:*

(1) $\phi^{-1}(0) \cong \mathbb{P}^{n-1}$*;*

(2) π *induces an isomorphism* $Bl_0\mathbb{A}^n \setminus \pi^{-1}(0) \cong \mathbb{A}^n \setminus \{0\}$.

Proof. It suffices to prove $\phi^{-1}(0) \cong \mathbb{P}^{n-1}$, the remaining claim is clear. To prove this, let $L \subset \mathbb{A}^n$ be a line through 0. Then $\phi(L \setminus \{0\}) = x$ is a single point and so $L \times x \subset \Gamma_\phi$, which shows that $x \in \mathbb{P}^{n-1}$, as claimed. □

5 Dimension

5.1 Definition and basic properties

Recall the following simple lemma from differential geometry.

Lemma 5.1. *Let X be a connected real manifold. Then $\dim X$ is the supremum over all $\ell \in \mathbb{N}$ such that there is a chain of connected closed submanifolds*

$$\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subset X$$

Because of singularities, we cannot literally use the above lemma as a definition for the dimension of an algebraic variety. However, after replacing "connected" by "irreducible", this perfectly works and gives rise to the following definition for the dimension of an arbitrary Noetherian topological space.

Definition 5.2. *Let X be a Noetherian topological space. Then the dimension of X is the supremum over all $\ell \in \mathbb{N}$ such that there is a chain of non-empty closed irreducible subsets*

$$\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subset X$$

This should be compared to the Krull dimension of a ring.

Definition 5.3. *Let R be a commutative ring. The Krull dimension of R is the supremum over all ℓ such that there is a chain of prime ideals*

$$\mathfrak{p}_\ell \subsetneq \mathfrak{p}_{\ell-1} \subsetneq \cdots \subsetneq \mathfrak{p}_0 \subsetneq R.$$

If X is an affine variety, then a closed subset $Z \subset X$ is irreducible if and only if $I(Z) \subset k[X]$ is prime. This shows the following.

Lemma 5.4. *Let X be an affine variety. Then,*

$$\dim(X) = \dim k[X].$$

From commutative algebra, we use (without proof) the following basic fact about the Krull dimension of finitely generated integral k-algebras.

Theorem 5.5 (Krullscher Hauptidealsatz)**.** *Let R be a finitely generated integral k-algebra. Then,*

(a) $\dim R < \infty$*;*

(b) if $0 \neq f \in R$ *is not a unit, then* $\dim(R/(f)) = \dim R - 1$.

Example 5.6. *Let $p \in \mathbb{A}^n$ be a point. Then $\dim(\{p\}) = 0$.*

Proof. This follows immediately from the definitions. □

Example 5.7. $\dim \mathbb{A}^n = n$

Proof. Proof by induction on n. For $n = 0$, the statement follows from the previous example. For $n > 0$, note that

$$\dim k[x_1, \ldots, x_n] - 1 = \dim k[x_1, \ldots, x_n]/(x_n) \cong k[x_1, \ldots, x_{n-1}]$$

by Theorem 5.5. Hence, $\dim \mathbb{A}^n = \dim \mathbb{A}^{n-1} + 1$ by Lemma 5.4. This proves the claim by induction on n. □

As a first application of the concept of dimension and the above result from commutative algebra, we prove the following weak version of Bezout's theorem.

Corollary 5.8. *Let $F_1, F_2 \in k[x_0, x_1, x_2]$ be two homogeneous non-constant polynomials and let $X_i := V(F_i) \subset \mathbb{P}^2$. Then $X_1 \cap X_2 \neq \emptyset$.*

Proof. We need to prove $V_{\mathbb{P}^2}(F_1, F_2) \neq \emptyset$, or equivalently $\{0\} \neq V_{\mathbb{A}^3}(F_1, F_2)$. The latter follows clearly if we can prove $\dim V_{\mathbb{A}^3}(F_1, F_2) \geq 1$. Replacing F_1 by one of its irreducible factors, we see that we may without loss of generality assume that F_1 is irreducible in $R := k[x_0, x_1, x_2]$. Hence, $R' := R/(F_1)$ is a finitely generated integral k-algebra and so Theorem 5.5 implies

$$\dim(R'/(\overline{F}_2)) = \begin{cases} 1 & \text{if } \overline{F}_2 \neq 0; \\ 2 & \text{if } \overline{F}_2 = 0, \end{cases}$$

where $\overline{F}_2 \in R'$ denotes the image of F_2 via the projection $R \to R'$, and where we note that $\overline{F}_2$ is not a unit in R', because it vanishes at $0 \in V_{\mathbb{A}^3}(F_1)$. This proves the corollary by Lemma 5.4, because the ring of regular functions of $V_{\mathbb{A}^3}(F_1, F_2)$ is nothing but $R'/(\overline{F}_2)$. □

Remark 5.9. *Corollary 5.8 says that the intersection of two hypersurfaces in $\mathbb{P}^2$ is non-empty. The example of two parallel lines $X_1 = V(x_1)$ and $X_2 = V(x_1 - 1)$ in $\mathbb{A}^2$ shows that the statement does not hold in affine space, i.e. two hypersurfaces in $\mathbb{A}^2$ do not need to meet nontrivially.*

Remark 5.10. *The proof of Corollary 5.8 easily generalizes to show more generally that the intersection of two hypersurfaces in $\mathbb{P}^n$ has dimension at least $n - 2$. (To see this one has to use that the affine cone of a projective algebraic set has dimension one more than the projective set – this assertion might appear on exercise sheet 5.)*

The definition of the dimension of a topological space implies easily the following lemma.

Lemma 5.11. *Let X be a nonempty irreducible Noetherian topological space and let $X = \bigcup_i U_i$ be an open covering. Then $\dim X = \sup_i \dim U_i$.*

Proof. Clearly, $\dim X \geq \dim U_i$ for all i. It thus suffices to prove $\dim X \leq \sup_i \dim U_i$. For this, let

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X$$

be a chain of irreducible closed subsets. We need to prove $n \leq \dim U_i$ for some i and we will do so by induction on n. If $n = 0$, we have $0 \leq \dim U_i$ whenever U_i is non-empty. In the induction step, we assume $n > 0$ and assume that we have proven our claim for chains of length at most $n - 1$ and for arbitrary open coverings of nonempty irreducible Noetherian topological spaces. In particular, applying this to the open covering of Z_{n-1} that is induced by the U_i's, we know by induction that there is an index i_0 such that

$$n - 1 \leq \dim(Z_{n-1} \cap U_{i_0}).$$

Since X is irreducible, the non-empty open subset $U_{i_0} \subset X$ is dense and so it cannot be contained in Z_{n-1}. We thus find that

$$Z_{n-1} \cap U_{i_0} \subsetneq U_{i_0}.$$

Hence, $\dim(U_{i_0}) \geq 1 + \dim(Z_{n-1} \cap U_{i_0}) \geq n$, which proves our claim. This concludes the proof of the lemma. □

The following result collects some important technical properties of the dimension of quasi-projective varieties.

Theorem 5.12. *Let X be a quasi-projective algebraic variety. Then,*

(a) if X is affine and $f \in k[X]$ such that $\emptyset \neq V_X(f) \subsetneq X$, then

$$\dim(V_X(f)) = \dim X - 1;$$

(b) in the notation of item (a), any irreducible component of $V_X(f)$ has dimension $\dim X - 1$;

(c) if $U \subset X$ is a non-empty open subset, then $\dim X = \dim U$.

Proof. Item (a) follows from Lemma 5.4 and Theorem 5.5.

Item (b) follows from items (a) and (c), as follows: Let $V_X(f) = Z_1 \cup \cdots \cup Z_r$ be the decomposition into irreducible components, i.e. each Z_i is irreducible and Z_i is not contained in Z_j for $i \neq j$. Up to renumeration, it suffices to prove $\dim(Z_1) = \dim(X) - 1$.

It is easy to see that Z_1 is not contained in $\bigcup_{i=2}^r Z_i$ (otherwise $Z_1 \subset Z_i$ for some $i \geq 2$, as Z_1 is irreducible, which is a contradiction) and so there is a function $g \in I(\bigcup_{i=2}^r Z_i)$ which does not vanish identically on Z_1. Hence, $U := X \setminus V_X(g)$ is a non-empty open subset which is isomorphic to an affine variety (by exercise sheet 3). Moreover,

$$V_U(f|_U) = V_X(f) \cap U = Z_1 \cap U$$

is irreducible. We thus find from item (a) that

$$\dim(Z_1 \cap U) = \dim U - 1.$$

Moreover, by item (c), we have $\dim U = \dim X$ and $\dim(Z_1 \cap U) = \dim Z_1$, which proves

$$\dim Z_1 = \dim X - 1$$

as we want.

It remains to prove item (c). Clearly, for any non-empty open subset $U \subset X$ we have $\dim U \leq \dim X$. The converse is a little bit tricky and we will prove it by induction on $\dim X$. Note first that we can cover X by affine open subsets $X = \bigcup_i U_i$. By Lemma 5.11, we get $\sup_i \dim U_i = \dim X$. We may thus without loss of generality assume that X is affine (because we know that the dimension can at most go down if we shrink our open subset).

Let $U \subset X$ be non-empty open, where X is affine and denote the complement of U in X by $Z := X \setminus U$. Let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of Z into its irreducible components. By induction on r we may assume that Z is irreducible. Pick a point $x \in U$. Since $x \notin Z$, there is a function $f \in k[X]$ which vanishes at x but which is not contained in $I(Z)$, i.e. it does not vanish identically along Z. (Draw a picture!) Hence, Z is not a component of $V_X(f)$ and

$$V_X(f) \cap U \neq \emptyset.$$

We can choose an irreducible component V of $V_X(f)$ with

$$\dim V = \dim V_X(f) = \dim X - 1,$$

where the last equality follows from Lemma 5.4 and Theorem 5.5. By assumptions, Z is not contained in V and we claim that also the converse holds, i.e. V is not contained in Z. Indeed, if it was, then $V \subsetneq Z$ and so $\dim X - 1 = \dim V < \dim Z$, because V is closed and irreducible. Hence, $\dim Z = \dim X$, which is absurd, because $Z \subsetneq X$ and Z is closed and irreducible. So by induction, (using that V is irreducible), we find

$$\dim V = \dim(V \cap U).$$

Since $\dim V = \dim X - 1$, we find that

$$\dim X - 1 = \dim(V \cap U) \leq \dim U - 1,$$

where the last inequality uses that $V \cap U \subsetneq U$ is an irreducible closed subset. Hence,

$$\dim X \leq \dim U,$$

as we want. This concludes the proof of the theorem. □

5.2 Relation to the transcendence degree of field extensions

The next results relates the concept of dimension to field theory. For this we recall that for any field extension K of k, one can find a transcendence basis, i.e. a collection of algebraically independent elements $(z_i)_{i\in I}$ with $z_i \in K$ such that K is an algebraic extension of the subfield $k(z_i \mid i \in I) \subset K$. Moreover, the cardinality of I is independent of the chosen basis and it is called the transcendence degree of K over k:

$$trdeg_k(K) = |I|.$$

We will use the following fact from algebra, see [4, Theorem 4.7A] and the references therein.

Theorem 5.13 (Existence of separating transcendence basis)**.** *Let k be a perfect field (e.g. algebraically closed) and let K/k be a finitely generated field extension. Then there is a transcedence basis $z_1, \ldots, z_n$ of K/k such that K is separable over $k(z_1, \ldots, z_n)$. In particular, K is a simple algebraic extension of $k(z_1, \ldots, z_n)$.*

This theorem from algebra has the following consequence.

Corollary 5.14. *Let X be a quasi-projective algebraic variety (as always over an algebraically closed field k), and let $n := trdeg_k(k(X))$. Then X is birational to an irreducible hypersurface in $\mathbb{A}^{n+1}$. In particular,*

$$\dim X = n = trdeg_k(k(X)).$$

Proof. By Theorem 5.13, we can find a separating transcedence basis $z_1, \ldots, z_n \in k(X)$ of $k(X)/k$. Then, $k(X)$ is a finite separable extension of $k(z_1, \ldots, z_n)$ and so it is normal, that is, it is generated by a single element $z_{n+1} \in k(X)$. Let $f \in k(z_1, \ldots, z_n)[t]$ be the minimal polynomial of z_{n+1}. Then, $k(X)$ is isomorphic to the fraction field of $k(z_1, \ldots, z_n)[t]/f(t)$. Note that f is a polynomial in t whose coefficients are fractions of polynomials in $k[z_1, \ldots, z_n]$. Multiplying through the denominators of all those fractions, we may assume that there is a polynomial $F \in k[t_1, \ldots, t_{n+1}]$ with $f(t) = F(z_1, \ldots, z_n, t)$. We may further assume that F is irreducible. This implies that the hypersurface $V(F) \subset \mathbb{A}^{n+1}$ is irreducible (because F is) and has the same function field as X. Hence, X is birational to $V(F)$ (see Exercise sheet 3), which proves the first claim in the corollary. It follows in particular that X and $V(F)$ have isomorphic non-empty open subsets, see Proposition 3.12. Hence, $\dim X = \dim V(F)$ by Theorem 5.12. Since $\dim V(F) = n + 1 - 1$ by Lemma 5.4 and Theorem 5.5, we get that $\dim X = n$, as we want. This completes the proof of the corollary. □

Example 5.15. *Let $X \subset \mathbb{P}^n$ be a quasi-projective algebraic set. Then,* $\dim X = 0$ *if and only if X is a finite collection of points.*

Proof. If X consists of finitely many points, then $\dim X = 0$ is clear. For the converse, assume that $\dim X = 0$. Up to replacing X by an irreducible component with top dimension, we may assume that X is irreducible. By Corollary 5.14, X is birational to an irreducible hypersurface in $\mathbb{A}^1$, hence to a point. This implies that X is already isomorphic to a point (e.g. because two birational varieties have isomorphic non-empty open subsets by Proposition 3.12), which proves our claim. □

5.3 The theorem on the dimension of fibres

The following theorem uses the concept of dimension to describe some of the most basic properties of regular maps $f : X \to Y$ between quasi-projective algebraic varieties. Up to replacing Y by the closure of the image of f (which must be irreducible because X is), we may assume that f is dominant. For a point $y \in Y$ we then define the fibre

$$X_y := f^{-1}(y)$$

of f above y as preimage of y under f.

Here is the main theorem of this section.

Theorem 5.16. *Let $f : X \to Y$ be a dominant regular map between quasi-projective varieties of dimensions $d := \dim X$ and $e := \dim Y$. Then,*

(1) for any $y \in f(X)$, $\dim X_y \geq d - e$;

(2) there is a non-empty open subset $U \subset Y$, such that $\dim X_y = d - e$ for all $y \in U$;

Recall the example of $Bl_0\mathbb{A}^n \to \mathbb{A}^n$ (see Example 4.16), which shows that the dimension of the fibres can indeed jump in general.

Proof of Theorem 5.16. We first prove item (1). For this, let $d = \dim X$, $e = \dim Y$ and fix a point $y \in f(X)$. Choose a preimage $x \in X_y$ of y. We want to prove $\dim X_y \geq d - e$. Up to replacing Y by an affine open neighbourhood V of y and X by an open neighbourhood of x in $f^{-1}(V)$, we may assume without loss of generality that X and Y are affine. The idea is now to prove the theorem by induction on $\dim Y$. For this we choose a nonzero regular function $\phi \in k[Y]$ on Y which vanishes on y. Choose an irreducible component S of

$$f^{-1}X_Y(\phi) = V_X(f^*\phi)$$

with $\dim S_y = \dim X_y$. By Theorem 5.12, $\dim S = d - 1$. Let $T \subset Y$ be the closure of $f(S)$. Then,

$$f|_S : S \to T$$

is a dominant regular map between quasi-projective varieties. By induction, $\dim S_y \geq \dim S - \dim T$. Since $\dim S = d - 1$ and $\dim T \leq e - 1$, we find

$$\dim X_y = \dim S_y \geq d - 1 - (e - 1) = d - e.$$

This proves item (1).

We prove item (2) next. This is done in several steps.

Lemma 5.17. *Wlog X and Y are affine.*

Proof. We may clearly assume that Y is affine. Next, we can choose $Z \subsetneq X$ closed such that $W := X \setminus Z$ is non-empty and affine. Let $Z = Z_1 \cup \cdots \cup Z_r$ be the irreducible components of Z. We may assume that $f(Z_i)$ is dense in Y for $i = 1, \ldots, s$ and it is not dense for $i \geq s + 1$, where $s \leq r$ is some integer.

For $i \leq s$, we consider the dominant morphism $f_i : Z_i \to Y$ induced by f. By induction on $\dim X$, $\exists\ U_i \subset Y$ open such that

$$\dim f_i^{-1}(y) = \dim Z_i - \dim Y < \dim X - \dim Y$$

for all $y \in U_i$.

Hence, if we can prove the theorem for $f|_W : W \to Y$, then there is an open subset $U \subset Y$ with

$$\dim(f|_W^{-1}(y)) = \dim W - \dim Y = \dim X - \dim Y$$

for all $y \in U$. It then follows that the following open subset will work for $f : X \to Y$:

$$U \cap \bigcap_{i=1}^{s} U_i \cap \bigcap_{i=s+1}^{r} (Y \setminus \overline{f}(Z_i))$$

This proves the lemma. □

From now on we assume that X and Y are affine and $f : X \to Y$ is a dominant morphism. Since f is dominant,

$$f^* : k[Y] \to k[X]$$

is injective and so we get an injection of fields $k(Y) \subset k(X)$. Let $d = \dim X$ and $e := \dim Y$. Since $k(X)$ has transcendence degree d over k, while $k(Y)$ has transcendence degree e over k, we can find elements $\phi_1, \dots, \phi_{d-e} \in k(X)$ that are algebraically independent over $k(Y)$. Up to shrinking X (which we can by the proof of the above lemma), we may assume $\phi_i \in k[X]$ for all $i = 1, \dots, d-e$. Consider the subring $k[\phi_1, \dots, \phi_{d-e}] \subset k[X]$, generated by $\phi_1, \dots, \phi_{d-e}$. Since $k[X]$ is finitely generated over k, there are finitely many elements $\phi_{d-e+1}, \dots, \phi_m \in k[X]$ which generate $k[X]$ over $k[\phi_1, \dots, \phi_{d-e}]$. That is,

$$k[\phi_1, \dots, \phi_m] = k[X].$$

Let $y \in Y$ and let M be an irreducible component of $f^{-1}(y)$. Then M is an affine variety with

$$k[M] = k[\phi_1|_M, \dots, \phi_m|_M].$$

Since $\phi_{d-e+1}, \dots, \phi|_d$ are algebraic over $k(Y)(\phi_1, \dots, \phi_{d-e})$, we find (after multiplication with denominators) that $\phi_{d-e+1}, \dots, \phi|_d$ are algebraic over the ring $k[Y][\phi_1, \dots, \phi_{d-e}]$, i.e. there are algebraic relations

$$F_j(\phi_1, \dots, \phi_{d-e})(\phi_j) = 0$$

where

$$F_j \in k[Y][t_1, \dots, t_{d-e}][t]$$

are polynomials whose coefficients are functions on Y.

Idea: The hope is now that these algebraic relations remain nontrivial (i.e. not all coefficients of the polynomials vanish) for all y contained in some Zariski open subset U of Y. If we can show this, then the field extension $k(M)/k(\phi_1|_M, \dots, \phi_{d-e}|_M)$ is algebraic for all $y \in U$. If so, then $\operatorname{trdeg}(k(M)/k) \leq d-e$ and so $\dim M \leq d-e$. Hence, $\dim f^{-1}(y) \leq d-e$ for all $y \in U$ and so equality holds by item (1). We have thus seen that item (2) of the theorem follows once we can prove the following lemma.

Lemma 5.18. *There is some dense open subset $U \subset Y$, such that for all $y \in Y$ and for all components M of $f^{-1}(y)$ with $\dim M = \dim f^{-1}(y)$, the field extension*

$$k(M)/k(\phi_1|_M, \dots, \phi_{d-e}|_M)$$

is algebraic for all $y \in U$.

Proof. To prove the lemma, we need to find some dense open $U \subset Y$ such that for all $j \geq d - e + 1$, the polynomial

$$F_j|_M(\phi_1|_M, \dots, \phi_{d-e}|_M)(t)$$

is nonzero for all $y \in U$.

Note first that the restriction of functions on Y to M are constant and so the coefficients of $F_j|_M(t_1, \dots, t_{d-e})(t)$ are constant. Let then $\alpha_j \in k(\phi_1, \dots, \phi_{d-e})$ be the leading term of

$$F_j|_M(\phi_1|_M, \dots, \phi_{d-e}|_M)(t),$$

viewed as a polynomial in t.

The lemma follows then from the following

Claim 1. *There is some open dense subset $U_j \subset Y$, such that $\alpha_j|_M \neq 0$ for all $y \in U_j$.*

To prove the above claim, we use an argument similar to the one used in the proof of (1). Let $Z_1, \dots, Z_r$ be the irreducible components of $V_X(\alpha_j)$ and assume that $f(Z_i) \subset Y$ is dense for $i \leq s$ and not dense for $i \geq s+1$. By Theorem 5.12, $\dim Z_i = \dim X - 1$. By induction on $\dim X$, for all $i \leq s$, there is some open dense subset $V_i \subset X$ such that for all $y \in V_i$,

$$\dim(f|_{Z_i}^{-1}(y)) = \dim Z_i - \dim Y < \dim X - \dim Y.$$

Hence,

$$U_j := \bigcap_{i=1}^{s} V_i \cap \bigcap_{i=s+1}^{r} (Y \setminus \overline{f(Z_i)})$$

does the job. This proves the above claim and so the lemma follows. □

This concludes the proof of the theorem. □

The above theorem has several interesting consequences; we collect some of those in what follows.

Corollary 5.19. *Let $f : X \dashrightarrow Y$ be a dominant rational map between quasi-projective varieties. Then $\dim X \geq \dim Y$.*

Proof. This follows from Theorem 5.16, applied to the morphism $\phi : \mathrm{dom}(\phi) \to Y$, because $\mathrm{dom}(\phi) \subset X$ is non-empty open and so it has the same dimension as X by Theorem 5.12. Alternatively, we can also argue that $\phi^* : k(Y) \to k(X)$ is a nontrivial field homomorphism, hence an injection of fields and so

$$\dim X = trdeg_k k(X) \geq trdeg_k k(Y) = \dim Y.$$

□

Remark 5.20. *The above corollary shows in particular that in algebraic geometry there is nothing like space filling curves.*

Corollary 5.21. *The image of a quasi-projective variety under a regular map is a finite disjoint union of quasi-projective varieties.*

Proof. We prove this by induction on $\dim X$. Let $f : X \to Y$ be a regular map between quasi-projective varieties. Up to replacing Y by the closure of $f(X)$ (which is automatically irreducible), we may assume that f is dominant. By Theorem 5.16, $f(X)$ contains an open subset $U \subset Y$. Let $Z = Y \setminus U$ be its complement. It has finitely many irreducible components and we let $Z' \subset Z$ be one such component. The subset $f^{-1}(Z') \subset X$ is closed and so it has finitely many irreducible components $X_1, \ldots, X_r$. Since $X_i \subsetneq X$ is closed and irreducible, $\dim X_i < \dim X$ and so the induction hypothesis applies to the morphism $f|_{X_i} : X_i \to f(X_i) \subset Z'$ that is induced by f. This proves the corollary by induction. □

Corollary 5.22. *Let $f : X \to Y$ be a surjective regular map between quasi-projective varieties. Then the subset*

$$P_l := \{y \in Y \mid \dim(X_y) \geq l\}$$

is closed in Y for all $l \geq 0$. In other words, the function $y \mapsto \dim(X_y)$ is upper semi-continuous.

Proof. Let $d = \dim X$ and $e = \dim Y$. By Theorem 5.16, $P_{d-e} = Y$ and $P_{d-e-1} = \emptyset$. By induction on l, it then suffices to prove that P_{d-e+1} is closed in Y. That is, we need to prove that $Y \setminus P_{d-e+1}$ is open. By Theorem 5.16, this set contains at least an open subset U and the openness can then be proven by applying Theorem 5.16 to the map induced by f on the components of $f^{-1}(Y \setminus U)$, noting that by induction on $\dim X$, we may assume that we know the corollary in lower dimension. The details are not difficult but slightly cumbersome and so we leave them to the reader. □

6 Images of projective varieties are closed

The main result of this section is the following theorem.

Theorem 6.1. *Let X be a projective variety and let $f : X \to Y$ be a regular map, where Y is a quasi-projective variety. Then $f(X) \subset Y$ is closed.*

This has the following important consequences.

Corollary 6.2. *Let $f : X \to Y$ be a regular map between quasi-projective varieties. If X is a projective variety, then the image $f(X)$ is a projective variety as well.*

Proof. Since Y is quasi-projective, there is an injective regular map $Y \hookrightarrow \mathbb{P}^m$. The corollary therefore follows from the above theorem, applied to the composition $X \to Y \hookrightarrow \mathbb{P}^m$. □

Corollary 6.3. *Any regular function on a projective variety is constant.*

Proof. If $f \in k[X]$ is a regular function, then we get a regular map $f : X \to \mathbb{A}^1$. By Theorem 6.1, $f(X)$ is closed. Since X is irreducible, so is $f(X)$. Hence, $f(X)$ is either a point or $\mathbb{A}^1$. The latter is impossible, because the image of the composition $X \to \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ must be closed as well. This proves the corollary. □

Corollary 6.4. *Let $f : X \to Y$ be a regular map from a projective variety X to an affine variety Y. Then f is constant, i.e. $f(X)$ is a single point of Y.*

Proof. This follows immediately from Corollary 6.3, applied to $x_i \circ F$, where x_i is the i-th coordinate function on $Y \subset \mathbb{A}^n$. □

6.1 Products

Recall from the Exercise sheet, that there is an inclusion

$$\iota : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}, \quad ([x_0, \ldots, x_n], [y_0, \ldots, y_m]) \mapsto [..., x_i y_j, ...].$$

The image of this inclusion is closed, which allows us to put the structure of a (quasi-) projective algebraic set on the product of (quasi-) projective algebraic sets.

Lemma 6.5. *(1) The closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are those that are cut by a bunch of polynomials*

$$F(x_0, \ldots, x_n, y_0, \ldots, y_m)$$

that are homogeneous in the x-, as well as in the y-coordinates.

(2) The closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$ are those that are cut out by a bunch of polynomials

$$F(x_0, \ldots, x_n, y_1, \ldots, y_m)$$

that are homogeneous in the x coordinates.

Proof. The second item follows from the first by restriction to $\mathbb{A}^m \subset \mathbb{P}^m$. To prove the first item, note that by definition, any closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ is cut out by a bunch of functions of the form $F \circ \iota$, where $F \in k[z_0, \ldots, z_{nm-1}]$ is homogeneous. Clearly, the function $F \circ \iota([x], [y])$ can be written as a polynomial in the coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$ and $y_0, \ldots, y_m$ of $\mathbb{P}^m$, which is homogeneous and of the same degree in both sets of variables.

On the other hand, if $G(x, y)$ is a polynomial in the x_i and y_j which is homogeneous in both sets of variables of degrees d and e with $d \neq e$, then this still defines a subset of $\mathbb{P}^n \times \mathbb{P}^m$. Such a subset is algebraic, as we can replace G by the set of polynomials $y_j^{d-e} G$ for $j = 0, \ldots, m$, where we assumed wlog that $d > e$. □

Corollary 6.6. *The image $\iota(\mathbb{A}^n \times \mathbb{A}^m)$ carries the natural structure of an affine variety on $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$. In particular, the product on affine varieties $X \subset \mathbb{A}^n$ and $Y \subset A^m$ constructed as above coincides with the naive product*

$$X \times Y \subset \mathbb{A}^{n+m}.$$

Corollary 6.7. *Let X any Y be projective algebraic sets and let $S \subset X$ and $T \subset Y$ be closed. Then $S \times Y$ and $X \times T$ are closed in $X \times Y$. In particular, the topology on $X \times Y$ is finer than the product topology.*

Proof. This is an immediate consequence of the previous lemma. □

Corollary 6.8. *If X and Y are irreducible, then so is $X \times Y$.*

Proof. Suppose that $X \times Y = Z \cup Z'$ for closed subsets $Z, Z' \subset X \times Y$. Let

$$U := \{y \in Y \mid X \times \{y\} \subset Z\} \text{ and } U' := \{y \in Y \mid X \times \{y\} \subset Z'\}.$$

By the previous corollary, for any $y \in Y$, $Z \cap X \times \{y\}$ and $Z' \cap X \times \{y\}$ are closed in $X \cong X \times \{y\}$. Since X is irreducible the equation

$$X \times \{y\} = (Z \cap X \times \{y\}) \cup (Z' \cap X \times \{y'\})$$

thus shows that $Y = U \cup U'$. Hence, $Y = \overline{U} \cup \overline{U}'$. Since Y is irreducible, we may wlog assume $\overline{U} = Y$. Since $X \times U \subset Z$ and Z is closed, we get

$$\overline{X \times U} \subset Z.$$

We thus conclude $Z = X \times Y$ if we can prove

$$\overline{X \times U} = X \times Y.$$

But if that was false, then there is a point $(a, b) \in X \times Y$ that is not contained in $\overline{X \times U}$. That means that there is a bihomogeneous polynomial $F(x, y)$ with $F|_{X \times U} = 0$ but $F(a, b) \neq 0$. Hence, $F(a, y)$ is a homogeneous polynomial in the y coordinates which vanishes on U but not on b. Hence, $\overline{U} \subsetneq Y$, which is a contradiction. This proves the corollary. □

In the above notation, set $U := X \setminus S$ and $V := Y \setminus T$, then $U \times V \subset X \times Y$ is open. If $U \subset \mathbb{A}^n$ and $V \subset \mathbb{A}^m$ are affine, then $U \times V$ has a natural structure of an affine variety and one checks that this structure coincides with the one defined as open subset of $X \times Y$.

Example 6.9. *Note that the topology on $X \times Y$ is not the product topology! For instance,*

$$\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$$

has many more proper closed subsets than only unions of sets of the form $x \times \mathbb{A}^1$ and $\mathbb{A}^1 \times y$.

6.2 Proof of Theorem 6.1

Before we prove the theorem, we need the following lemma.

Lemma 6.10. *Let $f : X \to Y$ be a regular map between quasi-projective varieties. Then the graph*

$$\Gamma_f := \{(x, f(x)) \in X \times Y \mid x \in X\}$$

is closed in $X \times Y$.

Proof. We have $Y \subset \mathbb{P}^m$ for some m. Since closed subsets of $X \times \mathbb{P}^m$ pullback to closed subsets in $X \times Y$, it suffices to prove the statement in the case where $Y = \mathbb{P}^m$. Consider the map

$$g : X \times \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^m, \quad (x, p) \mapsto (f(x), p).$$

Clearly, g is regular and the preimage of the diagonal $\Delta_{\mathbb{P}^m} := \{(p, p) \mid p \in \mathbb{P}^m\}$ is nothing but Γ_f. It thus suffices to prove that $\Delta_{\mathbb{P}}^m$ is closed in $\mathbb{P}^m \times \mathbb{P}^m$, which is simple to check. This concludes the lemma. □

We are now ready to prove the theorem.

Proof of Theorem 6.1. Consider the map $f : X \to Y$ and let $\Gamma_f \subset X \times Y$ be the graph of f. By Lemma 6.10, Γ_f is closed in $X \times Y$. Consider the projection

$$p : X \times Y \to Y, \quad (x, y) \mapsto y.$$

Note that we have $p(\Gamma_f) = f(X)$. The theorem follows therefore from the following stronger statement. □

Theorem 6.11. *Let X, Y be quasi-projective varieties. If X is a projective variety, then the projection $p : X \times Y \to Y$ is a closed map, i.e. it takes closed sets to closed sets.*

Proof. Since X is projective, there is some embedding $X \subset \mathbb{P}^n$ such that X is closed in $\mathbb{P}^n$. It follows that it suffices to prove the theorem in the case where $X = \mathbb{P}^n$. Since closedness is a local property, and since any quasi-projective variety can be covered by open subsets, it is furthermore enough to prove the theorem in the case where Y is affine. In that case, $Y \subset \mathbb{A}^m$ is a closed subset of some affine space and so it is actually enough to prove the theorem in the special case $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.

Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be a closed subset. By Lemma 6.5, Z is cut out by finitely many polynomials

$$F_i(x_0, \ldots, x_n, y_1, \ldots, y_m) \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$$

with $i = 1, \ldots, r$, which are homogeneous in the x-coordinates. Therefore, a point $a \in \mathbb{A}^n$ does not lie in $p(Z)$ if and only if

$$\{x \in \mathbb{P}^n \mid F_i(x, a) = 0 \quad \forall i\}$$

is empty. This is equivalent to asking that the homogeneous ideal

$$J_a := \langle F_i(x_0, \ldots, x_n, a_1, \ldots, a_m) \mid i = 1, \ldots, r\rangle \subset k[x_0, \ldots, x_n]$$

satisfies $V_{\mathbb{P}^n}(J) = \emptyset$. By Proposition 4.4, the latter is equivalent to

$$S_+ := \bigoplus_{d \geq 1} S_d \subset \sqrt{J}_a,$$

where S_d denotes the degree d part of the graded ring $S := k[x_0, \ldots, x_n]$. This in turn is equivalent to

$$S_d \subsetneq J_a$$

for all $d \geq 1$. (Exercise!)

To prove the theorem, it thus suffices to show that the set of points $y \in \mathbb{A}^m$ with

$$S_d \subset J_a$$

for some d is open in $\mathbb{A}^m$. Since arbitrary unions of open sets are open, it is in fact enough to prove that for fixed $d \geq 1$, the set of points $y \in \mathbb{A}^m$ with

$$S_d \subset J_a$$

is open in $\mathbb{A}^m$.

Let d_i be the homogeneous degree of $F_i(x, y)$ in the x variables. For any partition $\mathfrak{s} := (s_0, \ldots, s_n)$ of s by nonnegative integers $s_i \geq 0$, we consider the monomial

$$x^{\mathfrak{s}} := \prod_i x_i^{s_i}$$

and denotes its degree by $|\mathfrak{s}| = \sum s_i$. The piece S_d is generated by all monomials of degree $|\mathfrak{s}| = d$ as above.

We thus find that $S_d \subset J_a$ is equivalent to the fact that the map

$$L(a) := \bigoplus_{i=1}^{r} S_{d-d_i} \longrightarrow S_d$$

which on S_{d-d_i} is given by multiplication with $F_i(x, a)$ is surjective. The above map is a linear map of vector spaces which is given by a matrix whose coefficients depend polynomially on the point $a \in \mathbb{A}^n$. The linear map $L(a)$ is surjective if its rank is full, i.e. if at least one of the $N \times N$ minors ($N = \dim_k S_d$) has nonzero determinant. Each such determinant is a polynomial equation in the coordinates of the point $a \in \mathbb{A}^n$. Asking that at least one of these equations is nonzero is an open condition on $a \in \mathbb{A}^n$. This concludes the proof of the Theorem. □

7 Local properties and smoothness

7.1 Localization and local rings

We start by recalling the following concept from commutative algebra.

Definition 7.1. *Let A be a commutative ring and let $S \subset A$ be a multiplicatively closed subset, i.e. $1 \in S$ and $s, t \in S \Rightarrow st \in S$. Then the localization $S^{-1}A$ of A at S is defined as*

$$S^{-1}A := (A \times S)/\sim$$

where $(x, s) \sim (y, t)$ if and only if there is some $u \in S$ with $u(xt - ys) = 0$.
The equivalence class of (x, s) is denoted by $\frac{x}{s}$.

Note that the element u in the definition of the relation $\sim$ is needed when you want to check that the relation is transitive.
Note that $S^{-1}A$ is a commutative ring with ring structure

$$\frac{x}{s}\frac{y}{t} = \frac{xy}{st} \quad \text{and} \quad \frac{x}{s} + \frac{y}{t} = \frac{xt + ys}{st}.$$

Example 7.2. *If A is a domain, i.e. has no zero-divisors, then $S := A\backslash\{0\}$ is a multiplicative set and*

$$S^{-1}A \cong \mathrm{Frac}(A).$$

Example 7.3. *If S contains 0, then $S^{-1}A = \{\frac{0}{1}\}$ is the zero ring.*

Example 7.4. *If $f \in A$, then $S := \{f^n \mid n \in \mathbb{Z}_{\geq 0}\}$ is a multiplicatively closed set (because $f^0 = 1$ by definition). The localization $S^{-1}A$ is usually denoted by*

$$A_f := S^{-1}A.$$

Its elements are of the form $\frac{x}{f^i}$ with $i \geq 0$.

Example 7.5. *If $\mathfrak{p} \subset A$ is a prime ideal, then $S := A \setminus \mathfrak{p}$ is a multiplicatively closed set and the localization $S^{-1}A$ is denoted by*

$$A_{\mathfrak{p}} := S^{-1}A.$$

The next example is a special case of the previous one. It is of fundamental importance to algebraic geometry!

Example 7.6. *If X is an affine algebraic set and $Z \subset X$ is a closed and irreducible subset (e.g. a point), then the local ring of X at Z is defined as the localization of $k[X]$ at the prime (!) ideal $I(Z)$:*

$$\mathcal{O}_{X,Z} := k[X]_{I(Z)}.$$

That is, the elements of $\mathcal{O}_{X,Z}$ are fractions $\frac{f}{g}$, where $f, g \in k[X]$ are regular functions on X and g does not vanish identically along Z, i.e. $g \notin I(Z)$.

If X is irreducible, then $\mathcal{O}_{X,Z} \subset k(X)$ is the subring of all rational functions $\varphi \in k(X)$ with $\mathrm{dom}(\varphi) \cap Z \neq \emptyset$. That is, we are looking at all rational functions on X that are regular (hence actual functions) along some non-empty open subset of Z.

Remark 7.7. *In commutative algebra, you learn that there is a natural correspondence*

$$\{\textit{prime ideals } \mathfrak{q} \subset A \textit{ with } \mathfrak{q} \subset \mathfrak{p}\} \leftrightarrow \{\textit{prime ideals of } A_{\mathfrak{p}}\},$$

given by mapping a prime ideal $\mathfrak{q} \subset A$ contained in $\mathfrak{p}$ to the ideal in $A_{\mathfrak{p}}$ that is generated by $\frac{x}{1}$ with $x \in \mathfrak{p}$. In geometric terms, if $A = k[X]$ and $Z \subset X$ is closed and irreducible, the prime ideals of $\mathcal{O}_{X,Z}$ correspond to those closed irreducible subsets $S \subset X$ that contain Z.

Since $\mathcal{O}_{X,Z}$ is not a finitely generated k-algebra in general, there is no affine k-variety Y with $k[Y] = \mathcal{O}_{X,Z}$. This is unfortunate, as it would be good to have a geometric object that corresponds to the localization of X at Z. We will see next semester that the language of schemes allows us to associated with any ring A a space (called affine scheme)

$$\operatorname{Spec} A$$

whose points are the prime ideals of A. In particular, the points of $\operatorname{Spec}(\mathcal{O}_{X,Z})$ will correspond to the subvarieties of X that contain Z.

Definition 7.8. *Let X be a quasi-projective algebraic set, $x \in X$. The local dimension of X at x is given by*

$$\dim_x X := \max_{x \in Z \subset X} \dim Z,$$

where $Z \subset X$ runs through all irreducible components that contain x.

Lemma 7.9. *Let X be an affine algebraic set. Then,*

(a) there is a natural isomorphism

$$\mathcal{O}_{X,x} \longrightarrow \{(\psi, U) \mid x \in U, U \subset X \textit{ is open}, \psi \in k[U]\}/\sim$$

where $(\psi, U) \sim (\varphi, V)$ if there is an open subset $W \subset U \cap V$ which contains x and such that $\psi|_W = \varphi|_W$.

(b) If $U \subset X$ is open, $x \in U$, then $\mathcal{O}_{U.x} \cong \mathcal{O}_{X,x}$.

(c) $\dim(\mathcal{O}_{X,x}) = \dim_x X$.

Proof. Item (a) is easy. Item (b) follows from (a). Item (c) follows from the previous remark. □

7.2 Zariski tangent space

Definition 7.10. *Let $X \subset \mathbb{A}^n$ be an affine algebraic set, $x = (x_1, \ldots, x_n) \in X$, $I(X) = \langle f_1, \ldots, f_r \rangle$. Then the tangent space $T_{X,x} \subset \mathbb{A}^n$ of X at $x \in X$ is given by*

$$T_{X,x} := V(d_x f_1, \ldots, d_x f_r),$$

where $d_x f_j \in k[t_1, \ldots, t_n]$ is the linear polynomial that appears as linear term of f_j in its Taylor expansion around x, i.e. $d_x f_j := \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(x)(t_j - x_j)$.

Note that by definition, $T_{X,x}$ is a linear subspace of $\mathbb{A}^n$ for all $x \in X$.

Example 7.11. *If $X := V(f) \subset \mathbb{A}^n$ with $f \in k[t_1, \ldots, t_n]$ irreducible, $x \in X$, then*

$$T_{X,x} = V\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)(t_j - x_j)\right),$$

has dimension $\geq n - 1$. Moreover, equality holds if and only if $\frac{\partial f}{\partial x_j}(x) \neq 0$ for at least one j.

Lemma 7.12. *In the above example, the set of points*

$$X^{sm} := \{x \in X \mid \dim(T_{X,x}) = n-1\}$$

is open and dense in X.

Proof. Note that $X \setminus X^{sm}$ is given by $V_X(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ and so it is closed. That is, X^{sm} is open. To prove that it is dense, we need to show that it is nonempty. If not, then

$$V(f) \subset V(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}).$$

Hence,

$$\operatorname{rad}((\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})) \subset \operatorname{rad}((f)) = (f)$$

where the last equality holds because f is irreducible. This implies $\frac{\partial f}{\partial x_i} \in (f) = 0$ for all i by degree reasons. This is already a contradiction in characteristic zero. However, in characteristic p, polynomials like $\sum x_i^p$ are nonzero but have zero derivative. In fact, $\frac{\partial f}{\partial x_i} \in (f) = 0$ tells us that

$$f(x_1, \dots, x_n) = g(x_1^p, \dots, x_n^p)$$

for some polynomial g, where p is the characteristic of k. But then $f(x) = \tilde{g}(x)^p$, where $\tilde{g}$ is given by replacing the coefficients of g by p-th roots (which is possible because $k = \overline{k}$), and so f is not irreducible. This concludes the lemma. □

Proposition 7.13. *Let X be an affine algebraic set, $x \in X$. Then the map*

$$\psi : I(x) \longrightarrow T^*_{X,x}, \quad f \mapsto d_x f$$

induces an isomorphism

$$I(x)/I(x)^2 \longrightarrow T^*_{X,x}.$$

Proof. To see that ψ is well-defined, we need to check that $\psi(f) = 0$ for any $f \in I(X)$. But this follows immediately from the definition of $T_{X,x}$ and the product rule $d_x(fg) = f(x)d_x g + g(x) d_x f$.

Once we know that ψ is well-defined, it is clear that it is surjective, because the linear maps given by restricting the coordinate functions t_i to $T_{X,x}$ lie in the image of ψ.

It remains to check that $\ker(\psi) = I(x)^2$. The inclusion $I(x)^2 \subset \ker(\psi)$ follows again from the product rule $d_x(fg) = f(x)d_x g + g(x)d_x f$. It remains to check $\ker(\psi) \subset I(x)^2$. For this, let $f \in \ker(\psi)$. That is, $f \in I(x)$ with $d_x f = 0$. We can write $f = L + g$, where l is a linear polynomial and $g \in I(x)^2$. Then $d_x f = d_x L = L$ vanishes by assumption. That is,

$$L \in I(V(d_x f_1, \dots, d_x f_r)) = (d_x f_1, \dots, d_x f_r)$$

where $I(X) = \langle f_1, \dots, f_r \rangle$. Hence,

$$L = \sum \lambda_j d_x f_j.$$

But then consider

$$L - \sum_j \lambda_j f_j.$$

This is a regular function on X which lies in $I(x)^2$ and so $L \in I(x)^2$, because $\sum_j \lambda_j f_j$ restricts to zero on X. Hence, $f \in I(x)^2$, which concludes the proof of the proposition. □

Definition 7.14. *Let X be a quasi-projective algebraic set, $x \in X$.*

(1) $T_{X,x} := T_{U,x}$ for any affine open neighbourhood $U \subset X$ of $x \in X$;

(2) $\mathcal{O}_{X,x} := \mathcal{O}_{U,x}$ for any affine open neighbourhood $U \subset X$ of $x \in X$;

(3) x is a smooth point of X if $\dim(T_{X,x}) = \dim_x X$; the set of smooth points on X is denoted by X^{sm};

(4) x is a singular point of X, if it is not a smooth point; the set of singular points on X is denoted by X^{sing}.

Note that the above definition of $T_{X,x}$ is well-defined up to isomorphism because of the intrinsic description of the tangent space given in Proposition 7.13 above. Similarly, $\mathcal{O}_{X,x}$ is well defined by the description of the local ring of an affine variety in Lemma 7.9 above.

Let us also recall the following fact from commutative algebra, see e.g. [1, Proposition 2.8].

Theorem 7.15 (Nakayama's Lemma)**.** *Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated A-module. If the images of $x_1, \dots, x_n \in M$ generate the $A/\mathfrak{m}$ vector space $M/\mathfrak{m}M$, then $x_1, \dots, x_n$ generate M.*

Corollary 7.16. *Let X be a quasi-projective algebraic set, $x \in X$. Then,*

$$\dim_x X \le \dim(T_{X,x}).$$

Proof. Let $n = \dim_k(T_{X,x})$. Then there are n regular functions $f_1, \dots, f_n \in I(x)$ such that the images of f_i in $I(x)/I(x)^2$ form a basis of

$$T^*_{X,x} \cong I(x)/I(x)^2.$$

By Nakayama's lemma, the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$ is generated by $f_1, \dots, f_n$. Using Krull's Hauptidealsatz (which applies to non-units in integral domains), we find

$$\dim(\mathcal{O}_{X,x}) = 1 + \dim(\mathcal{O}_{X,x}/(f_1)).$$

The right hand side is the local ring of $Z := V_X(f_1) \subset X$ at x. Moreover, $d_x f_2, \dots, d_x f_n$ generate $T_{Z,x} = T_{X,x} \cap V(d_x f_1)$. By induction on n, we may thus assume that

$$\dim Z \le \dim(T_{Z,x}) = n - 1.$$

By Krulls Hauptidealsatz, $\dim X = 1 + \dim Z$, and so $\dim X \le \dim(T_{X,x})$, as we want. □

Definition 7.17. *A local ring A with maximal ideal $\mathfrak{m}$ is regular if $\mathfrak{m}$ can be generated by* $\dim A$ *many elements.*

Fact from algebra: A regular local ring is a UFD.

Corollary 7.18. *Let X be a quasi-projective variety. Then $x \in X$ is smooth if and only if the local ring $\mathcal{O}_{X,x}$ is regular.*

Proof. After shrinking we may assume that X is affine. The corollary then follows from Nakayama's lemma and Proposition 7.13 above, as in the proof of the Corollary above. More precisely, if $\mathcal{O}_{X,x}$ is regular, then the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ can be generated by $\dim X$ many elements $f_1, \dots, f_n$, then the images of $f_1, \dots, f_n$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$ generate the cotangent space $T^*_{X,x}$ and so $\dim T_{X,x} \le n$ and we see that equality holds by the above corollary. Conversely, if $\mathfrak{m}_x/\mathfrak{m}_x^2$ has dimension $n = \dim X$, then let $f_1, \dots, f_n \in \mathfrak{m}_x$ be elements whose images generate that vector space. By Nakayama's lemma, $\mathfrak{m}_x = (f_1, \dots, f_n)$ and so $\mathcal{O}_{X,x}$ is regular. □

Remark 7.19. *Next semester, we will define a concept of smoothness that works in greater generality, and in particular also over fields that are not algebraically closed. It is however important to note that this generalization will not be equivalent to the local rings being regular, i.e. the above corollary will not hold anylonger. In fact, smoothness will be a stronger condition than asking that all local rings are regular. However, after all, the two notions will differ only over non-perfect fields.*

7.3 Varieties are generically smooth

Theorem 7.20. *Let X be a quasi-projective variety. Then*

$$X^{sm} := \{x \in X \mid x \text{ is a smooth point of } X\}$$

is an open dense subset of X.

Proof. Since the problem is local on X, we may assume that $X \subset \mathbb{A}^n$ is affine. Since X is birational to a hypersurface $V(f) \subset \mathbb{A}^{d+1}$ with f irreducible and $d = \dim X$, we deduce from Lemma 7.12 that $X^{sm} \neq \emptyset$.

Let $T_X := \{(x, v) \in X \times \mathbb{A}^n \mid v \in T_{X,x}\}$. This is an algebraic set, as it is cut out by the equations

$$d_x f = \sum_i^n \frac{\partial f}{\partial t_i}(x)(t_i - x_i),$$

where f runs through generators of $I(X)$. Projection to the first factor yields a regular map

$$p : T_X \to X,$$

whose fibre above $x \in X$ is $T_{X,x}$. If T_X was irreducible, then the theorem would immediately follow from the fact that

$$\{x \in X \mid \dim(p^{-1}(x)) \geq \dim X + 1\}$$

is closed in X. However, T_X will not be irreducible in general (think about a curve X with a singular point). Nonetheless, the argument is very similar to this original idea and reduces the problem to the theorem on the dimension of fibres.

Let Z be an irreducible component of $\overline{X^{sing}} \subset X$. We have to show that every $z \in Z$ is a singular point of X.

Lemma 7.21. *There is an irreducible component W of $p^{-1}(Z)$ such that $T_{X,x} \subset W$ for a dense set of points $x \in Z$ with $\dim(T_{X,x}) < \dim X$. Moreover, $p|_W : W \to Z$ is surjective.*

Proof. Let $W_1, \ldots, W_r$ be the irreducible components of $p^{-1}(Z) \subset T_X$. Let further $A_i := \{x \in Z \mid T_{X,x} \subset W_i\}$. Since $T_{X,x}$ is irreducible for all x, it must be contained in at least one W_i and so

$$Z = \bigcup A_i.$$

Since Z is an irreducible component of $\overline{X^{sing}}$, we know that it contains a dense set $S \subset Z$ of points $x \in Z$ with $\dim(T_{X,x}) > \dim X$. Hence,

$$S = \bigcup (A_i \cap S).$$

Taking closures, we get

$$Z = \overline{S} = \bigcup \overline{(A_i \cap S)}.$$

Since Z is irreducible, $Z = \overline{(A_i \cap S)}$ for some i. This proves that there is a component $W = W_i$ which has the first property claimed in the lemma.

To prove that $p(W) = Z$, it suffice to show that $Z \times \{0\} \subset W$. To prove this, note that

$$Z \times \{0\} \cap W$$

is closed, as it is given by intersecting two closed subsets of $X \times \mathbb{A}^n$ and it contains a dense set of $Z \times \{0\}$, as it contains any $x \times \{0\}$ with the property that $T_{X,x} \subset W$. Hence, $Z \times \{0\} \cap W$ and so $p(W) = Z$, as we want. This proves the lemma. □

Let $W \subset p^{-1}(Z)$ be the component of the lemma. Consider

$$p : W \to Z.$$

This is a surjective regular map between quasi-projective varieties and so we know that

$$\dim W_z \geq \dim W - \dim Z$$

for all $z \in Z$ and equality holds over some non-empty open subset $V \subset Z$. By construction, V contains a point $z \in V$ such that $W_z = T_{X,z}$ has dimension greater than $\dim X$. Hence,

$$\dim W - \dim Z > \dim X$$

and we deduce that for all $x \in X$,

$$\dim(T_{X,x}) \geq \dim W_x \geq \dim W - \dim Z > \dim X.$$

Hence, $Z \subset X^{sing}$ and the theorem is proven. □

Example 7.22. *Let $f = \sum_{i=0}^n x_i^d \in k[x_0, \ldots, x_{n+1}]$. Then the hypersurface $X := V(f) \subset \mathbb{P}^{n+1}$ is smooth of dimension n, provided that the characteristic of k does not divide d.*

Proof. Exercise. □

Example 7.23. *Let $f = \sum_{i=0}^n x_i^d \in k[x_1, \ldots, x_{n+1}]$ and let $d \geq 1$ be an integer which is coprime to the characteristic of k. Consider the hypersurface $X := V(f) \subset \mathbb{A}^{n+1}$. Then $X \setminus \{0\}$ is smooth of dimension n, provided that the characteristic of k does not divide d. Consider the blow-up $\tau : Bl_0\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ and let $X' := \overline{\tau^{-1}(X \setminus \{0\})}$. Then X' is smooth and birational to X.*

Proof. Exercise. □

Example 7.24 (Nodal singularity)**.** *Let $X = V(x_2^2 - x_1^3 - x_1^2)$ and assume that the characteristic of k is different from 2 and 3. Then $X^{sing} = \{(0, 0)\}$.*

Proof. Note that $x_2^2 = x_1^2(x_1 - 1)$. Use this to draw a picture of X.

Let $f = x_2^2 - x_1^3 - x_1^2$. A point $p \in X$ is singular if and oly if $\frac{\partial f}{\partial x_i}(p) = 0$ for $i = 1, 2$. We have

$$\frac{\partial f}{\partial x_1} = -3x_1^2 - 2x_1$$

and

$$\frac{\partial f}{\partial x_2} = 2x_2.$$

Since we assume that k has characteristic different from 2 and 3, $p = (p_1, p_2) \in X^{sing}$ implies $p_2 = 0$ and $p_1(3p_1 + 2) = 0$. Since $p = (0, 0) \in X$, we deduce that $(0, 0) \in X^{sing}$. The only other possibility is $p = (0, \frac{-2}{3})$, but this point satisfies

$$f(0, \frac{-2}{3}) = \frac{2^3 - 2^2 3}{3^3} = \frac{-4}{3^3} \neq 0$$

because the characteristic is different from 2 and 3. This proves $X^{sing} = \{(0, 0)\}$, as claimed. □

Example 7.25 (Elliptic curve)**.** *Let $X := V(f) \subset \mathbb{P}^2$ be a plane curve where f is irreducible of degree 3. If X is smooth, then it is called an elliptic curve. An explicit example is given for instance by $f = x_0x_2^2 - x_1^3 - x_0^3$, or by $f = x_0^3 + x_1^3 + x_2^3$ as in one of the previous examples.*

Example 7.26 (The locus of smooth hypersurfaces is open and dense among all hypersurfaces)**.** *Let $S_d \subset k[x_0, x_1, \dots, x_{n+1}]$ denote the vector space of homogeneous polynomials of degree d. Then the projectivization $\mathbb{P}(S_d)$ is a natural parameter space for all hypersurfaces $X \subset \mathbb{P}^{n+1}$, i.e. for any point $[f] \in \mathbb{P}(S_d)$, we get a hypersurface $X = V(f) \subset \mathbb{P}^{n+1}$. The locus of those points $[f] \in \mathbb{P}(S_d)$ such that $V(f)$ is singular is closed, see theorem below. To show that the complement of this locus is open and dense, it thus suffices to find one hypersurface of given degree that is smooth. We have done so already in the case where d and the characteristic of k are coprime, but the statement is true in general.*

Theorem 7.27. *Let n, d be positive integers and let $S_d \subset k[x_0, x_1, \dots, x_{n+1}]$ denote the vector space of homogeneous polynomials of degree d. The locus those points $[f] \in \mathbb{P}(S_d)$ such that $V(f)$ is singular is closed in $\mathbb{P}(S_d)$.*

Proof. Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}(S_d)$ be the universal hypersurface, given by the equation

$$\mathcal{X} = V(\sum_I a_I x^I) \subset \mathbb{P}^{n+1} \times \mathbb{P}(S_d),$$

where $x_0, \dots, x_{n+1}$ denote the coordinates on $\mathbb{P}^{n+1}$, for any partition $I = (i_0, \dots, i_{n+1})$ of d by non-negative integers, we write $x^I = \prod_l x_l^{i_l}$ and the a_I denote the homogeneous coordinates on $\mathbb{P}(S_d)$. By construction, the fibre of the projection $p : \mathcal{X} \to \mathbb{P}(S_d)$ above $[f]$ is given by $V(f) \subset \mathbb{P}^{n+1}$.

For $j = 0, \dots, n + 1$, we may consider the hypersurface

$$\mathcal{X}_j := V(\sum_I a_I \frac{\partial x^I}{\partial x_j}) \subset \mathbb{P}^{n+1} \times \mathbb{P}(S_d).$$

The intersection

$$\mathcal{X}^{sing} := \mathcal{X} \cap \bigcap_{j=0}^{n+1} \mathcal{X}_j$$

is closed subset of $\mathbb{P}^{n+1} \times \mathbb{P}(S_d)$ and so it is a projective algebraic set. It follows that the image of $\mathcal{X}^{sing}$ under the second projection

$$\mathrm{pr}_2 : \mathbb{P}^{n+1} \times \mathbb{P}(S_d) \to \mathbb{P}(S_d)$$

is closed. This proves the theorem, because $V(f) \subset \mathbb{P}^{n+1}$ is singular if and only if $[f] \in \mathrm{pr}_2(\mathcal{X}^{sing})$. □

Remark 7.28. *One can show that $\mathcal{X}^{sing}$ has codimension $n+2$ in $\mathbb{P}^{n+1} \times \mathbb{P}(S_d)$, even though it is cut out by $n+3$ equations. The point is that we have Euler's formula, which tells us that*

$$d \cdot f = \sum_{j=0}^{n+1} x_j \cdot \frac{\partial f}{\partial x_j}(x)$$

and so $\mathcal{X} \subset \bigcap_{j=0}^{n+1} \mathcal{X}_j$.

8 Finite maps and normal varieties

Definition 8.1. *Let X, Y be quasi-projective algebraic sets. A regular map*

$$f : X \to Y$$

is finite, if for all $y \in Y$ there is some affine open neighbourhood $V \subset Y$ of y, such that $U := f^{-1}(V)$ is affine and $f|_U : U \to V$ induces a finite map of rings $f^* : k[V] \to k[U]$, i.e. the pullback map f^* makes $k[U]$ a finitely generated $k[V]$-module.

Example 8.2. *If X, Y are irreducible and $f : X \to Y$ is surjective and finite, then $\dim X = \dim Y$. Indeed, let $y \in Y$ and let $x \in f^{-1}(y)$ be a preimage of y. Let $V \subset Y$ be an affine open neighbourhood of y, such that $U := \mathrm{pr}_1^{-1}(V)$ is affine and $k[U]$ is a finite $k[V]$ module. Since f is surjective $k[V] \hookrightarrow k[U]$ is a subring. Since $k[U]$ is a finite $k[V]$-module, $k(U)$ is a finite field extension of $k(V)$ and so both fields have the same transcendence degree over k. Hence $\dim X = \dim U = \dim V = \dim Y$, as claimed.*

Example 8.3. *The projection $\mathrm{pr}_1 : \mathbb{A}^2 \to \mathbb{A}^1$ is not finite.*

Example 8.4. *Let ℓ be a positive integer, then*

$$f : \mathbb{P}^1 \to \mathbb{P}^1, \quad [x_0 : x_1] \mapsto [x_0^\ell : x_1^\ell]$$

is finite.

Proof. Covering $\mathbb{P}^1$ with the standard open subsets, we see that it suffices to show that $\mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^\ell$ is finite. This is clear, because $k[t]$ is a finite $k[t^\ell]$-module. □

Definition 8.5. *Let R be an integral domain, $K := \mathrm{Frac}\, R$ the fraction field of R. We say that R is normal, if it is integrally closed in K, i.e. if $f \in R[t]$ is a monic polynomial and $\alpha \in K$ is a zero of that polynomial, then $\alpha \in R$.*

Example 8.6. *The ring $\mathbb{Z}[\sqrt{5}]$ is not integrally closed, because the monic polynomial $x^2 - x - 1$ has the zero $\alpha = \frac{1+\sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5})$ which does not lie in $\mathbb{Z}[\sqrt{5}]$.*

For an integral domain R, we have the following facts from commutative algebra.

- R is normal $\Leftrightarrow$ $R_\mathfrak{p}$ is normal $\forall$ primes $\mathfrak{p} \subset R$ $\Leftrightarrow$ $R_\mathfrak{m}$ is normal $\forall$ maximal ideals $\mathfrak{m} \subset R$.
- If R is a UFD (e.g. a regular ring), then it is normal. (easy!)
- If $(R, \mathfrak{m})$ is a local ring of dimension one which is noetherian, then R is normal if and only if R is a DVR, i.e. if and only if it is a principal ideal domain which has exactly one non-zero maximal ideal $\mathfrak{m}$ (in particular $\mathfrak{m} = (\pi)$ is principal).

Definition 8.7. *A quasi-projective variety X is normal, if $\mathcal{O}_{X,x}$ is normal for all $x \in X$.*

Example 8.8. *If X is smooth, then it is normal.*

Theorem 8.9. *Let X be a quasi-projective variety. If X is normal, then*

$$\dim X^{sing} \leq \dim X - 2.$$

Proof. W.l.o.g. X is affine and $n = \dim X$. For a contradiction, suppose there is an irreducible component $Z \subset X^{sing}$ of dimension $n-1$.

Claim 2. *After shrinking X, we may assume that the maximal ideal of $\mathcal{O}_{X,Z}$ is generated by a regular function $u \in k[X]$.*

Proof. Let $\mathfrak{p} := I(Z) \subset K[X]$. Since X is normal, the local ring $\mathcal{O}_{X,Z} = k[X]_{\mathfrak{p}}$ is a normal of dimension one, hence a DVR. That is, the maximal ideal $\mathfrak{p} \cdot k[X]_{\mathfrak{p}}$ is generated by one element $\frac{f}{g} \in k[X]_{\mathfrak{p}}$. Here, g does not vanish identically along Z. Replacing X by the complement of $V_X(g)$, we may then assume that there is an element $u \in k[X]$ whose image in $k[X]_{\mathfrak{p}}$ generates $\mathfrak{p} \cdot k[X]_{\mathfrak{p}}$. □

Note that the variety Z is generically smooth and so there is a smooth point $x \in Z^{sm}$. Hence, the maximal ideal of $\mathcal{O}_{Z,x}$ is generated by $n-1 = \dim Z$ many elements $\overline{g}_1, \dots, \overline{g}_{n-1} \in \mathcal{O}_{Z,x}$. Since X is affine, the restriction map $k[X] \to k[Z]$ is onto and induces a surjection on local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{Z.x}$. We may thus choose elements $g_i \in \mathcal{O}_{X,x}$ which maps to $\overline{g}_i \in \mathcal{O}_{Z,x}$. It then follows that the maximal ideal of $\mathcal{O}_{X,x}$ is generated by

$$g_1, \dots, g_{n-1}, u$$

and so $\mathcal{O}_{X,x}$ is a regular local ring. Hence, $x \in X^{sm}$, as we want. □

Definition 8.10. *Let X and X' be quasi-projective varieties. A regular map $f : X' \to X$ is a normalization, if X' is normal and f is birational and finite.*

Theorem 8.11. *Let X be an affine variety. Then there is a unique normalization $f : X' \to X$.*

Proof. Let $R = k[X]$ and let $S \subset k(X)$ be the integral closure of $R \subset k(X)$. Then $\operatorname{Frac} S = \operatorname{Frac} R = k(X)$. Moreover, S is a finitely generated R-module and so it is a finitely generated k-algebra, since R is. That is,

$$S \cong k[x_1, \dots, x_n]/I$$

for some ideal I. In particular, there is an affine variety X' with $k[X'] = S$. There is a regular map $f : X' \to X$ such that $f^* : k[X] \to k[X']$ corresponds to the inclusion $R \subset S$. Hence, f is finite and birational and X' is normal, as we want. □

Proposition 8.12. *Let $f : X \dashrightarrow Y$ be a rational map from a normal variety X to a projective variety $Y \subset \mathbb{P}^m$. Then, f is defined in codimension one. That is, $\dim(X \setminus \operatorname{dom}(f)) \leq \dim X - 2$.*

Proof. We know that $\operatorname{dom}(f) \subset X$ is open. Let Z be a component of $X \setminus \operatorname{dom}(f)$. We need to show that $\dim Z \leq \dim X - 2$. For a contradiction, we assume that this is not the case. Then $\dim Z = 1$.

Since $Y \subset \mathbb{P}^m$ is closed, the rational map f is given by

$$f = [f_0 : \cdots : f_m]$$

where $f_i \in k(X)$ are rational function ons X. Since X is normal, the local ring $\mathcal{O}_{X,Z}$ is normal of dimension one, hence a DVR with maximal ideal $\mathfrak{m}_{X,Z} = (\pi)$. Moreover, $\mathcal{O}_{X,Z} \subset k(X)$

is the subring of rational functions on X that are defined at some point of Z. In particular, $\operatorname{Frac} \mathcal{O}_{X,Z} = k(X)$ and so we can write

$$f_i = \epsilon_i \cdot \pi^{a_i}$$

for some unit $\epsilon_i \in \mathcal{O}_{X,Z}$ and integers $a_i \in \mathbb{Z}$. Up to reordering, we may assume that $a_0 \leq a_1 \leq \cdots \leq a_n$. We then have

$$f = [f_0 : \cdots : f_m] = [f_0\pi^{-a_0} : \cdots : f_m\pi^{-a_0}] = [\epsilon_0 : \epsilon_1\pi^{a_1-a_0} : \ldots \epsilon_m\pi^{a_m-a_0}].$$

Since each ϵ_i and π are regular functions on some open subset which intersects Z nontrivially, and since ϵ_i does not vanish identically along Z, we see that f is regular at some point of Z (namely where each ϵ_i and π is defined and where ϵ_0 does not vanish). This contradicts the assumption that Z consists of points where f is not defined. □

Corollary 8.13. *Let X and Y be two projective normal (or smooth) curves. If X and Y are birational, then they are isomorphic.*

Proof. Let $f : X \dashrightarrow Y$ be a birational map with rational inverse $g : Y \dashrightarrow X$. Since X and Y are projective and normal, f and g are defined away from a subset of codimension two by the above proposition. Hence f and g are defined everywhere because X and Y are curves. Hence, f is a regular map with regular inverse g and so it is an isomorphism, as we want. □

9 Divisors, Class groups and Bezout's Theorem

9.1 Prime divisors and valuations

Definition 9.1. *Let X be a quasi-projective variety of dimension $n > 0$.*

(1) A prime divisor is an irreducible codimension one subvariety $D \subset X$.

(2) $\operatorname{Div} X :=$ free abelian group, freely generated by the prime divisors on X.

(3) $D \in \operatorname{Div} X$ is effective (denoted by $D \geq 0$) if $D = \sum_i a_i D_i$ with $a_i \geq 0$.

Recall the following important definition from commutative algebra.

Definition 9.2. *Let K/k be a field extension. A discrete valuation of K/k is a non-zero map*

$$\nu : K^* \to \mathbb{Z}$$

such that

- *$\nu(f) = 0$ if $f \in k$;*
- *$\nu(fg) = \nu(f) + \nu(g)$;*
- *$\nu(f + g) \geq \min(\nu(f), \nu(g))$.*

We have the following basic lemma.

Lemma 9.3. *In the above notation, the subring $\mathcal{O}_\nu := \{f \in K^* \mid \nu(f) \geq 0\} \cup \{0\}$ is a DVR, i.e. it is a principal ideal domain with exactly one non-zero maximal ideal $\mathfrak{m}_\nu := \{f \in K^* \mid \nu(f) > 0\} \cup \{0\}$. Conversely, if $R \subset K$ is a DVR with maximal ideal $\mathfrak{m} = (\pi)$, then there is a unique discrete valuation $\nu : K^* \to \mathbb{Z}$ whose valuation ring is R.*

Proof. The first claim, saying that $\mathcal{O}_\nu$ is DVR is an easy fact from commutative algebra (see e.g. [1]). It essentially boils down to the observation that an element in $\mathcal{O}_\nu \setminus \{0\}$ is invertible if and only if it has trivial valuation, and so $\mathcal{O}_\nu$ has a unique maximal ideal, which is zero if ν is the zero map and it is principal, generated by any element with minimal positive valuation, otherwise.

For the second claim, note first that any element in R can be uniquely written as $\epsilon \cdot \pi^r$ for some unit $\epsilon \in R$ and some $r \geq 0$. We may then define $\nu(\epsilon \cdot \pi^r) = r$. Since $\mathrm{Frac}(R) = K$, any $\varphi \in K$ can be written as $\varphi = \frac{f}{g}$ with $g \neq 0$ and $f, g \in R$ and so we can extend the above definition to arbitrary nonzero $\varphi = \frac{f}{g} \in K$ via

$$\nu\left(\frac{f}{g}\right) = \nu(f) - \nu(g).$$

It is clear from the definition, that $R \setminus \{0\}$ consists of exactly those elements of K^* which have non-negative valuation. Hence $R = \mathcal{O}_\nu$, as we want. This concludes the lemma. □

An important example of discrete valuations are given by divisors on normal varieties as follows.

Example 9.4. *Let X be a normal quasi-projective variety, D a prime divisor on X. Then there is a discrete valuation*

$$\nu_D : k(X)^* \to \mathbb{Z}$$

which measures the order of zeros or poles of a rational function generically along D.

Proof. To begin with, let us pick an open affine subset $U \subset X$ which intersects D. Then $\mathcal{O}_{X,D} = k[U]_{I(D\cap U)}$. Since X is normal, this local ring is normal of dimension one, hence it is a DVR. In particular, its maximal ideal $\mathfrak{m}_{X,D}$ is generated by a single element $\pi \in \mathcal{O}_{X,D}$. If $f \in \mathcal{O}_{X,D}$, i.e. f is a regular function on some open subset of X which meets D, then we have $f = \epsilon \cdot \pi^r$ for some unit $\epsilon \in \mathcal{O}_{X,D}$ and some $r \geq 0$. We then define

$$\nu_D(f) = \nu_D(\epsilon \cdot \pi^r) = r$$

Recall that $\mathcal{O}_{X,D} = k[U]_{I(D\cap U)}$ and so

$$\mathrm{Frac}(\mathcal{O}_{X,D}) = \mathrm{Frac}(k[U]_{I(D\cap U)}) = k(U) = k(X).$$

That is, any rational function $\varphi \in k(X)$ can be written as

$$\varphi = \frac{f}{g}$$

with $f, g \in \mathcal{O}_{X,D}$, $g \neq 0$ and we define

$$\nu_D(\varphi) = \nu_D\left(\frac{f}{g}\right) = \nu_D(f) - \nu_D(g).$$

That is, ν_D measures the order of poles or zero of a rational function (generically) along D. It is clear from the definition, that

$$\nu_D : k(X)^* \to \mathbb{Z}$$

is a discrete valuation with valuation ring $\mathcal{O}_{X,D} \subset k(X)$. Since $\mathcal{O}_{X,D}$ does not depend on the affine open subset $U \subset X$ which meets D from above, we see that ν_D does not depend on U. This concludes the proof. □

9.2 The divisor class group

Definition 9.5. *Let X be a normal quasi-projective variety. For $f \in k(X)^*$ we set*

$$\mathrm{Div}(f) := \sum_{D \subset X} \nu_D(f) D,$$

where $D \subset X$ runs through all prime divisors on X.

The next lemma shows that the above sum is finite; equivalently, $\mathrm{Div}(f) \in \mathrm{Div}(X)$.

Lemma 9.6. *Let X be a normal quasi-projective variety. For any $f \in k(X)^*$, we have $\mathrm{Div}(f) \in \mathrm{Div}(X)$.*

Proof. We need to show that the sum in the definition of $\mathrm{Div}(f)$ is finite. Since X can be covered by finitely many affine varieties, it suffices to treat the case where X is affine. Then $f = \frac{g}{h}$ for regular functions $g, h \in k[X]$ with $h \neq 0$. The algebraic set $V_X(g) \cup V_X(h)$ has finitely many irreducible components $D_1, \dots, D_r$. We have shown earlier that any component of an algebraic subset that is cut out by one equation has codimension one and so each D_i is a prime divisor on X. Moreover, since $\nu_D(f)$ measures the zeros, resp. poles of f along D, we find that it vanishes for all $D \neq D_i$ and so $\mathrm{Div}(f)$ is a finite sum, as we want. This concludes the proof of the lemma. □

The discussion so far allows us to define the following very important invariant of quasi-projective varieties.

Definition 9.7. *Let X be a quasi-projective variety. The class group of X is defined by*

$$\mathrm{Cl}(X) := \mathrm{Div}(X)/\sim,$$

where $D_1 \sim D_2 \Leftrightarrow$ there is some rational function $f \in k(X)^*$ with $\mathrm{Div}(f) = D_1 - D_2$.

Note that $\mathrm{Cl}(X)$ is an abelian group which often allows us to distinguish non-isomorphic varieties. Indeed, if $f : X \to Y$ is an isomorphism, then it maps codimension one subvarieties on X to those on Y and we get an isomorphism

$$f_* : \mathrm{Div}(X) \xrightarrow{\sim} \mathrm{Div}(Y), \quad D \mapsto f_* D$$

which respects the above equivalence relation and so it descends to an isomorphism

$$f_* : \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(Y).$$

More generally, for any regular map $f : X \to Y$ beweteen quasi-projective varieties, we may define $f_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ and this will descend to class groups. By linearity, it suffices to define f_* on a prime divisor $D \subset X$. Here we set $f_* D = 0$ if $f(D) \subset Y$ is not of codimension one and

$$f_* D = \deg(D \to f(D)) \cdot f(D)$$

otherwise, where $\deg(D \to f(D))$ denotes the degree of the field extension $k(f(D)) \subset k(D)$.

Example 9.8. $\mathrm{Cl}(\mathbb{A}^n) = 0$.

Proof. Let $D \subset \mathbb{A}^n$ be a prime divisor. Choose an irreducible element $f \in I(D) \subset k[t_1, \dots, t_n]$ (possible because $I(D)$ is prime). Then $D \subset V(f)$, but $V(f)$ is irreducible because

$$k[t_1, \dots, t_n]/(f)$$

is a domain and so $D = V(f)$. Hence, $D \sim 0$. This proves $\mathrm{Cl}(\mathbb{A}^n) = 0$. □

Example 9.9. *Let $X \subset \mathbb{A}^n$ be an affine variety such that $k[X]$ is a UFD. Then* $\mathrm{Cl}(X) = 0$.

Proof. Same proof as above. □

Remark 9.10. *Let $X \subset \mathbb{A}^n$ be an affine variety. If X is smooth, then $\mathcal{O}_{X,x}$ is regular, hence a UFD for all $x \in X$, but this does not imply that $k[X]$ is a UFD. In particular,* $\mathrm{Cl}(X)$ *might be large. An explicit example is given by any affine piece of a smooth elliptic curve, e.g. $X = V(x_2^2 - (x_1+1)x_1(x_1-1)) \subset \mathbb{A}^2$. (We will probably prove this later.)*

Lemma 9.11. *Let X be a normal quasi-projective variety, $f \in k(X)$. Then f is regular on X if and only if* $\mathrm{Div}(f) \geq 0$.

Proof. Since both assertions are local, we may wlog assume that X is affine. If f is regular, then $\mathrm{Div}(f) \geq 0$ is clear as f does not have any poles. Conversely, suppose that $\mathrm{Div}(f) \geq 0$. Then $\nu_D(f) \geq 0$ for all $D \subset X$. Hence, $f \in k(X)$ lies in the localization $k[X]_{I(D)}$ for all prime divisors $D \subset X$. But the prime divisors on X are via $D \mapsto I(D)$ ins one to one correspondence to the prime ideals of height one in $k[X]$ and so

$$f \in \bigcap_{\mathfrak{p}} k[X]_{\mathfrak{p}},$$

where the intersection runs through all prime ideals of height one. It is a deep result from commutative algebra that the latter intersection coincides with $k[X]$, and so f is regular. □

Example 9.12. *Let $n \geq 1$. Then we have $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}[H]$, generated by the class of a hyperplane $H \subset \mathbb{P}^n$.*

Proof. Let $H := V(x_0) \subset \mathbb{P}^n$ be a hyperplane. Then $H \cong \mathbb{P}^{n-1}$ is irreducible of codimension one. Thus H is a prime divisor on $\mathbb{P}^n$.

We show now that the class of H generates $\mathrm{Cl}(\mathbb{P}^n)$. For this, let $D \subset \mathbb{P}^n$ be a prime divisor. Pick an irreducible homogeneous polynomial $F \in I(D) \subset k[x_0, \dots, x_n]$. One easily checks that $V(F) \subset \mathbb{P}^n$ is irreducible and contains D, hence $D = V(F)$. But then the rational function $\varphi = \frac{F}{x_0^{\deg F}}$ has divisor

$$\mathrm{Div}(\varphi) = D - \deg F \cdot H$$

and so $D \sim \deg F \cdot H$, as we want.

Next, we show that the class of H generates $\mathrm{Cl}(\mathbb{P}^n)$ freely. To see this, assume that $m \cdot H \sim 0$ for some $m \in \mathbb{Z}$. We need to prove that $m = 0$. For a quick proof, we may after possibly replacing φ by φ^{-1} assume that m is non-negative and so φ is regular by the previous lemma. But any regular function on $\mathbb{P}^n$ is constant and so $m = 0$.

We give a second proof, which does not rely on the above lemma (and hence not on the hard fact from commutative algebra, that we used there). By assumptions, there is a rational function $\varphi \in k(\mathbb{P}^n)$ with $\mathrm{Div}(\varphi) = mH$. We can find homogeneous polynomials F and G of the same degree such that $\varphi = \frac{F}{G}$. We may assume here that F and G have no factor in common and so we can write

$$F = \prod F_i^{a_i} \quad \text{and} \quad G = \prod G_j^{b_j}$$

where $a_i, b_j \geq 1$ and the F_i and G_j are irreducible homogeneous polynomials which are mutually coprime. Hence,

$$\mathrm{Div}(\varphi) = \mathrm{Div}\left(\frac{\prod F_i^{a_i}}{\prod G_j^{b_j}}\right) = \sum_i a_i V(F_i) - \sum_j b_j V(G_j).$$

Here, $V(F_i)$ and $V(G_j)$ are prime divisors which are mutually distinct because the F_i and G_j are mutually coprime. Since $\mathrm{Div}(\varphi) = mH$, we find that

$$mH = \sum_i a_i V(F_i) - \sum_j b_j V(G_j).$$

This is an equality in $\mathrm{Div}(\mathbb{P}^n)$ and so we conclude $b_j = 0$ for all j. Hence, $\deg G = 0$. But then also $\deg F = 0$ and so φ is constant. Since $\mathrm{Div}(\varphi) = mH$, we conclude $m = 0$, as we want. □

9.3 Bezout's theorem

Let X be a smooth projective curve, i.e. a smooth projective variety of dimension one. A divisor D on X is nothing but a $\mathbb{Z}$-linear combination of points on X:

$$D = \sum_i a_i [x_i],$$

with $a_i \in \mathbb{Z}$ and $x_i \in X$. We define the degree of D as

$$\deg D = \sum a_i.$$

This yields a group homomorphism

$$\deg : \mathrm{Div}\, X \longrightarrow \mathbb{Z}.$$

Theorem 9.13. *Let X be a smooth projective curve and let $\varphi \in k(X)$ be a rational function on X. Then $\deg(\mathrm{Div}(\varphi)) = 0$. In particular, the degree of a divisor induces a well-defined homomorphism*

$$\deg : \mathrm{Cl}(X) \longrightarrow \mathbb{Z}.$$

Proof. The rational function φ corresponds to a rational map

$$\varphi : X \dashrightarrow \mathbb{P}^1.$$

Since X is normal, φ is defined in codimension one (exercise!) and so it is a morphism $\varphi : X \to \mathbb{P}^1$. We may assume that φ is non-constant and so it is surjective, because its image is closed in $\mathbb{P}^1$ since X is projective. We have the following fact that we will use without proof.

Fact 1. *Let $f : X \to Y$ be a surjective regular map between projective curves. For $y \in Y$ let $f^{-1}(y) = \{x_1, \dots, x_r\}$ be its preimages. Let $\pi \in \mathcal{O}_{Y,y}$ be a local parameter, i.e. a generator of $\mathfrak{m}_{Y,y}$. For each i, consider $f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x_i}$ and let*

$$a_i := \nu(f^*\pi)$$

be the vanishing order of $f^*\pi$ at x_i. Then,

$$\sum_{i=1}^{r} a_i = \deg([k(X) : k(Y)])$$

does not depend on $y \in Y$.

The fact implies that $\varphi^{-1}([0:1])$ and $\varphi^{-1}([1:0])$ have the same degree, where each point in the preimage is counted with the correct multiplicity. In particular,

$$\deg(\mathrm{Div}(\varphi)) = 0,$$

as we want. □

We aim to apply the above theorem to prove a version of Bezout's theorem. For this, let $F, G \in k[x_0, x_1, x_2]$ be non-constant irreducible (or more general, square-free, which means that in the decomposition of F and G into powers of irreducible factors, each irreducible factor appears with exponent one) homogeneous polynomials, and consider the corresponding plane curves

$$X := V(F) \subset \mathbb{P}^2 \quad \text{and} \quad Y := V(G) \subset \mathbb{P}^2.$$

We assume that X and Y have no component in common and aim to compute the number of intersection points $X \cap Y$, counted with the correct multiplicities. For simplicity, we assume that X is smooth (for the general case, one may pass to the normalization of X which is a smooth projective model of X). We then define $\sharp(X \cap Y)$ as follows. Let $E \in k[x_0, x_1, x_2]$ be a homogeneous polynomial of degree $\deg G$, such that

$$V_{\mathbb{P}^2}(F, G, E) = \emptyset.$$

Then

$$f := \frac{G}{E}|_X \in k(X)$$

is a rational function whose divisor of zeros and poles

$$\mathrm{Div}(f) = D - D'$$

with $D, D' \geq 0$ and such that D and D' have no point in common, has the property that D does not depend on E. We may then define

$$\sharp(X \cap Y) = \deg D.$$

Theorem 9.14 (Bezout's Theorem)**.** *In the above notation*

$$\sharp(X \cap Y) = \deg F \cdot \deg G.$$

Proof. Let $L \in k[x_0, x_1, x_2]$ be a linear homogeneous polynomial such that $V(L)$ is not tangent to X at any $x \in X$. (This is possible, because the lines in $\mathbb{P}^2$ are parametrized by $\mathbb{P}(k[x_0, x_1, x_2]_{(1)}) \cong \mathbb{P}^2$, while the lines that are tangent to X correspond to the image of the regular map

$$X \to \mathbb{P}(k[x_0, x_1, x_2]_{(1)}), \quad x \mapsto d_x F$$

and so they form a subset of dimension at most one of $\mathbb{P}(k[x_0, x_1, x_2]_{(1)}) \cong \mathbb{P}^2$.) We may additionally assume that

$$V(F, G, L) = \emptyset.$$

Hence, in the definition of $\sharp(X \cap Y)$ we can take $E = L^{\deg G}$. Then,

$$f := \frac{G}{E}|_X \in k(X)$$

is a rational function on X and we can write

$$\mathrm{Div}(f) = D - D'$$

where $D, D' \geq 0$ are effective and have no points in common. By the previous theorem,

$$\sharp(X \cap Y) = \deg D = \deg D'.$$

Since L is not tangent to X at any point, we have for all $x \in X \cap V(L)$ that

$$d_x L : T_{X,x} \to k$$

is surjective. For $x \in X \cap V(L)$, we conclude that the image of L in $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is a generator and so L generates the maximal ideal $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ by Nakayama's lemma. Since $V(F, G, L) = \emptyset$, G does not vanish at $x \in X \cap V(L)$ and so

$$\frac{G}{L^{\deg G}} \in \mathrm{Frac}(\mathcal{O}_{X,x}),$$

where $G \in \mathcal{O}_{X,x}$ is a unit and $L \in \mathcal{O}_{X,x}$ is a uniformizer. Hence, $x \in X \cap V(L)$ appears in $\mathrm{Div}(f)$ with coefficient $-\deg G$ and so we conclude

$$\sharp(X \cap Y) = \deg D' = \deg G \cdot \sharp(V(L) \cap X),$$

where $\sharp(V(L) \cap X)$ denotes the number of intersection points of $V(L)$ and X. Since for all $x \in V(L) \cap X$, the image of L in $\mathcal{O}_{X,x}$ is a uniformizer, we find that L vanishes of order one at x. Equivalently, the homogeneous polynomial F which cuts out X vanishes of order one at $x \in V(L)$. That is, the restriction of F to $V(L) \cong \mathbb{P}^1$ is a polynomial of degree $\deg F$ without multiple zeros and so it has exactly $\deg F$ many zeros. That is,

$$\sharp(V(L) \cap X) = \deg F$$

and so

$$\sharp(X \cap Y) = \deg F \cdot \deg G,$$

as we want. □

10 Sheaves

Let X be a topological space.

Definition 10.1. *A pre-sheaf $\mathcal{F}$ of abelian groups on X consists of an abelian group $\mathcal{F}(U)$ for each open subset $U \subset X$ and a group homomorphism $\tau_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ for any nested open subsets $V \subset U$ of X, with the following properties:*

(a) $\mathcal{F}(\emptyset) = 0$,

(b) $\tau_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$;

(c) $\tau_{U,V} \circ \tau_{V,W} = \tau_{U,W}$ *for any open subsets* $W \subset V \subset U$ *of* M.

Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U. The $\tau_{U,V}$ are called restriction maps; one often writes $\tau_{U,V}(s) = s|_V$ for $s \in \mathcal{F}(U)$.

A pre-sheaf of vector spaces, rings, etc. is defined in an analogous way.

The most important examples of presheaves of abelian groups are functions on some space with values in an abelian group. Before we list some examples, note that a presheaf $\mathcal{F}$ of functions on some space satisfies the following properties:

For any open subset $U \subset X$ and any open covering $U = \bigcup_{i \in I} U_i$, all of the above examples have the following crucial properties:

(i) If $f, g \in \mathcal{F}(U)$ with $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$;

(ii) If $f_i \in \mathcal{F}(U_i)$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there is a unique section $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$.

Definition 10.2. *A pre-sheaf $\mathcal{F}$ of abelian groups (or vector spaces, rings, etc.) on X is called sheaf, if (i) and (ii) above hold.*

Examples.

(1) Let G be an abelian group.

(a) The (locally) constant presheaf $\underline{G}_X$ on X with values in G is given by

$$\underline{G}_X(U) = \{f : U \to G \mid f \text{ is locally constant}\}$$

for all open subsets $U \subset X$. This presheaf is a sheaf. Note that $\underline{G}_X(U) = G^r$, where r denotes the number of connected components of U.

(b) The naive constant presheaf $\underline{G}_X^{naiv}$ on X with values in G, given by

$$\underline{G}_X^{naiv}(U) = \{f : U \to G \mid f \text{ is constant}\}.$$

This presheaf is not a sheaf.

(c) The skyscraper presheaf $\underline{G}_{\{x\}}$ with values in G and supported on a point $x \in X$ is given by

$$\underline{G}_{\{x\}}(U) = \begin{cases} 0 & \text{if } x \notin U \\ G & \text{if } x \in U. \end{cases}$$

This is a sheaf of abelian groups on X.

(d) If G carries a topology, e.g. $G = \mathbb{R}$ with the euclidean topology, then the presheaf $\mathcal{C}^0_{X,G}$ of continuous functions on X with values in G is given by

$$\mathcal{C}^0_{X,G}(U) := \{f : U \to G \mid f \text{ is continuous}\}.$$

This is a sheaf of abelian groups on X.

(2) Let X be a smooth manifold, then the presheaf $\mathcal{C}^\infty_X$ of smooth real valued functions on X is given by

$$\mathcal{C}^\infty_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\}.$$

This presheaf is a sheaf of rings on X.

(3) Let X be a quasi-projective variety, then the presheaf $\mathcal{O}_X$ of regular functions on X is given by

$$\mathcal{O}_X(U) = k[U] = \{f : U \to k \mid f \text{ is regular}\}.$$

This presheaf is a sheaf of rings on X.

To understand sheaves locally at a point, it is important to consider their stalks, defined as follows.

Definition 10.3. *Let X be a topological space, $\mathcal{F}$ a presheaf of abelian groups on X and let $x \in X$ be a point. The stalk of $\mathcal{F}$ at x is defined as the direct limit*

$$\mathcal{F}_x := \lim_{x \in U} \mathcal{F}(U).$$

That is, elements in $\mathcal{F}_x$ are represented by pairs (U, s), where $U \subset X$ is an open neighbourhood of x and $s \in \mathcal{F}(U)$ is a section of $\mathcal{F}$ over U. Two pairs (U, s) and (U', s') represent the same element if there is an open subset $x \in V \subset U \cap U'$ such that $s|_V = s'|_V$.

Lemma 10.4. *The stalk $\mathcal{F}_x$ is an abelian group, with group law*

$$[(U,s)] + [(V,t)] = [(U \cap V, s|_{U\cap V} + t|_{U\cap V})]$$

Proof. We essentially only have to check well-definedness of the group law, which is easy. Indeed, once this is done, the neutral element is $[(U,0)]$ for any $x \in U \subset X$ and the inverse of $[(U,s)]$ is $[U,-s]$. □

Example 10.5. *Let X be a quasi-projective variety. Then the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at a point $x \in X$ is nothing but the local ring of X at x.*

Definition 10.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of abelian groups on a topological space X. A presheaf homomorphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of group homomorphisms $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ with $\varphi_U(s)|_V = \varphi_V(s|_V)$ for all $V \subset U$ and all $s \in \mathcal{F}(U)$. The homomorphism φ is an isomorphism if there is a presheaf homomorphism $\psi : \mathcal{G} \to \mathcal{F}$ with $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$.*

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then a sheaf homomorphism φ is a homomorphism of the underlying presheaves. More over, φ is an isomorphism of sheaves if it is an isomorphism of presheaves.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a homomorphism of presheaves, then there is an induced homomorphism of abelian groups

$$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x, \quad [(U,s)] \mapsto [(U, \varphi_U(s))].$$

Definition 10.7. *A sequence of homomorphisms of presheaves/sheaves*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact if for all $x \in X$ the induced sequence of stalks

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is an exact sequence of abelian groups. In particular, $\varphi : \mathcal{F} \to \mathcal{G}$ is injective/surjective if the induced homomorphism on all stalks is injective/surjective.

The most important difference between sheaves and presheaves is that sheaves are determined by their stalks, while presheaves are not. More precisely, there are homomorphisms of presheaves (as we will see later) that are isomorphisms on all stalks (i.e. they are injective and surjective), but which are not isomorphisms of presheaves. One of the main advantages of sheaves is that this does not happen for sheaves.

Lemma 10.8. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves. Then φ is an isomorphism if and only if it is injective and surjective.*

Proof. If φ is an isomorphism, then there is an inverse ψ of φ and so for each $x \in X$, ψ_x is an inverse of φ_x. Hence, φ_x is an isomorphism.

Let us conversely assume that φ is injective and surjective. Then $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism of abelian groups for all $x \in X$. We aim to show that for all open subsets $U \subset X$,

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is an isomorphism. For this we need to check injectivity and surjectivity and both statements are easy consequences of the sheaf axioms. This concludes the lemma. □

If $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a sheaf homomorphism, then the kernel $\ker(\varphi)$, defined via

$$\ker(\varphi)(U) = \ker(\varphi_U),$$

is a sheaf. The analogous definitions for the image and the cokernel do in general not give sheaves, but only presheaves. This makes it necessary to introduce sheafifications, which is a canonical way of passing from a presheaf to a sheaf without changing the corresponding stalks.

Proposition 10.9. *Let $\mathcal{F}$ be a presheaf on a topological space X. There is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ of presheaves, with the following universal property: for any sheaf $\mathcal{G}$ and for any morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism of sheaves $\psi : \mathcal{F}^+ \to \mathcal{G}$ with $\varphi = \psi \circ \theta$.*

The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism; it is called the sheafification of $\mathcal{F}$. Moreover, θ induces an isomorphism on stalks.

Proof. We construct $\mathcal{F}^+$ as follows. For any open subset $U \subset X$, let $\mathcal{F}^+(U)$ be the set of functions

$$s : U \to \bigcup_{x \in U} \mathcal{F}_x$$

with $s(x) \in \mathcal{F}_x$ such that for each $x \in U$ there is some open subset $x \in V \subset U$ and a section $t \in \mathcal{F}(V)$ with $s(y) = t_y$ for all $y \in V$.

One checks easily that $\mathcal{F}^+$ with the natural restriction maps is a sheaf and that the natural morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ has the properties claimed in the proposition. The uniqueness of $(\mathcal{F}^+, \theta)$ is a formal consequence of the uniqueness of ψ in the universal property of $(\mathcal{F}^+, \theta)$. □

Definition 10.10. *A subsheaf $\mathcal{F} \subset \mathcal{G}$ of a sheaf $\mathcal{G}$ is a sheaf such that for all open subsets $U \subset X$, $\mathcal{F}(U) \subset \mathcal{G}(U)$ is a subgroup and the restriction maps of $\mathcal{F}$ are given by restricting the restriction maps of $\mathcal{G}$ to these subgroups.*

Definition 10.11. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topological space X.*

(1) the kernel of φ is the subsheaf of $\mathcal{F}$, given by

$$\ker(\varphi)(U) = \ker(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U));$$

(2) the image of φ is the sheaf given as sheafification of the presheaf

$$U \mapsto \operatorname{im}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U));$$

(3) the cokernel of φ is the sheaf given as sheafification of the presheaf

$$U \mapsto \operatorname{coker}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U));$$

(4) if $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ and φ is the natural inclusion, then the quotient sheaf $\mathcal{G}/\mathcal{F}$ is given by $\operatorname{coker}(\varphi)$*, i.e. it is given as sheafification of*

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U).$$

Lemma 10.12. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topological space X.*

(1) φ is injective if and only if $\ker(\varphi) = 0$*;*

(2) φ is surjective if and only if $\mathrm{im}(\varphi) = 0$.

Proof. This is an immediate consequence of the fact that injectivity/surjectivity are defined on the level of stalks and sheafifications do not change stalks. □

Definition 10.13. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) Let $\mathcal{F}$ be a sheaf on X. The pushforward sheaf, or direct image sheaf $f_*\mathcal{F}$ is the sheaf which on an open subset $V \subset Y$ is given by

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)).$$

Note that this is well-defined because $f^{-1}(V) \subset X$ is open, as f is continuous. Note also that this definition yields indeed a sheaf.

(2) Let $\mathcal{G}$ be a sheaf on Y. The inverse image sheaf $f^{-1}\mathcal{G}$ is the sheaf on X associated to the presheaf

$$U \mapsto \lim_{V \supset f(U)} \mathcal{G}(V)$$

on X.

Examples.

(1) The stalks of $f^{-1}\mathcal{G}$ are given by

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}.$$

(2) If $f : Y \hookrightarrow X$ is an embedding (e.g. open or closed), then we write $\mathcal{G}|_Y := f^{-1}\mathcal{G}$. If f is an open embedding, then for all $U \subset Y$ open, we have $\mathcal{G}|_Y(U) = \mathcal{G}(U)$.

(3) If $Y = \{pt.\}$ is a single point, then $f_*\mathcal{F}$ is isomorphic to the constant sheaf with value $\mathcal{F}(X)$ on Y.

(4) If $\mathcal{G} \cong \underline{G}_Y$ is a constant sheaf, then $f^{-1}\mathcal{G} \cong \underline{G}_X$ is constant as well.

(5) If $\mathcal{F}$ is constant, then $f_*\mathcal{F}$ is not necessarily constant. For instance, let $f : \mathbb{R} \to \mathbb{R}$ be the map $t \mapsto t^2$ in the Eucledian topology. Then

$$(f_*\underline{\mathbb{Z}}_{\mathbb{R}})_t = \begin{cases} 0 & \text{if } t < 0; \\ \mathbb{Z} & \text{if } t = 0; \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } t > 0. \end{cases}$$

11 $\mathcal{O}_X$-modules: quasi-coherent, coherent, locally free

Definition 11.1. *Let X be a quasi-projective algebraic variety.*

(1) A sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is a sheaf of abelian groups so that for all open subsets $U \subset X$ the group $\mathcal{M}(U)$ has the structure of an $\mathcal{O}_X$-module which is compatible with the corresponding restriction morphisms, i.e. for any open subsets $V \subset U \subset X$ and any sections $s \in \mathcal{M}(U)$ and $f \in \mathcal{O}_X(U)$ we have

$$(f \cdot s)|_V = f|_V \cdot s|_V.$$

(2) A morphism $\varphi : \mathcal{M} \to \mathcal{M}'$ of $\mathcal{O}_X$-modules is a morphism of sheaves such that

$$\varphi_U : \mathcal{M}(U) \to \mathcal{M}'(U)$$

is a morphism of $\mathcal{O}_X(U)$-modules for all $U \subset X$ open.

(3) A $\mathcal{O}_X$-module is locally free of rank r if for any $x \in X$ there is an open neighbourhood $x \in U \subset X$ such that there is an isomorphism of $\mathcal{O}_U$-modules:

$$\mathcal{M}|_U \cong \mathcal{O}_U^{\oplus r}.$$

The most important example of a $\mathcal{O}_X$-module is constructed as follows.

Example 11.2. *Let X be an affine algebraic variety with ring of regular functions $R := k[X]$. Let M be an R-module. We define a sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules on X as follows.*
For any $x \in X$, consider the localization $M_{I(x)}$ of M at the maximal ideal $I(x) \subset R$. For $U \subset X$ open, let $\widetilde{M}(U)$ be the set of all functions

$$s : U \to \bigcup_{x \in U} M_{I(x)}$$

with $s(x) \in M_{I(x)}$ and such that for any $x \in U$ there is an element $m_x \in M$ and a function $f_x \in R$ with $f_x(x) \neq 0$ such that on some open neighbourhood $x \in V \subset U \setminus V_U(f_x)$ of x, we have $s(y) = \frac{m}{f}$ for all $y \in U$. Since each $M_{I(x)}$ is an $R_{I(x)}$-module, pointwise addition and scalar multiplication endows $\widetilde{M}$ with the structure of an $\mathcal{O}_X$-module.

Proposition 11.3. *Let X be an affine algebraic variety with ring of regular functions $R := k[X]$ and let M be an R-module.*

(a) The stalk of $\widetilde{M}$ at $x \in X$ is isomorphic to $M_{I(x)}$.

(b) Let $f \in R$ be a nonzero function and consider the open subset $U_f := X \setminus \{V_X(f)\}$. Then,

$$\widetilde{M}(U_f) \cong M_f$$

is the localization of M at the multiplicative system $S = \{f^n \mid n \in \mathbb{N}\}$.

(c) $\widetilde{M}(X) = M$.

Proof. For a reference, see the argument in [4, Chapter II, Proposition 2.2].
Item (c) is an immediate consequence of (b). Similarly, item (a) is a consequence of (b), because the open subsets of the form U_f form a basis in the topology and so it suffices to use them in the computation of the direct limit

$$\widetilde{M}_x = \lim_{x \in U \subset X} \widetilde{M}(U).$$

That is,

$$\widetilde{M}_x = \lim_{x \in U_f \subset X} \widetilde{M}(U_f) = \lim_{x \in U_f \subset X} M_f = \lim_{f \in R, f(x) \neq 0} M_f = M_{I(x)}.$$

It remains to prove item (b). Any element of M_f is of the form $\frac{m}{f^n}$ for some $m \in M$ and $n \in \mathbb{N}$. It follows immediately from the definition, that such an element gives rise to a section of $\widetilde{M}$ over U_f. We thus get a map

$$\varphi : M_f \to \widetilde{M}(U_f).$$

We claim that φ is injective. Indeed, suppose that $\varphi(\frac{m}{f^n}) = 0$. This means that

$$\frac{m}{f^n} = 0 \in M_{I(x)}$$

for all $x \in U$. But for $x \in U$, $M_{I(x)}$ is isomorphic to the localization of the R_f-module M_f at the maximal ideal $I(x)R_f$. That is, the morphism of R_f-modules

$$R_f \to M_f, \quad 1 \mapsto \frac{m}{f^n}$$

is zero when localized at any maximal ideal of R_f. But for a morphism of modules over a ring, being zero is a local property which can checked at all maximal ideals this implies that the above homomorphism is already zero, and so $\frac{m}{f^n} = 0$, as we want.

The hard part is to show that φ is surjective. Conversely, let $s \in \widetilde{M}(U_f)$. By assumptions, there is an open cover $U_f = \bigcup_{i \in I} V_i$ such that $s|_{V_i} = \frac{m_i}{g_i}$ for some $m_i \in M$ and $f_i \in R$ with $V_{V_i}(g_i) = \emptyset$. Shrinking V_i further, we may assume $V_i = U_{f_i} := X \setminus V(f_i)$ for some $f_i \in R$. Since g_i does not vanish on V_i, $V(g_i) \subset V(f_i)$ and so $f_i \in I(V(f_i)) \subset I(V(g_i)) = \sqrt{(g_i)}$. So up to replacing f_i by some power f_i^n, we get $f_i \in (g_i)$ and so $f_i = c_i g_i$ with $c_i \in R$. Replacing m_i by $c_i m_i$, we may thus assume $f_i = g_i$. That is, s is given on U_{f_i} by $\frac{m_i}{f_i}$. On overlaps $U_{f_i} \cap U_{f_j} = U_{f_i f_j}$, these elements need to coincide and so

$$\frac{m_i}{f_i} = \frac{m_j}{f_j} \in M_{f_i f_j}$$

by the injectivity proven above. The above equality means

$$(f_i f_j)^n (f_j m_i - f_i m_j) = 0$$

for some $n > 0$ which depends on i and j.

Since U is a noetherian topological space, we may also assume that the index set I is finite. hence, we can choose n in the above equality so large that it works for all i and j simultaneously. That is,

$$f_i^n f_j^{n+1} m_i - f_i^{n+1} f_j^n m_j = 0$$

for all i, j. Replacing f_i by f_i^{n+1} and m_i by $f_i^n m_i$, we do not change the elements $\frac{m_i}{f_i}$ and the above identity simplifies to

$$f_j m_i - f_i m_j = 0.$$

for all i, j. Since $U_f = \bigcup U_{f_i}$, we have that

$$f \in I(V(f)) = I(V(f_1, \ldots, f_r)) = \sqrt{(f_1, \ldots, f_r)}.$$

Up to replacing f by some power, we may thus assume that

$$f = \sum c_i f_i$$

for some $c_i \in R$. For

$$m := \sum c_i m_i,$$

we then have

$$f_j m = \sum_i c_i m_i f_j = \sum_i c_i f_i m_j = f m_j$$

and so

$$\frac{m}{f} = \frac{m_j}{f_j} \in M_{f_j}$$

This shows that the section $s \in \widetilde{M}(U)$ coincides with $\varphi(\frac{m}{f})$, as we want. □

Corollary 11.4. *In the notation of the previous proposition, the $\mathcal{O}_X$-module $\widetilde{R}$ is isomorphic to $\mathcal{O}_X$.*

Proof. There is a natural map of sheaves

$$\psi : \widetilde{R} \to \mathcal{O}_X,$$

which is an isomorphism because it is an isomorphism on all stalks, since $\mathcal{O}_{X,x} = R_{I(x)}$ by definition and $\widetilde{M}_x = R_{I(x)}$ by the previous proposition. □

Definition 11.5. *Let X be a quasi-projective variety. A $\mathcal{O}_X$-module $\mathcal{M}$ is quasi-coherent, if for each $x \in X$ there is an affine open neighbourhood $x \in U \subset X$, such that $\mathcal{M}|_U \cong \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M. If M is finitely generated, then $\mathcal{M}$ is called coherent.*

Examples

(1) $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module by the corollary above. It follows from this that any locally free $\mathcal{O}_X$-module is coherent. By the Exercise sheet 12, we thus see that the sheaf of regular sections of an algebraic vector bundle of rank r on X is a coherent $\mathcal{O}_X$-module for any $r \in \mathbb{N}$.

(2) Let $Y \subset X$ be a closed subvariety of a quasi-projective variety X. Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the subsheaf of regular functions on X that vanish along Y. Then $\mathcal{I}_Y$ is coherent, because if X is affine, then $\mathcal{I}_Y \cong \widetilde{I(Y)}$ is the coherent sheaf associated to the $k[X] = \mathcal{O}_X(X)$-module $I(Y) \subset k[X]$, which is finitely generated because $k[X]$ is noetherian.

We have the following technical result, whose proof we will have to skip, even though it is not hard, see [4, Chapter II, Proposition 5.4].

Proposition 11.6. *Let X be a quasi-projective variety and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module on X. Then for any open affine subset $U \subset X$, the natural map*

$$\widetilde{\mathcal{M}(U)} \to \mathcal{M}|_U$$

is an isomorphism of $\mathcal{O}_X$-modules.

With the help of the above result, we can easily see that quasi coherent $\mathcal{O}_X$-modules on affine varieties have a very pleasant behaviour.

Corollary 11.7. *Let X be an affine variety and let*

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then the induced sequence on global sections

$$0 \to \mathcal{M}_1(X) \to \mathcal{M}_2(X) \to \mathcal{M}_3(X) \to 0$$

is exact.

Proof. By the above proposition, $\mathcal{M}_i \cong \widetilde{M_i}$ for some $k[X]$-modules M_i. The exactness of the above sequence of $\mathcal{O}_X$-modules means that the corresponding sequence on stalks is exact. By Proposition 11.3, we have $(\mathcal{M}_i)_x \cong (\mathcal{M}_i(X))_{I(x)}$ for all $x \in X$. Hence, the sequence of $k[X]$-modules

$$0 \to \mathcal{M}_1(X) \to \mathcal{M}_2(X) \to \mathcal{M}_3(X) \to 0$$

becomes exact after localization at all maximal ideals of $k[X]$, and so it was already exact to begin with, because this is a local property, see [1, Chapter 3, page 40]. □

12 Differential forms

Let R be a k-algebra. We define the R-module $\Omega^1_{R/k}$ of Kähler differentials of R (over k) as quotient $\Omega^1_{R/k} = M/N$ of the free R-module M with basis given by symbols df where $f \in R$ by the submodule $N \subset M$ that is generated by the elements

$$d(f+g) - df - dg, \quad d(fg) - fdg - gdf, \quad da$$

for all $f, g \in R$ and $a \in k$.

Lemma 12.1. *If R is a finitely generated k-algebra, generated by elements $t_1, \dots, t_n \in R$, (e.g. $R = k[X]$ for an affine variety $X \subset \mathbb{A}^n$), then the symbols $dt_1, \dots, dt_n$ generate $\Omega^1_{R/k}$ as an R-module.*

Proof. By assumption, any element $f \in R$ can be expressed as polynomial in the t_i. The product rule shows that $df = \sum \frac{\partial f}{\partial t_i} dt_i$, where $\frac{\partial f}{\partial t_i}$ denotes the formal derivative of a polynomial with respect to the symbol t_i. This proves the lemma. □

Lemma 12.2. *Let $\omega = \sum f_i dg_i \in \Omega^1_{R/k}$. Then for all $x \in X$ the element $\omega(x) = \sum f_i(x) d_x g_i \in T^*_{X,x}$ is well-defined, i.e. independent of the representative $\sum f_i dg_i$ of ω.*

Proof. Clear. □

Lemma 12.3. *Let X be an affine variety with ring of regular functions $R = k[X]$ and let $f \in R$ be a nonzero regular function on X. Then there is a natural isomorphism of R_f-modules $(\Omega^1_{R/k})_f \cong \Omega^1_{R_f/k}$.*

Proof. There is a natural map

$$\varphi : (\Omega^1_{R/k})_f \longrightarrow \Omega^1_{R_f/k}, \quad \frac{\sum_i g_i dh_i}{f^m} \mapsto \sum_i \frac{g_i}{f^m} dh_i$$

of R_f-modules. This is surjective, because of the relation

$$d\frac{g}{f^n} = \frac{f^n dg + gdf^n}{f^{2n}}.$$

To prove injectivity, assume that

$$\sum_i \frac{g_i}{f^m} dh_i = 0 \in \Omega^1_{R_f/k}.$$

This means that the symbol $\sum_i \frac{g_i}{f^m} dh_i$ can be written as a linear combination of symbols

$$d(\frac{h}{f^m} + \frac{g}{f^n}) - d\frac{h}{f^m} - d\frac{g}{f^n}, \quad d(\frac{h}{f^m}\frac{g}{f^n}) - \frac{h}{f^m} d\frac{g}{f^n} - \frac{g}{f^n} d\frac{h}{f^m}, \quad da.$$

Using the above relation $d\frac{g}{f^n} = \frac{f^n dg + gdf^n}{f^{2n}}$, we find that $\sum_i \frac{g_i}{f^m} dh_i = 0$ in the localization of $\Omega^1_{R/k}$ at f, as we want. □

Corollary 12.4. *In the above notation, $(\Omega^1_{R/k})_{I(x)} \cong \Omega^1_{R_{I(x)}/k}$*

Proof. This is a formal consequence of the above lemma:

$$(\Omega^1_{R/k})_{I(x)} = \lim_{f\in R,\ f(x)\neq 0} (\Omega^1_{R/k})_f \cong \lim_{f\in R,\ f(x)\neq 0} \Omega^1_{R_f/k} = \Omega^1_{R_{I(x)}/k}.$$

□

Definition 12.5. *Let X be a quasi-projective variety. For $x \in X$, we put $\Omega^1_{X,x} := \Omega^1_{\mathcal{O}_{X,x}/k}$. We then define the sheaf Ω^1_X of regular differential forms on X via*

$$\Omega^1_X(U) = \{\omega : U \to \sqcup_{x\in U}\Omega^1_{X,x} \mid \omega(x) \in \Omega^1_{X,x}, \quad \textit{locally } \omega = \sum f_i dg_i\}.$$

Natural multiplication with regular functions turns Ω^1_X into an $\mathcal{O}_X$-module.

Proposition 12.6. *The $\mathcal{O}_X$-module Ω^1_X has the property that for any affine open subset $U \subset X$,*

$$\Omega^1_X|_U \cong \widetilde{\Omega^1_{k[U]/k}}.$$

In particular, Ω^1_X is a coherent $\mathcal{O}_X$-module.

Proof. W.l.o.g. $X = U$ is affine. Let $R = k[X]$. By the above corollary, we have $\Omega^1_{R_{I(x)}/k} = (\Omega^1_{R/k})_{I(x)}$. To prove the proposition, it now suffices to see that the local condition for the functions

$$s : U \to \sqcup_{x\in U}\Omega^1_{R_{I(x)}/k} = \sqcup_{x\in U}(\Omega^1_{R/k})_{I(x)}$$

that we used in the definition of Ω^1_X and $\widetilde{\Omega^1_{R/k}}$ coincide. To see the latter, we only have to note that if $\omega = \sum g_i df_i$, then $g_i = \frac{a_i}{b_i}$ and $f_i = \frac{c_i}{d_i}$ for regular functions $a_i, b_i, c_i, d_i \in R$ on X and the product rule shows that

$$\sum \frac{a_i}{b_i} d\frac{c_i}{d_i} = \sum \frac{a_i}{b_i} \cdot \frac{d_i dc_i + c_i dd_i}{d_i} = \frac{a_i d_i dc_i + a_i c_i dd_i}{b_i d_i}.$$

This sum of fractions can be rewritten as a single fraction of an element of $\Omega^1_{R/k}$ by a function on X that does not vanish locally at the given point, which is exactly the local condition that appeared in the definition of $\widetilde{\Omega^1_{R/k}}$. □

Example 12.7. $\Omega^1_{\mathbb{A}^n} \cong \mathcal{O}^{\oplus n}_{\mathbb{A}^n}$

Proof. Let $R = k[t_1, \ldots, t_n]$. Then $\Omega^1_{R/k}$ is a free R-module with basis $dt_1, \ldots, dt_n$. Hence the result, because $\Omega^1_{\mathbb{A}^n} = \widetilde{\Omega^1_{R/k}}$. □

Proposition 12.8. *Let X be a smooth quasi-projective variety of dimension n. Then Ω^1_X is a locally free $\mathcal{O}_X$-module of rank n on X.*

Proof. Pick a point $x_0 \in X$. We need to show that Ω^1_X is free in some neigbourhood of x_0. After shrinking X assume that it is affine. Hence, $X = V(f_1, \ldots, f_m) \subset \mathbb{A}^N$ for some polynomials $f_1, \ldots, f_m \in k[t_1, \ldots, t_N]$. Consider the Jacobian matrix

$$J(f_1, \ldots, f_m)(x) := \left(\frac{\partial f_i}{\partial t_j}(x)\right)_{i,j}.$$

The module $\Omega^1_{R/k}$ is generated by the symbols $dt_1, \dots, dt_N$. On the other hand, since $df_i = 0$ for all i, we have the relations

$$\sum_j \frac{\partial f_i}{\partial t_j} dt_j = 0$$

for all i. Putting all these equations into a single one, it is convenient to consider the Jacobian matrix

$$J(f_1, \dots, f_m) = \left(\frac{\partial f_i}{\partial t_j} \right)_{i,j}.$$

This is a $N \times m$-matrix with

$$J(f_1, \dots, f_m) \cdot \begin{pmatrix} dt_1 \\ dt_2 \\ \vdots \\ dt_N \end{pmatrix} = 0.$$

Recall also that by definition, for every $x \in X$ the tangent space $T_{X,x}$ is a translate of the kernel of the linear map given by $J(f_1, \dots, f_N)$. Since X is smooth of dimension n, we know that for all $x \in X$,

$$\mathrm{rk}(J(f_1, \dots, f_m)(x)) = N - n.$$

Hence, up to reordering of the coordinates t_i, we may assume that

$$J(f_1, \dots, f_m)(x) = \begin{pmatrix} * & A(x) \\ * & * \end{pmatrix}$$

where $A(x)$ is an $(N-n) \times (N-n)$-matrix which is invertibel at our given point $x_0 \in X$. Being invertible is a Zariski open condition and so we may after shrinking X assume that $A(x)$ is invertible for all $x \in X$.

But then

$$0 = \begin{pmatrix} A(x)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} * & A(x) \\ * & * \end{pmatrix} \cdot \begin{pmatrix} dt_1 \\ dt_2 \\ \vdots \\ dt_N \end{pmatrix} = \begin{pmatrix} * & \mathbb{1} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dt_1 \\ dt_2 \\ \vdots \\ dt_N \end{pmatrix}.$$

Hence, for all $i = n+1, \dots, N$,

$$dt_i = \sum_{i=1}^{n} \lambda_i(x) dt_i$$

for some regular functions λ_i on X. This shows that the R-module $\Omega^1_{R/k}$ is generated by $dt_1, \dots, dt_n$. It remains to prove that there are no relations among these symbols. Indeed, suppose that

$$\sum_{i=1}^{n} g_i dt_i = 0$$

for some regular functions $g_i \in R$. By the first part, $d_x t_1, \dots, d_x t_n$ form a basis of the cotangent space $T^*_{X,x}$ for all $x \in X$. Evaluating the above identity at $x \in X$ we get

$$\sum_{i=1}^{n} g_i(x) d_x t_i = 0$$

and so $g_i(x) = 0$ for all $x \in X$ and all i. Hence, $g_i = 0$ for all i, as we want. This completes the proof of the proposition. $\square$

If $M = R^r$ is a free R-module, we can define $\Lambda^p_R M$ as R-module with generated by symbols $m_1 \wedge \cdots \wedge m_r$ which satisfy the usual conditions. This way, we can define a for any locally free $\mathcal{O}_X$-module the exterior product $\Lambda^p_{\mathcal{O}_X}\mathcal{M}$ as (the sheafification of)

$$U \mapsto \Lambda^p_{\mathcal{O}_X(U)}\mathcal{M}(U).$$

Definition 12.9. *Let X be a smooth quasi-projective variety. Then $\Omega^p_X := \Lambda^p\Omega^1_X$ is the sheaf of regular p-forms.*
If $n = \dim X$, then $\omega_X := \Omega^n_X$ is the canonical sheaf (or canonical line bundle).

Note that ω_X is a locally free $\mathcal{O}_X$-module of rank 1 on X.

Definition 12.10. *Let X be a smooth projective variety. Then its geometric genus is defined by*

$$p_g(X) := \dim_k(\Gamma(X, \omega_X)),$$

where $\Gamma(X, \omega_X) = \omega_X(X)$ denotes the space of global sections of ω_X. If X is a curve, we call the geometric genus simply genus and denote it by $g(X) := \dim_k(\Gamma(X, \omega_X))$.

Example 12.11. *Let $X = \mathbb{P}^1$. Then $g(X) = 0$.*

Proof. We need to prove that $\Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$. For this let $\omega \in \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})$. Consider the standard open covering $\mathbb{P}^1 = U_0 \cup U_1$ with $U_i = \mathbb{P}^1 \setminus V(t_i = 0)$. The rational functions $u_0 = \frac{t_1}{t_0}$ and $u_1 = \frac{t_0}{t_1}$ satisfy

$$k[U_0] = k[u_0] \text{ and } k[U_1] = k[u_1].$$

On U_i, we have

$$\omega|_{U_i} = g_i(u_i)du_i$$

for some polynomial u_i. Note that $U_{01} = U_0 \cap U_1$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ with ring of regular functions $k[U_{01}] = k[u_0, u_0^{-1}] = k[u_1^{-1}, u_1]$ with $u_0 = u_1^{-1}$. Hence, on U_{01}, we get

$$g_0(u_0)du_0 = g_1(u_1)du_1 = g_1(u_0^{-1})d\frac{1}{u_0} = du_0$$

and so

$$g_0(u_0) = g_1(u_0^{-1})u_0^{-2}.$$

Since g_i is a polynomial for each i, the above equality is only possible if $g_i = 0$ for all i. Hence, $\omega = 0$, as we want. □

Example 12.12. *Let $X = V(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}^2$, where we assume that* $\mathrm{char}(k) \neq 3$. *Then $\omega_X \cong \mathcal{O}_X$ and so $g(X) = 1$.*

Proof. Let $U_{ij} := X \setminus V(x_i, x_j)$. Then $X = U_{01} \cup U_{02} \cup U_{12}$. On U_{01} we have the regular functions $u = \frac{x_1}{x_0}$ and $v = \frac{x_2}{x_0}$ and we consider the regular differential form

$$\omega_{01} := \frac{du}{v^2}$$

on $U_{0,1}$.
On U_{02}, u, v are regular as well. Since $u^3 + v^3 + 1 = 0$, we have

$$3u^2du + 3v^2dv = 0$$

and so using that 3 is invertible in k, we get

$$\frac{du}{v^2} = \frac{-dv}{u^2} =: \omega_{02}$$

where we note that ω_1 is regular on U_{02}. Finally, on U_{12}, we have the regular functions u^{-1} and $\frac{u}{v}$. On $U_{01} \cap U_{12}$, we have $du^{-1} = -u^{-2}du$

$$\frac{du}{v^2} = \frac{u^2}{v^2}d(u^{-1}) =: \omega_{12}$$

which extends to a regular differential form ω_{12} on U_{12}. Altogether, we have constructed a nontrivial global section ω of ω_X with

$$\omega|_{U_{ij}} = \omega_{ij}.$$

Looking at the local expressions above, one easily checks that ω_{ij} generates the $k[U_{ij}]$-module $\Omega^1_{U_{ij}/k}$. We thus get a surjective morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{O}_X \to \omega_X$, which on $U \subset X$ is given by

$$\varphi_U : \mathcal{O}_X(U) \to \omega_X(U), \quad f \mapsto f \cdot \omega.$$

Since X is a smooth curve, $\omega_X = \Omega^1_X$ is locally free of rank one and so the above surjection must be injective, hence an isomorphism. This concludes the proof of the example. □

13 Line bundles and divisors

Operations on $\mathcal{O}_X$-modules: Let $\mathcal{M}$ and $\mathcal{M}'$ be $\mathcal{O}_X$-modules. Then $\mathcal{M} \otimes \mathcal{M}'$ is the $\mathcal{O}_X$-module that is given as sheafification of

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{M}'(U).$$

Also, $\mathcal{H}om(\mathcal{M}, \mathcal{M}')$ is the $\mathcal{O}_X$-module, given by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}|_U, \mathcal{M}'|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}|_U, \mathcal{M}'|_U)$ denotes the $\mathcal{O}_X(U)$-module given by the group of homomorphisms between the $\mathcal{O}_U$-modules $\mathcal{M}|_U$ and $\mathcal{M}'|_U$. Moreover,

$$\mathcal{M}^\vee := \mathcal{H}om(\mathcal{M}, \mathcal{O}_X)$$

and if $\mathcal{M}'$ is locally free, then

$$\mathcal{H}om(\mathcal{M}, \mathcal{M}') \cong \mathcal{M}^\vee \otimes \mathcal{M}'$$

Definition 13.1. *Let X be a quasi-projective variety. By slight abuse of notation, we call a locally free $\mathcal{O}_X$-module $\mathcal{L}$ of rank one a line bundle on X.*

Let $\mathcal{L}$ and $\mathcal{L}'$ be line bundles on X. Then $\mathcal{L} \otimes \mathcal{L}'$ is again a line bundle on X. Similarly, $\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ is a line bundle on X. Since

$$\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{H}om(\mathcal{L}, \mathcal{L}),$$

this line bundle admits a section without zeros and so it is trivial:

$$\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X.$$

Altogether, we see that the set of isomorphism classes of line bundles on X, denoted by $\mathrm{Pic}\, X$, is an abelian group under $\otimes$.

Let now X be normal. Recall that for a divisor $D \in \mathrm{Div}(X)$, we have the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$, given by

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid \mathrm{Div}(f) + D \geq 0\}.$$

Definition 13.2. *Let $D \in \mathrm{Div}(X)$ be a divisor on X. We say that D is a Cartier divisor, if $\mathcal{O}_X(D)$ is locally free (of rank one).*

Proposition 13.3. *If X is smooth, then $\mathcal{O}_X(D)$ is locally free of rank one. That is, any divisor on X is Cartier.*

Proof. Let $x \in X$. We need to find a neighbourhood $x \in U \subset X$ of x, such that $\mathcal{O}_X(D)|_U \cong \mathcal{O}_U$. Since X is smooth, $\mathcal{O}_{X,x}$ is a regular local ring. Let $D = \sum_i a_i D_i$ be a decomposition into prime divisors $D_i \subset X$. Up to shrinking X, we may assume $x \in D_i$ for all i. Each prime divisor D_i thus corresponds to a prime ideal

$$\mathfrak{p}_i \subset \mathcal{O}_{X,x},$$

consisting of all functions defined in some neighbourhood of x, which vanish along D_i. Since $\mathcal{O}_{X,x}$ is a regular local ring, the height one prime ideal $\mathfrak{p}_i$ is principal:

$$\mathfrak{p}_i = (g_i)$$

for some $g_i \in \mathcal{O}_{X,x}$ (in fact, any $g_i \in \mathfrak{p}_i$ irreducible will do the job). Up to shrinking X, we may assume that X is affine and $g_i \in k[X]$ is regular on X for each i. Then

$$g := \prod_i g_i^{a_i} \in k(X)$$

satisfies $\mathrm{Div}(g) = D$ and so $\mathcal{O}_X(D) \cong \mathcal{O}_X$ by exercise 2c on sheet 11. This concludes the proposition. $\square$

Note that $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \cong \mathcal{O}_X(D + D')$. Hence, if X is smooth, we get a homomorphism of groups

$$\mathrm{Div}(X) \to \mathrm{Pic}(X), \quad D \mapsto \mathcal{O}_X(D)$$

and we know by Exercise 2 on sheet 11 that this descends to an injective group homomorphism

$$\mathrm{Cl}(X) \hookrightarrow \mathrm{Pic}(X), \quad [D] \mapsto \mathcal{O}_X(D).$$

We aim to prove that this map is an isomorphism. To this end we need to show that the above map is surjective, that is, any line bundle on a smooth quasi-projective variety comes from a divisor.

Definition 13.4. *Let $\mathcal{L}$ be a line bundle on a quasi-projective variety X. A rational section of $\mathcal{L}$ is (the equivalence class of) a section s of $\mathcal{L}$ over some non-empty open subset $U \subset X$.*

Let $\mathcal{L}$ be a line bundle on a normal quasi-projective variety X. Let $s \in \mathcal{L}(U)$ be a rational section of $\mathcal{L}$. Then

$$\mathrm{Div}(s) \in \mathrm{Div}(X)$$

is defined as follows. Let $X = \bigcup U_i$ be an affine open covering of X such that there are isomorphisms

$$\varphi_i : \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}_{U_i}.$$

Then

$$\mathrm{Div}(s)|_{U_i} := \mathrm{Div}(\varphi_i(s)).$$

These definitions are compatible, because on $U_{ij} := U_i \cap U_j$, the composition

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$$

is an isomorphism and so $\mathrm{Div}(f) = \mathrm{Div}(\varphi_{ij}(f))$. Hence,

$$\mathrm{Div}(\varphi_j(s))|_{U_{ij}} = \mathrm{Div}(\varphi_{ij}(\varphi_j(s)))|_{U_{ij}} = \mathrm{Div}(\varphi_i(s))|_{U_{ij}}.$$

This proves that the divisors $\mathrm{Div}(s)|_{U_i} := \mathrm{Div}(\varphi_i(s))$ glue together to give a divisor

$$\mathrm{Div}(s) \in \mathrm{Div}(X).$$

The surjectivity of

$$\mathrm{Cl}(X) \to \mathrm{Pic}(X), \quad [D] \mapsto \mathcal{O}_X(D).$$

in the case where X is smooth, then follows form the following.

Proposition 13.5. *Let $\mathcal{L}$ be a line bundle on a normal quasi-projective variety X, and let $s \in \mathcal{L}(U)$ be a rational section. Then the divisor $D := \mathrm{Div}(s)$ has the property that*

$$\mathcal{L} \cong \mathcal{O}_X(D).$$

Proof. Consider the map

$$\psi : \mathcal{O}_X(D) \to \mathcal{L}$$

of $\mathcal{O}_X$-modules, given by

$$\mathcal{O}_X(D)(U) \to \mathcal{L}(U), \quad f \mapsto f \cdot s.$$

Note that this is well-defined, because

$$\mathrm{Div}(f \cdot s)|_U = \mathrm{Div}(f)|_U + \mathrm{Div}(s)|_U \geq 0$$

and so $f \cdot s \in \mathcal{L}(U)$ is a regular section of $\mathcal{L}$ over U.
To see that ψ is an isomorphism, consider the map $\phi : \mathcal{L} \to \mathcal{O}_X(D)$ given by

$$\phi_U : \mathcal{L}(U) \to \mathcal{O}_X(D)(U), \quad t \mapsto f,$$

where $f \in k(X)$ is the rational function given by $t = fs$. Clearly, ϕ is an inverse of ψ and so the latter is an isomorphism, as claimed. This proves the proposition. □

Definition 13.6. *Let X be a smooth quasi-projective variety. Then the canonical divisor is given by*

$$K_X = \mathrm{Div}(s),$$

where s is a rational section of the canonical bundle ω_X.

Note that

$$\mathcal{O}_X(K_X) \cong \omega_X$$

and so K_X is unique up to linear equivalence. That is, the class of K_X in $\mathrm{Cl}(X)$ is unique.

Example 13.7. *$K_{\mathbb{P}^1} = -2 \cdot H$, where H is the class of a point on $\mathbb{P}^1$.*

Proof. This follows from the computation that proved $g(\mathbb{P}^1) = 0$:
Consider the standard open covering $\mathbb{P}^1 = U_0 \cup U_1$ with $U_i = \mathbb{P}^1 \setminus V(t_i = 0)$. The rational functions $u_0 = \frac{t_1}{t_0}$ and $u_1 = \frac{t_0}{t_1}$ satisfy

$$k[U_0] = k[u_0] \quad \text{and} \quad k[U_1] = k[u_1].$$

On U_0, we have the differential form

$$\omega = du_0$$

which has no zeros and poles on U_0.

Note that $U_{01} = U_0 \cap U_1$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ with ring of regular functions $k[U_{01}] = k[u_0, u_0^{-1}] = k[u_1^{-1}, u_1]$ with $u_0 = u_1^{-1}$. Hence, on U_{01}, we get

$$\omega = du_0 = du_1^{-1} = \frac{-du_1}{u_1^2}$$

and so

$$\mathrm{Div}(\omega) = -2 \cdot [0:1]$$

as claimed. $\square$

If X is a smooth projective curve, then

$$\deg : \mathrm{Cl}(X) \to \mathbb{Z}$$

is well-defined and so we get a new numerical invariant of smooth projective curves, the degree of K_X.

Proposition 13.8. *Let $X \subset \mathbb{P}^2$ be a smooth projective curve of degree d. Then*

$$\deg(K_X) = d \cdot (d-3).$$

Proof. By assumption, there is an irreducible homogeneous polynomial $F \in k[x_0, x_1, x_2]$ of degree d such that

$$X = V_{\mathbb{P}^2}(F).$$

Let $U_i := X \setminus V(x_i)$. On U_0 we have regular functions $y_1 = \frac{x_1}{x_0}$ and $y_2 = \frac{x_2}{x_0}$. Putting $f := F(1, y_1, y_2)$, we get

$$k[U_0] = k[y_1, y_2]/f$$

and

$$0 = df = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2.$$

Since X is smooth, the above partials have no common zero on X and so the rational differential form

$$\omega := \frac{1}{\frac{\partial f}{\partial y_1}} dy_2 = -\frac{1}{\frac{\partial f}{\partial y_2}} dy_1$$

on X satisfies $\mathrm{Div}(\omega)|_{U_0} = 0$.

On U_1 we have the regular functions $z_0 = \frac{x_0}{x_1}$ and $z_2 = \frac{x_2}{x_1}$. Putting $g := F(z_0, 1, z_2)$, we have

$$k[U_1] = k[z_0, z_2]/g.$$

Moreover,

$$F(x_0, x_1, x_2) = x_1^d g(z_0, z_2) = x_0^d f(y_1, y_2),$$

where $y_1 = z_0^{-1}$ and $y_2 = \frac{z_2}{z_0}$. Hence,

$$g(z_0, z_2) = z_0^d f(z_0^{-1}, \frac{z_2}{z_0})$$

and so

$$\frac{\partial g}{\partial z_2}(z_0, z_2) = z_0^{d-1} \frac{\partial f}{\partial y_2}(z_0^{-1}, \frac{z_2}{z_0})$$

Hence,

$$\omega = \frac{1}{\frac{\partial f}{\partial y_2}} dy_1 = z_0^{d-1} \frac{1}{\frac{\partial g}{\partial z_2}} dz_0^{-1} = -z_0^{d-3} \frac{1}{\frac{\partial g}{\partial z_2}} dz_0.$$

Since $\frac{\partial g}{\partial z_0} dz_0 = -\frac{\partial g}{\partial z_2} dz_2$, we also find

$$\omega = z_0^{d-3} \frac{1}{\frac{\partial g}{\partial z_0}} dz_2.$$

Since the partials of g have no common zero on U_1, we thus altogether conclude that

$$\mathrm{Div}(\omega)|_{U_1} = (d-3) \cdot V(z_0) \cap U_1.$$

Up to a linear change of coordinates, we may from the beginning assume that $[0:0:1] \notin X$ and so $X = U_1 \cup U_0$. Moreover, $V(z_0) \cap U_1 = V(x_0) \cap X$ and so

$$K_X = \mathrm{Div}(\omega) = (d-3) \cdot V(x_0) \cap X$$

By Bezout's theorem, $\deg(K_X) = (d-3)d$, as claimed. □

14 Riemann–Roch theorem for curves

Let X be a smooth projective curve and let D be a divisor on X. We then define

$$h^0(X, D) := h^0(X, \mathcal{O}_X(D)) := \dim_k(\Gamma(X, \mathcal{O}_X(D))).$$

Remark 14.1. *The above notation is motivated by the fact that for any sheaf $\mathcal{F}$ on a topological space X, $\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$ coincides with the 0-th sheaf cohomology of $\mathcal{F}$. In the case where X is a quasi-projective variety and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, we define these cohomology groups in Section 15 below.*

The most powerful tool in the study of divisors or line bundles on a smooth projective curve X is the Riemann-Roch theorem, which states the following.

Theorem 14.2. *Let X be a smooth projective curve, D a divisor on X. Then,*

$$h^0(X, D) - h^0(X, K_X - D) = \deg D + 1 - g(X)$$

We will now collect a few consequences of that theorem. To this end note that $h^0(X, D) = 0$ if $\deg D < 0$. Hence, the above theorem allows us to compute $h^0(X, D)$ explicitly if $\deg(K_X - D) = \deg K_X - \deg D < 0$.

(1) Applying the theorem to $D = K_X$, we find that $g(X) - 1 = \deg K_X + 1 - g(X)$ and so

$$g(X) = \frac{\deg K_X + 2}{2}$$

In particular, this implies that a smooth projective curve $X \subset \mathbb{P}^2$ of degree d has genus

$$g(X) = \frac{d(d-3)+2}{2} = \frac{(d-1)(d-2)}{2}.$$

(2) If $D \geq 0$ is an effective divisor on $\mathbb{P}^1$, then

$$h^0(\mathbb{P}^1, D) = \deg D + 1.$$

(3) If X is a smooth projective curve of genus 0, then $X \cong \mathbb{P}^1$.

Proof. Let $x, y \in X$ and consider the divisor $D = x - y \in \mathrm{Div}(X)$. Since $g(X) = 0$, we have $\deg K_X = -2$ and so $h^0(X, K_X - D) = 0$. Hence, Riemann–Roch yields

$$h^0(X, D) = \deg D + 1 - g(X) = 1.$$

That is, there is a rational function $\varphi \in k(X)$ with

$$\mathrm{Div}(\varphi) + D \geq 0$$

Since D as well as $\mathrm{Div}(\varphi)$ are divisors of degree zero, we must have

$$\mathrm{Div}(\varphi) = -D.$$

If $x \neq y$, then φ yields a rational map $X \dashrightarrow \mathbb{P}^1$ of degree one and one checks that this is an isomorphism (see some previous Exercise sheet). Hence, $X \cong \mathbb{P}^1$ as claimed. □

(4) An elliptic curve is a smooth projective curve X with $g(X) = 1$. We claim that any elliptic curve X has the following properties:

$$\omega_X \cong \mathcal{O}_X \quad \text{and} \quad h^0(X, D) = \deg D \text{ for all } D \geq 0.$$

Proof. First note that $\omega_X \cong \mathcal{O}_X$ is equivalent to $K_X \sim 0$. Since $g(X) = 1$, we have $h^0(X, K_X) = 1$. That is, there is a rational function $\varphi \in k(X)$ with

$$\mathrm{Div}(\varphi) + K_X \geq 0.$$

On the other hand, $\deg K_X = 0$ because $g(X) = 1$ and so the above inequality must be an equality:

$$\mathrm{Div}(\varphi) + K_X = 0.$$

Hence, $K_X \sim 0$, as claimed. For the second claim, note that $D \sim 0$ if $\deg D = 0$ and $D \geq 0$. We may thus assume $\deg D > 0$. Then $h^0(X, K_X - D) = 0$ and so $h^0(X, D) = \deg D + 1$, as claimed. □

(5) Let X be an elliptic curve, pick a point $x_0 \in X$ and let $\mathrm{Cl}^0(X) := \ker(\deg : \mathrm{Cl}(X) \to \mathbb{Z})$. Then there is a natural bijection

$$\phi : X \to \mathrm{Cl}^0(X), \quad x \mapsto x - x_0.$$

In particular, X carries the structure of a group.

Proof. Note that ϕ is injective, as otherwise there was a rational function $\varphi \in k(X)$ with $\mathrm{Div}(\varphi) = x - x_0$ for $x \neq x_0$ and this implies $X \cong \mathbb{P}^1$, hence $g(\mathbb{P}^1) = 0$. Next, we need to prove surjectivity. That is, for any divisor D of degree zero we need to find $x \in X$ with $D \sim x - x_0$. To this end, consider the divisor $D' := D + x_0$ of degree one. By Riemann-Roch, $h^0(X, D') = 1$ and so there is a rational function $\varphi \in k(X)$ with

$$D'' := \mathrm{Div}(\varphi) + D' \geq 0.$$

Note that $\deg(D'') = \deg(D') = 1$. Since D'' is effective, we find that $D'' = x$ for a point $x \in X$. But then

$$x - x_0 = \mathrm{Div}(\varphi) + D$$

and so $D \sim x - x_0$, as claimed. □

(6) Let X be an elliptic curve. Then there is an irreducible cubic polynomial $F \in k[x_0, x_1, x_2]$ such that X is isomorphic to the plane curve $V_{\mathbb{P}^2}(F) \subset \mathbb{P}^2$.

Sketch of Proof. Pick a point $x \in X$ and consider for $n \geq 0$ the divisor $D_n = 3 \cdot x$ on X. By Riemann-Roch, we have $h^0(X, D_n) = n$ and there are by definition natural inclusions

$$H^0(X, D_{n-1}) \subset H^0(X, D_n).$$

We deduce that $H^0(X, D_1)$ is generated by a constant rational function $\varphi_1 = \lambda \neq 0$. Moreover, $H^0(X, D_2)$ is generated by φ_1 and by a rational function φ_2 which has a pole of order two at x and is regular otherwise. Finally $H^0(X, D_3)$ is generated by φ_1, φ_2 and by a rational function φ_3 which has a pole of order three at x and is regular otherwise.

Consider now the rational map

$$\varphi := [\varphi_1 : \varphi_2 : \varphi_3] : X \dashrightarrow \mathbb{P}^2.$$

Since each φ_i is regular away from x, φ is regular away from x. Moreover, in some njeighbourhood of x, we have

$$\varphi = [\frac{\varphi_1}{\varphi_3} : \frac{\varphi_2}{\varphi_3} : 1]$$

which shows that φ is regular at x. This shows that φ is a regular map, as claimed.

Next we aim to show that φ is an embedding. For this we need to see that φ separates points and tangent directions and both statements follow from the fact that

$$h^0(X, 3x - D') = 1$$

for any degree two divisor D'.

Now that we know that φ embeds X as a plane curve in $\mathbb{P}^2$, we deduce e.g. from the formula for the degree of the canonical divisor of smooth plane curves, that the degree of the respective curve must be three. This completes the proof. □

15 Cohomology of quasi-coherent sheaves

15.1 Definition and easy examples

Let X be a topological space, $\mathcal{F}$ a sheaf on X. Let $X = \bigcup_{i=1}^r U_i$ be a finite open covering , denoted by $\mathcal{U}$. For every open subset $\{i_0, i_1, \ldots, i_p\} \subset \{1, \ldots, r\}$, put

$$U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$$

We then define

$$C^p := C^p(\mathcal{U}, \mathcal{F}) := \oplus_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

together with maps

$$d^p : C^p \to C^{p+1}, \quad (s_{i_0,\ldots,i_p}) \mapsto (t_{j_0,\ldots,j_{p+1}}),$$

where

$$t_{j_0,\ldots,j_{p+1}} := \sum_{l=0}^{p+1} (-1)^l s_{j_0,\ldots,\hat{j}_l,\ldots,j_{p+1}}|_{U_{j_0,\ldots,j_{p+1}}}.$$

This way we get a complex

$$0 \to C^0 \to C^1 \to C^2 \to \ldots$$

The cohomology groups of this complex are called Cech cohomology groups of $\mathcal{F}$ with respect to the cover $\mathcal{U}$:

$$H^p(\mathcal{U}, \mathcal{F}) := \frac{\ker(d^p : C^p \to C^{p+1})}{\operatorname{im}(d^p : C^{p-1} \to C^p)}.$$

We will use the following theorem as a black box for now.

Theorem 15.1. *Let X be quasi-projective variety and let $\mathcal{U}$ and $\mathcal{U}'$ be finite coverings by affine open subsets of X. Then for any quasi-coherent sheaf $\mathcal{F}$, there are natural isomorphisms*

$$H^p(\mathcal{U}, \mathcal{F}) \cong H^p(\mathcal{U}', \mathcal{F}).$$

In the above theorem, the assumption that $\mathcal{F}$ is quasi-coherent is essential, as it guarantees that taking global sections on affine pieces is right exact. By the above theorem, we can make the following definition of sheaf cohomology in the case of quasi-coherent sheaves.

Definition 15.2. *Let X be a quasi-projective variety and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module on X. Then the sheaf cohomology of $\mathcal{F}$ is defined by*

$$H^i(X, \mathcal{F}) := H^i(\mathcal{U}, \mathcal{F}).$$

The next lemma explains the notation $h^0(X, \mathcal{F}) = \dim_k(\Gamma(X, \mathcal{F}))$, used before, where we recall that $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ denotes the space of global sections of $\mathcal{F}$.

Lemma 15.3. *We have $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.*

Proof. For any open covering $\mathcal{U}$ of X, we have

$$H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$$

as a consequence of the sheaf axioms. This concludes the proof of the lemma. □

Example 15.4. *Let $X = \mathbb{P}^1$, then $H^1(X, \mathcal{O}_X) = 0$ and $H^1(X, \omega_X) \cong k$.*

Proof. Consider the affine open cover $X = U_0 \cup U_1$ with $U_i = \mathbb{P}^1 \setminus V(t_i)$. On U_0 we have the affine coordinate $u_0 = \frac{t_1}{t_0}$ and on U_1, we have the affine coordinate $u_1 = u_0^{-1}$.

The Cech complex for $H^i(X, \mathcal{O}_X)$ reads:

$$0 \to \mathcal{O}_X(U_0) \oplus \mathcal{O}_X(U_1) \xrightarrow{d^0} \mathcal{O}_X(U_0 \cap U_1) \to 0,$$

where $d^0(f, g) = g - f$. Note that

$$\mathcal{O}_X(U_0) = k[u_0], \quad \mathcal{O}_X(U_1) = k[u_1] = k[\frac{1}{u_0}], \quad k[U_0, U_1] = k[u_0, u_0^{-1}].$$

Hence, $H^1(X, \mathcal{O}_X) = \operatorname{coker}(d^0) = 0$, as claimed.

Next, the Cech complex for $H^i(X, \omega_X)$ reads:

$$0 \to \Omega^1(U_0) \oplus \Omega^1(U_1) \xrightarrow{d^0} \Omega^1(U_0 \cap U_1) \to 0,$$

where $d^0(\alpha, \beta) = \beta - \alpha$. Note that

$$\Omega^1(U_0) = k[u_0]du_0, \quad \Omega^1(U_1) = k[u_1]du_1 = k[u_0^{-1}]\frac{-1}{u_0^2}du_0, \quad \Omega^1(U_0 \cap U_1) = k[u_0, u_0^{-1}]du_0.$$

Hence,

$$H^1(X, \omega_X) = \operatorname{coker}(d^0) \cong \frac{du_0}{u_0}k \cong k.$$

□

15.2 Four important theorems

The most important property of cohomology is the existence of long exact sequences associated to short exact sequences of sheaves on X, as follows.

Theorem 15.5. *Let X be a quasi-projective variety and let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then there is a long exact sequence*

$$\dots \to H^p(X, \mathcal{F}_1) \to H^p(X, \mathcal{F}_2) \to H^p(X, \mathcal{F}_3) \to H^{p+1}(X, \mathcal{F}_1) \to \dots$$

Sketch of proof. Fix an affine open covering $\mathcal{U}$ of X. As a matter of fact that we will prove next term, the intersections $U_{i_0,\dots,i_p}$ are all affine.
Let C_i^* be the Cech complex of $\mathcal{F}_i$ with respect to $\mathcal{U}$. The short exact sequence of sheaves then induces a short exact sequence of complexes

$$0 \to C_1^* \to C_2^* \to C_3^* \to 0;$$

the zero on the right is here due to the fact that taking global sections of quasi-coherent sheaves on affine varieties is exact, as we have seen on one of the exercise sheets. Note that the maps $C_i^* \to C_{i+1}^*$ respect the differentials and so they are really maps of complexes. The long exact sequence in cohomology is now a consequence the snake lemma, as we recall below. □

Lemma 15.6. *Let R be a commutative ring. Consider a commutative diagram of R-modules*

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0 \\ & & \downarrow d_1 & & \downarrow d_2 & & \downarrow d_3 & & \\ 0 & \longrightarrow & M_1' & \xrightarrow{f'} & M_2' & \xrightarrow{g'} & M_3' & & \end{array}$$

with exact rows. Then there is an induced exact sequence

$$\ker(d_1) \to \ker(d_2) \to \ker(d_3) \to \operatorname{coker}(d_1) \to \operatorname{coker}(d_2) \to \operatorname{coker}(d_3).$$

Proof. This is standard. The boundary map $\ker(d_3) \to \operatorname{coker}(d_1)$ is constructed via diagram chasing. The remaining maps are induced by f, g, f' and g' and exactness is checked directly via diagram chasing. □

Corollary 15.7. *Let R be a commutative ring. Let $0 \to C_1^* \xrightarrow{f} C_2^* \xrightarrow{g} C_3^* \to 0$ be an exact sequence of complexes of R-modules C_i^* that are bounded to the left, i.e. $C_i^j = 0$ for all $j << 0$. Then there is an induced long exact sequence in cohomology*

$$\dots \to H^i(C_1^*) \to H^i(C_2^*) \to H^i(C_3^*) \to H^{i+1}(C_1^*) \to \dots$$

Proof. Let $d_j^i : C_j^i \to C_j^{i+1}$ denote the differential in C_j. We construct

$$\dots \to H^i(C_1^*) \to H^i(C_2^*) \to H^i(C_3^*)$$

by induction on i. Since the complexes are bounded to the left, the induction start is clear. Suppose now that the above long exact sequence is constructed up to index i. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & \operatorname{coker}(d_1^{i-1}) & \xrightarrow{f^i} & \operatorname{coker}(d_2^{i-1}) & \xrightarrow{g^i} & \operatorname{coker}(d_3^{i-1}) & \longrightarrow & 0 \\ & & \downarrow d_1^i & & \downarrow d_2^i & & \downarrow d_3^i & & \\ 0 & \longrightarrow & \ker(d_1^{i+1}) & \xrightarrow{f^{i+1}} & \ker(d_2^{i+1}) & \xrightarrow{g^{i+1}} & \ker(d_3^{i+1}) & & \end{array}$$

with exact rows. The snake lemma then yields an exact sequence

$$\cdots \to H^i(C_1^*) \to H^i(C_2^*) \to H^i(C_3^*) \to H^{i+1}(C_1^*) \to H^{i+1}(C_2^*) \to H^{i+1}(C_3^*).$$

This concludes the proof of the corollary. □

In the remainder of this section, we state mostly without proofs some of the most important properties (besides the existence of long exact sequences mentioned above) of sheaf cohomology of quasi-coherent sheaves.

First of all, as a consequence of the definition, resp. the theorem above which lead us to make the definition, we find the following result, originally due to Serre.

Theorem 15.8. *Let X be an affine variety, then $H^k(X, \mathcal{F}) = 0$ for all $k > 0$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

We also have the following theorem of Grothendieck, which we state without proof.

Theorem 15.9. *Let X be a quasi-projective variety of dimension n. Then for any quasi-projective $\mathcal{F}$ on X and for any integer $p > n$, we have*

$$H^p(X, \mathcal{F}) = 0.$$

Next, we state without proof Serre-duality.

Theorem 15.10. *Let X be a smooth projective variety, $n = \dim X$ and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module on X. Then there are natural isomorphisms*

$$H^p(X, \mathcal{E}) \cong H^{n-p}(X, \omega_X \otimes \mathcal{E}^\vee)^*.$$

16 Proof of Riemann–Roch for curves

Let $\mathcal{F}$ be a quasi-coherent sheaf on a quasi-projective variety X. We write

$$h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F}).$$

If X is projective and $\mathcal{F}$ is coherent, then these dimensions are finite. We then define the Euler characteristic of $\mathcal{F}$ via

$$\chi(X, \mathcal{F}) := \sum (-1)^i h^i(X, \mathcal{F}).$$

This is a finite sum, because $h^i(X, \mathcal{F}) = 0$ for $i < 0$ or $i > \dim X$. If $\mathcal{F} = \mathcal{O}_X(D)$, we also write $h^i(X, D) := h^i(X, \mathcal{O}_X(D))$. By Serre duality, $h^i(X, D) = h^{n-i}(X, K_X - D)$ if X is smooth projective of dimension n.

With the above notation, the Riemann–Roch theorem for curve reads as follows.

Theorem 16.1. *Let X be a smooth projective curve and let $D \in \mathrm{Div}(X)$. Then,*

$$\chi(X, \mathcal{O}_X(D)) = h^0(X, D) - h^0(X, K_X - D) = \deg D + 1 - g(X).$$

Proof. **Step 1.** The case where $D = 0$.

Proof. Since $h^0(X, 0) = 1$ and $h^0(X, K_X) = g(X)$, the theorem is trivially true in this case. □

Step 2. Let $D' := D - x$. Then

$$\chi(X, D') = \chi(X, D) - 1$$

and so the theorem holds for D if and only if it holds for D'. In particular, the theorem holds for D if it holds for $D - x$ or $D + x$.

Proof. Consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \xrightarrow{\beta} \underline{k}_x \to 0,$$

where β is defined as follows. Let $a \in \mathbb{Z}$ be the maximal integer such that $D - ax \geq 0$. Then local sections of $\mathcal{O}_X(D)$ around x have a pole/zero of order at most a. If $\pi \in \mathcal{O}_{X,x}$ is a local parameter, i.e. a generator of the maximal ideal $\mathfrak{m}_{X,x}$, then for any local section $\varphi \in \mathcal{O}_X(D)(U)$ in a suitable neighbourhood $x \in U \subset X$ of x, we have that $\pi^a \cdot \varphi$ is regular at x and so

$$\beta(\varphi) := \pi^a \varphi(x) \in k$$

is well-defined. This defines β. Since $\lambda \pi^{-a}$ is for any $\lambda \in k$ a local section of $\mathcal{O}_X(D)$ at x, we find that β is injective and the kernel is obviously given by the subsheaf $\mathcal{O}_X(D') \subset \mathcal{O}_X(D)$. This proves that the above sequence is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules (note that the skyscraper sheaf $\underline{k}_x$ is quasi-coherent, as it corresponds to the module $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$), and so we get a long exact sequence

$$0 \to H^0(X, D') \to H^0(X, D) \to H^0(X, \underline{k}_x) \to H^1(X, D') \to H^1(X, D) \to 0$$

where we use that $H^1(X, \underline{k}_x) = 0$, because we can use an affine open covering of X such that x is not contained in the intersection of any two distinct open subsets and so $C^p = 0$ for all $p \geq 1$. The above exact sequence implies

$$0 = h^0(X, D') - h^0(X, D) + 1 - h^1(X, D') + h^1(X, D).$$

That is

$$\chi(X, D') + 1 = \chi(X, D)$$

and so the claim in step 2 follows. □

Step 3. The general case.

Proof. Write $D = D' - D''$ for effective divisors D', D'' that have no points in common. We prove the theorem by induction on $|D| := \deg D' + \deg D''$. If $|D| = 0$, then $D = 0$ and so the theorem follows from step 1. Otherwise, there is a point $x \in X$ such that $|D - x| < |D|$ or $|D + x| < |D|$. By step 2, the theorem holds for D if and only if it holds for $D - x$ or $D + x$ and so we find by induction that it holds for D. This finishes the proof of the theorem. □

□

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