

ZEROS OF HOLOMORPHIC ONE-FORMS AND TOPOLOGY OF KÄHLER MANIFOLDS

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APPENDIX WRITTEN JOINTLY WITH HSUEH-YUNG LIN

ABSTRACT. A conjecture of Kotschick predicts that a compact Kähler manifold X fibres smoothly over the circle if and only if it admits a holomorphic one-form without zeros. In this paper we develop an approach to this conjecture and verify it in dimension two. In a joint paper with Hao [HS19], we use our approach to prove Kotschick's conjecture for smooth projective threefolds.

1. INTRODUCTION

This paper is motivated by the following conjecture of Kotschick [Ko13].

Conjecture 1.1. *For a compact Kähler manifold X , the following are equivalent.*

- (A) X admits a holomorphic one-form without zeros;
- (B) X admits a real closed 1-form without zeros; or, by Tischler's theorem [Ti70] equivalently, the underlying differentiable manifold is a C^∞ -fibre bundle over the circle.

The implication (A) \Rightarrow (B) is clear; the possibility of the converse implication (B) \Rightarrow (A) is asked in [Ko13]. Condition (B) is equivalent to asking that the smooth manifold that underlies X is a quotient $M \times [0, 1] / \sim$, where M is a closed real manifold of odd dimension and $M \times 0$ is identified with $M \times 1$ via some diffeomorphism of M . Kotschick's conjecture relates this purely topological condition with the complex geometric condition that X has a holomorphic one-form without zeros.

The purpose of this paper is to related Kotschick's conjecture to the following condition

- (C) there is a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$, such that for any finite étale cover $\tau : X' \rightarrow X$, the sequence

$$H^{i-1}(X', \mathbb{C}) \xrightarrow{\wedge \omega'} H^i(X', \mathbb{C}) \xrightarrow{\wedge \omega'} H^{i+1}(X', \mathbb{C}),$$

given by cup product with $\omega' := \tau^* \omega$, is exact for all i .

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This is motivated by a theorem of Green and Lazarsfeld [GL87, Proposition 3.4], who proved the implication (A) \Rightarrow (C). Our first result is the following, which in view of Green and Lazarsfeld's theorem yields some positive evidence for Conjecture 1.1.

Theorem 1.2. *For any compact Kähler manifold X , we have (B) \Rightarrow (C).*

By the above theorem, in order to prove Kotschick's conjecture, it would be enough to show that (C) implies (A). Compared to the original implication (B) \Rightarrow (A), this has the major advantage that (C) and (A) are complex geometric conditions, while (B) is not. More precisely, it is natural to wonder whether a one-form $\omega \in H^0(X, \Omega_X^1)$ which satisfies condition (C) must be without zeros. This would have the remarkable implication that the question whether ω has zeros depends only on the de Rham class of ω and the homotopy type of X . We show that this is true for surfaces.

Theorem 1.3. *Let X be a compact Kähler surface. If $\omega \in H^0(X, \Omega_X^1)$ satisfies condition (C), then it has no zeros. In particular, Conjecture 1.1 holds for compact Kähler surfaces.*

The proof of Theorem 1.3 uses classification of surfaces. In the Appendix to this paper, written jointly with Lin, we give however a more general and direct argument which does not rely on classification results, see Theorem A.1 below.

In joint work with Hao [HS19], we use the approach developed here to prove Conjecture 1.1 for smooth projective threefolds.

The following theorem proves some partial results in arbitrary dimension.

Theorem 1.4. *Let X be a compact connected Kähler manifold with a holomorphic one-form ω such that the complex $(H^*(X, \mathbb{C}), \wedge \omega)$ given by cup product with ω is exact. Then the analytic space $Z(\omega)$ given by the zeros of $\omega \in H^0(X, \Omega_X^1)$ has the following properties.*

(1) *For any connected component $Z \subset Z(\omega)$ with $d = \dim Z$,*

$$H^d(Z, \omega_X|_Z) = 0.$$

In particular, ω does not have any isolated zero.

(2) *If $f : X \rightarrow A$ is a holomorphic map to a complex torus A such that $\omega \in f^*H^0(A, \Omega_A^1)$, then $f(X) \subset A$ is fibred by tori.*

Ein and Lazarsfeld [EL97, Theorem 3] showed that the image of a morphism $f : X \rightarrow A$ to a complex torus A is fibred by tori if $\chi(X, \omega_X) = 0$ and $\dim f(X) = \dim X$. In item (2) above we obtain the same conclusion without any assumption on f , but where we replace $\chi(X, \omega_X) = 0$ by the stronger condition on the exactness of $(H^*(X, \mathbb{C}), \wedge \omega)$.

Theorem 1.2 and item (2) in the above theorem imply for instance that a Kähler manifold X with simple Albanese torus $\text{Alb}(X)$ and with $b_1(X) > 2 \dim(X)$ does not admit a C^∞ -fibration over the circle. Similarly, we obtain the following corollary in the projective case.

Corollary 1.5. *Let X be a smooth complex projective variety such the manifold which underlies X fibres smoothly over the circle. Then there is a surjective holomorphic morphism $f : X \rightarrow A$ to a positive-dimensional abelian variety A .*

The following example of Debarre, Jiang and Lahoz shows that the étale covers in condition (C) are necessary to make Theorem 1.3 true.

Example 1.6 ([DJL17, Example 1.11]). *Let C_1, C_2 be smooth projective curves with $g(C_1) > 1$ and $g(C_2) = 1$ and automorphisms $\varphi_i \in \text{Aut}(C_i)$ of order two such that C_i/φ_i has genus one for $i = 1, 2$. Then the quotient*

$$X := (C_1 \times C_2)/(\varphi_1 \times \varphi_2)$$

has the same rational cohomology ring as an abelian surface, and so $\wedge \omega$ is exact on cohomology for any non-zero $\omega \in H^0(X, \Omega_X^1)$. However, if ω is obtained as pullback via the map $\pi : X \rightarrow C_1/\varphi_1$, then it vanishes along the multiple fibres of π , which lie above the branch points of $C_1 \rightarrow C_1/\varphi_1$.

Remark 1.7. *This paper raises the question whether condition (C) implies (A). In view of [GL87, Proposition 3.4] it is natural to wonder whether more generally, a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ such that for any finite étale cover $\tau : X' \rightarrow X$*

$$H^{i-1}(X', \mathbb{C}) \xrightarrow{\wedge \tau^* \omega} H^i(X', \mathbb{C}) \xrightarrow{\wedge \tau^* \omega} H^{i+1}(X', \mathbb{C})$$

is exact for all $i < c$ implies that $\text{codim}_X(Z(\omega)) \geq c$. This goes back to [BWY16], where it is asked whether equality always holds in [BWY16, Theorem 1.1]. However, blowing-up a point in $Z(\omega)$ easily produces counterexamples to this conjecture.

Why the Kähler assumption? The Kähler assumption in Conjecture 1.1 is essential. For instance, a Hopf surface X is a compact complex surface with $H^0(X, \Omega_X^1) = 0$, whose underlying differentiable manifold is diffeomorphic to $S^1 \times S^3$, and so it satisfies (B) but not (A).

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Notation. For a holomorphic one-form ω on a Kähler manifold X , we denote by $Z(\omega)$ the (possibly non-reduced) analytic space given by the zeros of ω , viewed as a section of the vector bundle Ω_X^1 .

2. PROOF OF THEOREM 1.2

Let X be a smooth connected manifold. We denote by $\text{Loc}(X)$ the group of local systems on X whose stalks are one-dimensional \mathbb{C} -vector spaces. Since local systems on the interval are trivial, the choice of a base point $s \in S^1$ induces a canonical isomorphism $\text{Loc}(S^1) \cong \mathbb{C}^*$. Hence, if we fix a base point $x \in X$, then for any $L \in \text{Loc}(X)$, any continuous map $\gamma : S^1 \rightarrow X$ with $\gamma(s) = x$ yields a canonical element $\gamma^*L \in \text{Loc}(S^1) \cong \mathbb{C}^*$, which, as one checks, depends only on the homotopy class of γ . This construction gives rise to the so called monodromy representation, which (since X is connected) induces an isomorphism between $\text{Loc}(X)$ and the character variety

$$\text{Char}(X) := \text{Hom}(\pi_1(X, x), \mathbb{C}^*) \cong H^1(X, \mathbb{C}^*).$$

If $L \in \text{Loc}(X)$, then the associated complex line bundle has locally constant transition functions, hence it admits a flat connection and so the first Chern class $c_1(L)$ must be torsion. The long exact sequence associated to the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^* \rightarrow 0$ of locally constant sheaves on X thus shows that $\text{Loc}(X)$ is isomorphic to an extension of a finite group by the connected subgroup $\text{Loc}^0(X) \subset \text{Loc}(X)$ which contains the trivial local system. Moreover,

$$\text{Loc}^0(X) \cong \frac{H^1(X, \mathbb{C})}{H^1(X, \mathbb{Z})} \cong (\mathbb{C}^*)^{b_1(X)}.$$

coincides with the subgroup $\{L \in \text{Loc}(X) \mid c_1(L) = 0\}$.

2.1. Local systems associated to closed 1-forms and Novikov's inequality. If α is a closed complex valued 1-form on X , then we can construct a local system $L(\alpha) \in \text{Loc}^0(X)$ as follows. Consider the twisted de Rham complex $(\mathcal{A}_{X, \mathbb{C}}^*, d + \wedge \alpha)$, where $\mathcal{A}_{X, \mathbb{C}}^k$ denotes the sheaf of complex valued C^∞ -differential k -forms on X , and where $\wedge \alpha$ acts on a k -form β via $\beta \mapsto \alpha \wedge \beta$. There is an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\alpha|_{U_i} = dg_i$ for some smooth function g_i on U_i . For a k -form β on U_i , we then have $(d + \wedge \alpha)(\beta) = 0$ if and only if $d(e^{g_i} \beta) = 0$. This shows that the twisted de Rham complex $(\mathcal{A}_{X, \mathbb{C}}^*, d + \wedge \alpha)$ is exact in positive degrees and it resolves a sheaf $L(\alpha)$ whose sections above U_i are given by all smooth functions f with $d(e^{g_i} f) = 0$, i.e. $f = e^{-g_i} c$ for some constant $c \in \mathbb{C}$. Hence, $L(\alpha) \in \text{Loc}(X)$ is a local system with stalk \mathbb{C} on X . Moreover, $c_1(L(\alpha)) = 0$ because the cocycle $(g_i - g_j) \in \check{C}^1(\mathcal{U}, (\mathcal{A}_X^0)^\times)$ maps to zero in $H^2(X, \mathbb{Z})$ and so

$$L(\alpha) \in \text{Loc}^0(X), \tag{1}$$

as we want.

Since $L(\alpha)$ is resolved by the Γ -acyclic complex $(\mathcal{A}_{X,\mathbb{C}}^*, d + \wedge\alpha)$, we find that

$$H^k(X, L(\alpha)) = H^k((A^*(X, \mathbb{C}), d + \wedge\alpha)), \quad (2)$$

where $A^k(X, \mathbb{C}) = \Gamma(X, \mathcal{A}_{X,\mathbb{C}}^k)$. In view of (2), we can define the Novikov Betti numbers $b_i(\alpha)$ of α as follows, cf. [Pa87] or [Fa04]:

$$b_k(\alpha) := \dim_{\mathbb{C}} H^k(X, L(\alpha)).$$

A closed 1-form α on X is Morse if locally at each zero $x \in Z(\alpha)$ of α , $\alpha = dh$ for some Morse function h . If α is Morse, its Morse index at a zero x is defined as the Morse index of h and we denote by $m_i(\alpha)$ the number of zeros of α of Morse index i . The Novikov inequalities then state the following, see [Pa87, Theorem 1]:

Theorem 2.1 (Novikov's inequalities). *Let X be a closed manifold and let α be a closed 1-form on X . Suppose that α is Morse in the above sense. Then for sufficiently large $t \in \mathbb{R}$, $m_i(\alpha) \geq b_i(t\alpha)$. In particular, if α has no zeros, then for $t \gg 0$,*

$$H^i(X, L(t\alpha)) = 0 \quad \text{for all } i.$$

2.2. Local systems associated to holomorphic 1-forms. Let now X be a compact Kähler manifold. For any holomorphic 1-form ω on X , ω is closed and so we get a local system $L(\omega)$ as above. This induces a short exact sequence

$$0 \longrightarrow H^0(X, \Omega_X^1) \longrightarrow \text{Loc}^0(X) \longrightarrow \text{Pic}^0(X) \longrightarrow 0, \quad (3)$$

where $\text{Loc}^0(X) \rightarrow \text{Pic}^0(X)$ is given by $L \mapsto L \otimes_{\mathbb{C}} \mathcal{O}_X$.

Lemma 2.2. *Let X be a compact Kähler manifold and let $\omega \in H^0(X, \Omega_X^1)$ be a holomorphic 1-form. Let $c \in \mathbb{Z} \cup \{\infty\}$ be maximal such that*

$$H^{i-1}(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^i(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^{i+1}(X, \mathbb{C})$$

is exact for all $i < c$. Then the local system $L(\omega)$ associated to ω satisfies $H^i(X, L(\omega)) = 0$ for all $i < c$. Moreover, if $c \neq \infty$, then $H^c(X, L(\omega)) \neq 0$.

Proof. The local system $L(\omega)$ is resolved by the following complex

$$(\Omega_X^*, \partial + \wedge\omega) := 0 \longrightarrow \Omega_X^0 \xrightarrow{\partial + \wedge\omega} \Omega_X^1 \xrightarrow{\partial + \wedge\omega} \dots \Omega_X^{n-1} \xrightarrow{\partial + \wedge\omega} \Omega_X^n \longrightarrow 0.$$

To see that this complex is exact in positive degrees, one uses that locally $\omega = dh$ and so for any local holomorphic form β , we have $de^h\beta = e^h(d\beta + dh \wedge \beta)$ and so $\partial\beta + \omega \wedge \beta = 0$ if and only if $de^h\beta = 0$ and we can use the holomorphic Poincaré lemma to prove the claim. Hence,

$$H^i(X, L(\omega)) = \mathbb{H}^i(X, (\Omega_X^*, \partial + \wedge\omega)).$$

There is a spectral sequence

$$'E_1^{p,q} := H^p(X, \Omega_X^q) \Rightarrow \mathbb{H}^{p+q}(X, (\Omega_X^*, \partial + \wedge\omega)).$$

The differential $d_1 : 'E_1^{p,q} \rightarrow 'E_1^{p,q+1}$ is induced by $\partial + \wedge\omega$. Since ∂ acts trivially on $'E_1^{p,q} := H^p(X, \Omega_X^q)$, we find that $d_1 = \wedge\omega$. It thus follows from [GL87, Proposition 3.7] that the above spectral sequence degenerates at the second page, i.e. $'E_2 = 'E_\infty$.

Our assumption implies $'E_2^{p,q} = 0$ for $p + q < c$ and so $H^i(X, L(\omega)) = 0$ for $i < c$. Let us now assume $c \neq \infty$. By the definition of c ,

$$H^{c-1}(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^c(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^{c+1}(X, \mathbb{C})$$

is not exact. Since $\omega \in H^{1,0}(X)$ is of type $(1,0)$, the above complex respects the Hodge decomposition and so we find that there must be some j such that

$$H^{j-1, c-j}(X) \xrightarrow{\wedge\omega} H^{j, c-j}(X) \xrightarrow{\wedge\omega} H^{j+1, c-j}(X)$$

is not exact. Hence $'E_2^{j, c-j} \neq 0$. Since $'E_2^{j, c-j} = 'E_\infty^{j, c-j}$, we get $H^c(X, L(\omega)) \neq 0$, as we want. This concludes the lemma. \square

2.3. Proof of Theorem 1.2. Let X be a compact Kähler manifold which admits a real closed one-form α without zeros, i.e. condition (B) in Conjecture 1.1 holds. Since the pullback of α via a finite étale cover is again a real closed one-form without zeros, in order to prove (C), it suffices to show that X carries a holomorphic one-form ω such that $\wedge\omega$ is exact on cohomology. For this, we may without loss of generality assume that X is connected.

Since α has no zero on X , Theorem 2.1 implies that there is a local system $L \in \text{Loc}^0(X)$ that has no cohomology. By the generic vanishing theorems [GL87, GL91, Ar92, Si93], the locus of those local systems that have some cohomology are subtori, translated by torsion points, see [Wa16, Theorem 1.3]. It follows that for general $\omega \in H^0(X, \Omega_X^1)$, the local system $L(\omega)$ has no cohomology. It thus follows from Lemma 2.2 that

$$H^{i-1}(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^i(X, \mathbb{C}) \xrightarrow{\wedge\omega} H^{i+1}(X, \mathbb{C})$$

is exact for all i , as we want. This finishes the proof of Theorem 1.2.

Remark 2.3. *Botong Wang points out that one can bypass the use of Theorem 2.1 in the above argument by showing directly that if X is a C^∞ -fibre bundle over the circle, then the pullback of a general local system on the circle has no cohomology on X .*

Remark 2.4. *Let X be a compact connected Kähler manifold. As we have used above, the results in [GL87] imply that $(H^*(X, \mathbb{C}), \wedge\omega)$ is exact if and only if $L(\omega)$ has no cohomology. The locus of such local systems is well understood by generic vanishing theory. In particular, [Wa16, Theorem 1.3] implies that the locus of those holomorphic*

one-forms $\omega \in H^0(X, \Omega_X^1)$ for which $(H^*(X, \mathbb{C}), \wedge\omega)$ is not exact is a finite union of linear subspaces of the form $f_i^* H^0(T_i, \Omega_{T_i}^1)$, where $f_i : X \rightarrow T_i$ is a finite collection of holomorphic maps to complex tori T_i . As a special case we see that if there is one holomorphic one-form ω on X which makes $(H^*(X, \mathbb{C}), \wedge\omega)$ exact, then this holds for all forms in a non-empty Zariski open subset of $H^0(X, \Omega_X^1)$.

3. THE CASE OF SURFACES

Proof of Theorem 1.3. Let X be a compact Kähler surface with a one-form $\omega \in H^0(X, \Omega_X^1)$ such that for any finite étale cover $\tau : X' \rightarrow X$,

$$H^{i-1}(X', \mathbb{C}) \xrightarrow{\wedge\omega'} H^i(X', \mathbb{C}) \xrightarrow{\wedge\omega'} H^{i+1}(X', \mathbb{C}) \quad (4)$$

is exact for all i , where $\omega' := \tau^*\omega$. This implies $\chi(X, \Omega_X^p) = 0$ for all p and so $c_2(X) = 0$.

Replacing X by its connected components, we may without loss of generality assume that X is connected. The classification of surfaces (see [BHPV04, Chapter VI.1]) thus shows that only the following cases occur.

Case 1. X is birational to a ruled surface over a curve C of positive genus.

Case 2. X is a minimal bi-elliptic surface or a complex 2-torus.

Case 3. X is a minimal properly elliptic surface.

In Case 1, exactness of (4) implies that X is birational to a ruled surface over an elliptic curve C . This implies $b_1(X) = 2$. Since $e(X) = 0$, we conclude $b_2(X) = 2$ and so X is a minimal ruled surface over an elliptic curve. In particular, since ω is nonzero, it must be a holomorphic one-form without zeros.

In Case 2, any nontrivial holomorphic one-form on X has no zeros and so we are done because exactness of (4) implies $\omega \neq 0$, as before.

In Case 3, the condition $c_2(X) = 0$ implies by [BHPV04, Proposition III.11.4] that X admits a fibration $\pi : X \rightarrow C$ to a curve C such that the reduction of any fibre of π is isomorphic to a smooth elliptic curve, but where multiple fibres are allowed. Let F be a general fibre of $\pi : X \rightarrow C$. Suppose for the moment that the one-form ω restricts to a nonzero form on F . In particular, the Albanese map $a : X \rightarrow \text{Alb}(X)$ does not contract F and the reduction of any fibre of a is isomorphic to F . Moreover, the restriction of ω to F does not depend on the fibre and so it is nonzero everywhere. That is, ω has no zeros.

It remains to deal with the case where ω restricts to zero on the fibres of $\pi : X \rightarrow C$. In this case, $\omega = \pi^*\alpha$ for a one-form α on C . Since cup product with ω is exact, C must be an elliptic curve. If π is smooth, then ω has no zeros. Otherwise, ω vanishes along the multiple fibres of π . We may thus assume that π has at least one multiple fibre.

The multiple fibres of π give rise to a orbifold structure on C . Since C is an elliptic curve, this orbifold is good and so there is a finite orbifold covering $C' \rightarrow C$ such that

the orbifold structure on C' is trivial, see e.g. [CHK00, Corollary 2.29]. Let X' be the normalization of the base change $X \times_C C'$. Then, X' is a smooth surface, $X' \rightarrow X$ is étale and $X' \rightarrow C'$ is an elliptic surface without singular fibres, see e.g. [BHPV04, Proposition III.9.1]. Since $\tau : X' \rightarrow X$ is finite étale, $(H^*(X', \mathbb{C}), \wedge \omega')$ is exact for $\omega' := \tau^* \omega$ by assumptions. On the other hand, since π has singular fibres, $C' \rightarrow C$ is a branched covering with nontrivial branch locus and so C' is a curve of genus ≥ 2 . This is a contradiction, because ω' is a pullback of a one-form from C' . This finishes the proof of Theorem 1.3. \square

Corollary 3.1. *Let X be a compact connected Kähler surface with a holomorphic one-form ω such that $(H^*(X, \mathbb{C}), \wedge \omega)$ is exact. Then ω has no zeros and (X, ω) is given by one of the following:*

- (a) X is a minimal ruled surface over an elliptic curve;
- (b) X is a complex 2-torus;
- (c) X is a minimal elliptic surface $f : X \rightarrow C$ such that one of the following holds:
 - (i) f is smooth, C is an elliptic curve and $\omega \in f^* H^0(C, \Omega_C^1)$;
 - (ii) f is quasi-smooth, i.e. all singular fibres are multiple fibres, and the restriction of ω to the reduction of any fibre of f is nonzero.

Proof. The classification into types (a), (b) and (c) follows directly from the proof of Theorem 1.3, where we note that bi-elliptic surfaces fall in the class (ci). The fact that ω has no zeros follows from this classification. \square

Corollary 3.2. *In the notation of Corollary 3.1, assume that X is projective. Then,*

- (d) X admits a smooth morphism to a positive-dimensional abelian variety;
- (e) if $\kappa(X) \geq 0$, then there is a finite étale cover $\tau : X' \rightarrow X$ which splits into a product $X' = A' \times S'$, where A' is a positive-dimensional abelian variety and S' is smooth projective.

Proof. Note that item (d) is clear in cases (a), (b) and (ci) of Corollary 3.1. It remains to deal with case (cii). In this case, since X and hence $\text{Alb}(X)$ are projective, $\text{Alb}(X)$ is isogeneous to $E \times \text{Jac}(C)$, where E is an elliptic curve which is isogeneous to the reduction of any fibre of f . It follows that there is a morphism $g : X \rightarrow E$ which restricts to an isogeny on the reduction of each fibre of $f : X \rightarrow C$. Since ω restricts non-trivially to the reduction of any fibre of f , the morphism $g : X \rightarrow E$ must be smooth, as we want.

It clearly suffices to prove item (e) in the case (c) of Corollary 3.1. In this case, there is a finite étale cover $X' \rightarrow X$, such that $\text{Alb}(X') \cong E \times \text{Jac}(C')$ for a smooth projective curve C' which maps finitely to C . Moreover, the Albanese map identifies X' to the product $E \times C'$, as we want. This concludes the corollary. \square

4. PROOF OF THEOREM 1.4

4.1. **Preliminaries.** We will use the following lemma.

Lemma 4.1. *Let K^* be a bounded complex of sheaves on a manifold X . Let $Z, Z' \subset X$ be closed subsets with $Z \cap Z' = \emptyset$, such that*

$$\text{supp } \mathcal{H}^i(K^*) \subset Z \cup Z'$$

for all i . Then the differentials $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ in the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(K^*)) \Rightarrow \mathbb{H}^{p+q}(X, K^*)$$

respect the natural decompositions

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q(K^*)|_Z) \oplus H^p(Z', \mathcal{H}^q(K^*)|_{Z'}).$$

Proof. Let $i : Z \rightarrow X$ and $j : Z' \rightarrow X$ be the inclusions. Then the natural map of complexes

$$K^* \rightarrow i_* i^{-1} K^* \oplus j_* j^{-1} K^*$$

is a quasi-isomorphism. This proves the lemma, because the spectral sequence depends only on the class of K^* in the derived category of sheaves on X . \square

4.2. **Item (1) of Theorem 1.4.** Let X be a compact connected Kähler manifold and let ω be a holomorphic one-form on X with associated local system $L(\omega)$. Recall the isomorphism

$$H^k(X, L(\omega)) \cong \mathbb{H}^k(X, (\Omega_X^*, \omega \wedge -)).$$

The above hypercohomology is computed by a spectral sequence with E_2 -page

$$E_2^{p,q} := H^p(X, \mathcal{H}^q(K^*)) \Rightarrow H^{p+q}(X, L(\omega)), \quad (5)$$

where $K^* := (\Omega_X^*, \omega \wedge -)$ and $\mathcal{H}^q(K^*)$ denotes the q -th cohomology sheaf of that complex. In particular, $\mathcal{H}^q(K^*) = 0$ if $\omega \wedge -$ is exact on holomorphic q -forms and the latter holds if ω has no zeros. More precisely, this shows that $\mathcal{H}^q(K^*)$ are sheaves that are supported on the zero locus $Z(\omega)$ of ω .

Lemma 4.2. *We have $\mathcal{H}^n(K^*) \cong \Omega_X^n|_Z$.*

Proof. Locally $\omega = \sum_{i=1}^n f_i dx_i$. We are interested in the cokernel of

$$\Omega_X^{n-1} \rightarrow \Omega_X^n, \quad \alpha \mapsto \sum_{i=1}^n f_i dx_i \wedge \alpha.$$

The image of the above map is clearly spanned by $f_i dx_1 \wedge \cdots \wedge dx_n$ with $i = 1, \dots, n$. Hence, $\mathcal{H}^n(K^*)$ is the quotient of Ω_X^n by the subsheaf $I_Z \otimes_{\mathcal{O}_X} \Omega_X^n$, where I_Z denotes the ideal sheaf of Z . Hence,

$$\mathcal{H}^n(K^*) \cong \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{O}_Z = \Omega_X^n|_Z.$$

This proves the lemma. \square

Proof of item (1) in Theorem 1.4. Let $Z \subset Z(\omega)$ be a connected component of the zero locus of ω . Then we can write $Z(\omega) = Z \cup Z'$, where Z and Z' are disjoint closed subsets of X .

Consider the spectral sequence (5). By Lemma 4.2, we have

$$H^d(X, \Omega_X^n|_Z) \hookrightarrow E_2^{d,n}.$$

Using Lemma 4.1, one easily checks that this term survives on the infinity page and we get

$$H^d(X, \Omega_X^n|_Z) \hookrightarrow E_\infty^{d,n}.$$

By Lemma 2.2, exactness of $(H^*(X, \mathbb{C}), \wedge \omega)$ implies $H^i(X, L(\omega)) = 0$ for all i . Hence, $E_\infty^{d,n} = 0$, and so $H^d(Z, \Omega_X^n|_Z) = 0$, as we want. \square

Corollary 4.3. *Let X be a compact Kähler manifold and let $\omega \in H^0(X, \Omega_X^1)$ such that the complex $(H^*(X, \mathbb{C}), \wedge \omega)$ given by cup product with ω is exact. Let $Z \subset Z(\omega)$ be a connected component of the zero locus of ω , and let $d = \dim Z$. Then*

$$H^d(Z', \omega_X|_{Z'}) = 0,$$

for any irreducible component Z' of the reduced scheme Z^{red} .

Proof. Consider the long exact sequence, associated to the short exact sequence

$$0 \longrightarrow \omega_X|_Z \otimes \mathcal{I}_{Z'} \longrightarrow \omega_X|_Z \longrightarrow \omega_X|_{Z'} \longrightarrow 0.$$

By item (1), $H^d(Z, \omega_X|_Z) = 0$. Moreover, $H^{d+1}(Z, \omega_X|_Z \otimes \mathcal{I}_{Z'}) = 0$ because of dimension reasons. This implies $H^d(Z', \omega_X|_{Z'}) = 0$, as we want. \square

Corollary 4.4. *Let X be a compact Kähler manifold with a holomorphic map $f : X \rightarrow A$ to a complex torus A . Let $\omega \in H^0(A, \Omega_A^1)$ such that the complex $(H^*(X, \mathbb{C}), \wedge f^*\omega)$ given by cup product with $f^*\omega$ is exact.*

Then the restriction of ω to $f(X) \subset \text{Alb}(X)$ does not vanish at a point $y \in f(X)$ such that the fibre $F := f^{-1}(y)$ is smooth with trivial normal bundle (the locus of such points $y \in f(X)$ is Zariski dense in $f(X)$).

Proof. Assume that ω vanishes at a point $y \in f(X)$ such that the fibre $F := f^{-1}(y)$ is smooth with trivial normal bundle. Then $F \subset Z(f^*\omega)^{\text{red}}$ is a connected component. This contradicts Corollary 4.3, because

$$H^{\dim F}(F, \omega_X|_F) = H^{\dim F}(F, \omega_F) \neq 0,$$

by Serre duality, where we used that F has trivial normal bundle. \square

4.3. Item (2) of Theorem 1.4. Let $f : X \rightarrow A$ be a holomorphic map to a complex torus A and assume that there is a one-form $\omega \in f^*H^0(A, \Omega_A^1)$ such that $(H^*(A, \mathbb{C}), \wedge\omega)$ is exact. Since exactness is an open property, $(H^*(A, \mathbb{C}), \wedge\omega')$ is exact for any general $\omega' \in f^*H^0(A, \Omega_A^1)$.

Let $Y := f(X)$ and fix a general point $y \in Y$. There are countably many non-trivial linear subspaces

$$\{0\} \neq W_i \subset T_{A,y}$$

such that there is a morphism of complex tori $\pi_i : A \rightarrow B_i$ with $\ker((d\pi_i)_y) = W_i$.

For a contradiction, we assume that Y is not fibred by tori. This implies that the tangent space $T_{Y,y}$ does not contain any of the W_i . We may thus choose a one-form $\omega' \in H^0(A, \Omega_A^1)$, such that ω' vanishes on $T_{Y,y} \subset T_{A,y}$, but which is non-trivial on each W_i . Let $Z \subset Z(\omega')$ be an irreducible component which contains y . Then ω' vanishes on Z and hence on the subtorus $\langle Z \rangle \subset A$, generated by Z . If Z was positive-dimensional, then $T_{\langle Z \rangle, y} = W_i$ for some i , which contradicts the fact that ω' does not vanish on W_i . Hence, Z is zero-dimensional and so y is an isolated zero of $\omega'|_Y$. But this implies that a small perturbation of $\omega'|_Y$ has an isolated zero in some neighbourhood of y . Hence, a general one-form $\omega \in H^0(A, \Omega_A^1)$ has the property that $Z(\omega|_Y)$ contains a general point of Y as a connected component. This contradicts Corollary 4.4, which finishes the proof.

APPENDIX, WRITTEN JOINTLY WITH HSUEH-YUNG LIN

In this appendix we prove the following.

Theorem A.1. *Let X be a compact connected Kähler manifold. Assume that $\omega \in H^0(X, \Omega_X^1)$ satisfies condition (C). Then $\dim Z(\omega) \leq \dim X - 2$.*

By Theorem 1.4, we also have $1 \leq \dim Z(\omega)$. If $\dim X = 2$, the above theorem thus implies $Z(\omega) = \emptyset$, which yields a new proof of Theorem 1.3, without using the Enriques-Kodaira classification.

We start with the following auxiliary result; the same argument appeared in the last two paragraphs in the proof of Theorem 1.3, as well as in [HS19, Proposition 6.4].

Lemma A.2. *Let X be a compact connected Kähler manifold with a morphism $f : X \rightarrow E$ to an elliptic curve E with irreducible fibres. Assume that there is a one-form $\alpha \in H^0(E, \Omega_E^1)$ such that $\omega := f^*\alpha$ satisfies condition (C). Then f has reduced fibres.*

Proof. Let Δ be the set of points $t \in E$ such that $f^{-1}(t)$ is a multiple fibre and let m_t be its multiplicity. This gives rise to an orbifold structure on E . Since E is an elliptic curve, this orbifold structure is good (see e.g. [CHK00, Corollary 2.29]) and so there is a finite cover $C \rightarrow E$ which locally above each point of $t \in \Delta$ is ramified of order m_t . A local computation shows that the normalization \tilde{X} of $X \times_E C$ is étale over X , cf. [BHPV04,

Proposition III.9.1]. There is a natural map $\tilde{f} : \tilde{X} \rightarrow C$ and our assumptions imply that there is a one-form $\omega \in H^0(C, \Omega_C^1)$ such that $(H^*(\tilde{X}, \mathbb{C}), \wedge \tilde{f}^* \omega)$ is exact. This implies $g(C) = 1$ and so $\Delta = \emptyset$, as we want. \square

Proof of Theorem A.1. Assume for the contrary that there is a prime divisor $D \subset Z(\omega)$. Let $f : X \rightarrow A$ be a morphism to a complex torus such that $\omega = f^* \alpha$ for some $\alpha \in H^0(A, \Omega_A^1)$, and assume that $\dim A$ is minimal with that property.

Since $\omega|_D = 0$, we have $\alpha|_{\langle f(D) \rangle} = 0$, where $\langle f(D) \rangle \subset A$ denotes the subtorus generated by $f(D)$. Hence, ω is the pullback of a one-form from $A/\langle f(D) \rangle$. Minimality of $\dim A$ thus shows that $f(D)$ is a point. It then follows from [HS19, Lemma 2.4] that A is an elliptic curve. Moreover, up to replacing f by its Stein factorization, we may by [HS19, Corollary 2.5] assume that all fibres of f are irreducible. Hence, f has reduced fibres by Lemma A.2. Since A is an elliptic curve, $Z(\omega)$ is contained in the singular locus of f , which has codimension at least two, because the fibres of f are reduced. This is a contradiction, which concludes the theorem. \square

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