QFT in curved space-time

1 Introduction

1.1 The Einstein-Hilbert action (31.01.2010)

Before we turn to quantum field theory in curved space-times, we shortly review the classical theory of gravity. Einstein's equations of classical general relativity can be derived from the Einstein-Hilbert action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g}(R - 2\Lambda).$$

with R the Ricci curvature and the cosmological constant Λ. Under a variation δg_{ab}, the variation of the action is given by:

$$\delta S = \frac{1}{16\pi G} \int d^4x \left(\delta \left(\sqrt{g}\right) (R - 2\Lambda) + \sqrt{g}\delta \left(R_{ab}\right) g^{ab} + \sqrt{g} R_{ab} \delta\big(g^{ab}\big) \right).$$

To analyze this expression, use:

$$\begin{aligned}
\delta g &= \det \left(g_{ab} + \delta g_{ab}\right) - \det g_{ab} = \det g_{ab} \left[\det \left({g^b}_c + \delta {g^b}_c \right) - 1\right] = \\
&= \det g_{ab} \cdot \mathrm{Tr} \delta {g^b}_c = g g^{ab} \delta g_{ab} = -g g_{ab} \delta g^{ab} \\
g^{ab} \delta R_{ab} &= g^{ab} \left(\partial_c \delta \Gamma^c_{ab} - \partial_b \delta \Gamma^c_{ac} + \Gamma \delta \Gamma\right) = g^{ab} \left(\nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ac}\right) \\
&= \nabla_c \left(g^{ab} \delta \Gamma^c_{ab} - g^{ab} \delta \Gamma^c_{bc} \right)
\end{aligned}$$

For the last derivation, we used normal coordinates around some point p, i.e. $\Gamma|_p = 0$ to get rid of terms quadratic in the Christoffel symbols. Next, we replaced the partial derivatives by covariant derivatives for the same reasons. Since the equation is now tensorial again, it holds everywhere. In the last step, we used the property of the Levi-Cività connection, $\nabla_a g^{bc} = 0$. A slightly different derivation is given in [15], appendix E.
The total derivative contributes only a boundary term, which can be eliminated adding an additional piece $2\int_{\partial M} K$ where K is the extrinsic curvature, $K = {h^a}_b \nabla_a n^b$, where $h_{ab} = g_{ab} \pm n_a n_b$ is the induced metric on the boundary, and n^a is a normal vector on the boundary. Ignoring these subtleties (for example for a manifold without boundary), we get:

$$\delta S = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \right) \delta g^{ab}$$

resulting in Einstein's equations in vacuum after setting the variation to zero.
To add matter simply add another term to the action, p.e. a field strength term of Yang-Mills gauge theory or electromagnetism:

$$S_M = \int d^4x \sqrt{g} \mathcal{L}_m = \frac{1}{4g^2} \int \mathrm{Tr}\, F \wedge \star F = \frac{1}{4g^2} \int d^4x \sqrt{g}\, \mathrm{Tr}\, F^{ab} F_{ab}.$$

Define the energy-momentum tensor by $T_{ab} = \frac{2}{\sqrt{g}} \frac{\delta\left(\sqrt{g}\mathcal{L}_m\right)}{\delta g^{ab}}$ and get Einstein's equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}.$$

1.2 Attempts to quantize general relativity

General relativity is a classical field theory, with metric field g_{ab}, which plays a dual rôle, both being the dynamical variable in the game, and describing the background. Over the last decades there were various attempts to find a quantum theory of gravity. One of the major lines are the following:

1. Covariant perturbation method: $g_{ab} = g_{ab}^{(0)} + h_{ab}$, where the first part solves the classical Einstein equations. Write down Feynman rules. The problem of this attempt is that the perturbation theory is non-renormalizable. Let's try to understand this.
   In natural units $c = \hbar = 1$, the action is dimensionless. In dimension n, R has dimension L^{-2}, while the measure contributes L^n, so Newton's constant G_N has dimension L^{n-2}. In fact, $G_N = l_p^{n-2}$ in these units, where l_p is the n dimensional Planck length.
   So our coupling parameter is:
   $$\frac{1}{\kappa^2} = \frac{1}{16\pi G_N} = \frac{1}{16\pi l_p^{n-2}} = \frac{M_p^{n-2}}{16\pi}$$
   with Planck mass M_p. So at a distance X-times smaller (or at momentum X-times larger), G_N seems X^{n-2}-times larger, so $G_N \sim p^{n-2}$ and the critical dimension is 2. So in 4d, gravity seems non-renormalizable, i.e. gets stronger at shorter distances. In 4 dimensions $G_N = l_p^2$.
   Pure GR is finite to one-loop order (i.e. the divergences cancel due to an analogue of the Ward identity) but not in higher orders or if coupled to matter.
   Next step: supergravity. Make the theory supersymmetric, add a gravitino (spin 3/2), to cancel some divergences. Highest dimension (with one time dimension) is 11, so this is the most natural setup. Get solutions in 4 dimensions by dimensional reduction (like in Kaluza-Klein theories).
   Finally: Perturbative String theory.

2. Canonical quantization method: Wheeler-DeWitt equation: $\widehat{H}\,|\Psi\rangle = 0$. Background independance as a principle.
   This approach leads to Loop Quantum Gravity after introducing Ashtekar's variables.

3. Path integral method: Try a generating functional:
   $$Z = \int \mathcal{D}\left[g_{ab}\right] \mathrm{e}^{-S}$$
   with the Einstein-Hilbert action S as above. It is not known how to define a measure on the moduli space of metrics. Another problem is again non-renormalizability. In LQG, the moduli space is reduced, one ends up with the spin foam formulation.

4. Alternate routes: Twistor theory, gravitational OR (Roger Penrose), non-commutative geometry (Alain Connes).

1.3 One step back: QFT in curved space-time

Lacking a full theory of gravity, let's see how far we get without it. We saw above, that (reinserting $\hbar$) the Einstein Hilbert action in four dimensions is proportional to $\frac{\hbar}{l_p^2}$. Looking at the formal path integral

$$\int D[g_{ab}]\,\mathrm{e}^{-S} = \int D[g_{ab}]\,\mathrm{e}^{-\frac{\hbar}{l_p^2}\int R}$$

it seems that quantum effects of gravity become important only at the order of the Planck length. A first step towards quantum electrodynamics is to consider matter in a classical background of an electromagnetic field. In analogy, we might hope to get some insight to quantum effects of gravity by considering quantized fields in a classical background, i.e. in a classical curved space. Still there are some issues with this idea. The principle of general covariance states that matter and mass couple equally strongly to gravity.

Nevertheless, if we consider for example a free field theory, only one-loop 1PI diagrams are possible (lacking interaction vertices). Up to order $\hbar$, we can therefore truncate gravity also to first-loop order.

For an interacting theory of matter fields, higher loops of these fields are possible, so we should also better include higher graviton loops. Recall the very different nature of the coupling constant in, say, electromagnetism, and in Einstein gravity. In the first case, it is given by the dimensionless fine-structure constant $\alpha \simeq \frac{1}{137}$, while for gravity, the coupling is dimensionfull, being proportional to the Planck length. While one single graviton loop is of zeroth order in G_N, higher loops come from graviton interactions, contributing powers of G_N. Hence for length scales l satisfying $\frac{G_N}{l^2} = \frac{l_p^2}{l^2} \ll \alpha = \frac{e^2}{4\pi}$, quantum effects of matter dominate, and we can truncate gravity at one-loop order. It seems that this should give reasonable results, though there is still much uncertainty about the domain of validity.

To sum up, there are several steps forward towards a quantum theory of gravity:

1. Quantum field theory in curved space-time: the background space-time is classical, meaning we work in zeroth order in $\hbar$. We ignore the back-reaction of the matter on space-time.

2. Semi-classical gravity: still treat the background as classical, but now take the back-reaction into account. One important element at this stage is the expectation value of the energy-momentum tensor (being quantized) in some state ψ. It contributes to Einstein's equation:
$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G \left\langle \hat{T}_{ab} \right\rangle_\psi .$$

3. Include also higher graviton loops.

We concentrate here on the first two stages, which already lead to interesting results, like particle creation, Unruh effect, and Hawking radiation.

2 Particle creation in a time-varying background (01.02.2010)

This subsection follows [15].

To do perturbative QFT (at least in the classical fashion), one needs a notion of particle states. Given such states, the S-matrix yields probabilities for a transition between a certain in-state as an element of some Fock space into an out-state in another Fock space.

In particular, we want to define a vacuum state (a state without particle excitations). The notion of a particle is no longer automatic in a curved background. As we will see later on, different observers might well disagree over the number of excitations. Only under certain assumptions, it is possible to define meaningful particle states. Possible setups are:

- asymptotically flat space-time, or
- non-vanishing curvature only in a compact region, or
- globally hyperbolic (space-time possesses a Cauchy hypersurface), or
- asymptotically stationary space-time (time-like Killing vector field gives rise to a split into positive and negative frequencies, see below), or
- Feynman propagator (i.e. Green's function) exists (propagating positive frequency in-states into positive frequency out-states).

To define positive and negative frequencies, we want a positive-definite inner product. For concreteness, consider a Klein-Gordon field. There exists a conserved current, which gives rise to an inner product, the so-called Klein-Gordon product. We will come back to details later on. If we restrict to fields of positive frequency (the Fourier transform $\widetilde{\Phi}(\omega, \vec{x})$ vanishes for $\omega < 0$), the product is positive-definite. The Hilbert space of possible states is given by the symmetric Fock space $\mathfrak{F}_S(\mathcal{H})$ with vacuum state $|0\rangle = (1, 0, 0, \ldots)$, where the i-th component in the infinite dimensional vector is in $\bigodot^{i-1} \mathcal{H}$. The field operator can now be given (in a distributional sense) by

$$\hat{\Phi}(x) = \sum_i \left(v_i^*(x) \hat{a}^- + v_i^*(x) \hat{a}^+ \right)$$

where a^+, a^- are creation and annihilation operators, and v_i denotes an orthonormal basis of $\mathcal{H}$. Specify Hilbert spaces $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$. The S-matrix describes, how a state evolves, p.e. for an in-state $|0\rangle \in \mathfrak{F}_S(\mathcal{H}_{\text{in}})$, the out-state $S|0\rangle \in \mathfrak{F}_S(\mathcal{H}_{\text{out}})$ tells about spontaneous creation of particle due to a time-dependant gravitational field. The latter state can be calculated. It vanishes for an odd tensor product of Hilbert spaces, hence particles can only be created in pairs. Particle creation only occurs, if some negative-frequency part is picked up.

3 Quantum fluctuations (13.02.2010)

We will from now on closely follow Mukhanov's book [13].
Let's start with an harmonic oscillator in a flat background with Hamiltonian $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$.
The classical equations of motion are $\ddot{x} + \omega^2 x = 0$.
The Schrödinger equation is: $\left(-\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2}{2}\right)|\psi\rangle = E|\psi\rangle$. Solve for the wavefunction:

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right), \qquad n = 0, 1, 2, \ldots.$$

In particular, the vacuum state is given by[1]:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right). \tag{3.1}$$

[1] alternative derivation: $a|0\rangle = 0$, where $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$. This leads to: $x\psi_0(x) + \frac{\hbar}{m\omega}\frac{d\psi_0(x)}{dx} = 0$.

The vacuum state has expectation value $\langle x \rangle = 0$. Nevertheless, there is a fluctuation $\delta q \sim \sqrt{\frac{\hbar}{m\omega}}$.
Now consider the field theory case. Here we have "one oscillator at each point". In momentum space: $\ddot{\Phi}_{\mathbf{k}} + (k^2 + m^2)\Phi_{\mathbf{k}} = 0$, leading to a wave functional[2]:

$$\Psi[\Phi] \sim \exp\left(-\frac{1}{2}\int d^3\mathbf{k}\ |\Phi_{\mathbf{k}}|^2\,\omega_{\mathbf{k}}\right).$$

In analogy, there occur vacuum fluctuations

$$\delta\Phi_{\mathbf{k}} = \sqrt{\langle|\Phi_{\mathbf{k}}|\rangle} \sim \frac{1}{\sqrt{\omega_{\mathbf{k}}}}.$$

If we consider fluctuations of a field in a box with length L, the fluctuations occur at $k \sim \frac{1}{L}$ and are given by:

$$\delta\Phi_L \sim \sqrt{(\delta\Phi_{\mathbf{k}})^2\,k^3} \sim \begin{cases} L^{-1} & \text{for } L \ll \frac{1}{m} \\ L^{-3/2} & \text{for } L \gg \frac{1}{m} \end{cases}.$$

One consequence of these fluctuations is the Casimir effect (see below).

4 Driven harmonic oscillator

In QFTCS, a time-varying background geometry leads to particle excitations. A lower-dimensional analogy is a harmonic oscillator coupled to a time-varying source term. Take the Lagrangian:

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t)q.$$

Assume further that $J(t)$ is non-vanishing for $t \in [0, T]$ only[3]. Now a^- satisfies:

$$\dot{a}^- = -i\omega a^- + \frac{i}{\sqrt{2\omega}}J(t),$$

which can be integrated. If we define

$$J_0 = \frac{i}{\sqrt{2\omega}}\int_0^T dt\ \mathrm{e}^{i\omega t}\,J(t),$$

a straight forward calculation shows, that the occupation number operator has expectation values[4]:

$$\langle 0_{\text{in}}|\hat{N}(t)|0_{\text{in}}\rangle = |J_0|^2$$
$$\langle 0_{\text{out}}|\hat{N}(t)|0_{\text{in}}\rangle = 0.$$

Hence the energy expectation value gets shifted due to the interactions:

$$\langle 0_{\text{in}}|\hat{H}(t)|0_{\text{in}}\rangle = \left(\frac{1}{2} + |J_0|^2\right)\omega.$$

[2] compare equation (3.1), with $m = 1$, $\hbar = 1$, $\omega \to \omega_{\mathbf{k}}$, $x \to \Phi_{\mathbf{k}}$.
[3] This corresponds later on to a time-dependant gravitational field, which is stationary in the beginning and the end, to get notions of particles.
[4] We are in the Heisenberg picture. If we start with the vacuum $|0_{\text{in}}\rangle$, the state stays in $|0_{\text{in}}\rangle$. So the first line indicates an exictation of the vacuum. However, after the interaction, the true vacuum is given by $|0_{\text{out}}\rangle$. This is reflected in the second equation.

5 Quantization of fields

Now we turn to quantum field theory. As stated above, there is "one harmonic oscillator at each point":

$$S[\phi] = \frac{1}{2}\int d^4x\, \dot{\phi}^2 - \int d^4x\, d^4y\, \phi(\mathbf{x})M(\mathbf{x},\mathbf{y})\phi(\mathbf{y}).$$

M gets diagonalized to $M(\mathbf{x},\mathbf{y}) = \left(-\Delta_{\mathbf{x}} + m^2\right)\delta(\mathbf{x}-\mathbf{y})$. Due to the Laplacian, the oscillators are coupled. We can decouple them by a Fourier transformation. Next we perform a mode expansion in eigenfunctions in momentum space.
The quantized solution for the field operator $\hat{\phi}$ is given by:

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left(v^*_{\mathbf{k}}(t)\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{a}^-_{\mathbf{k}} + v_{\mathbf{k}}(t)\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{a}^+_{\mathbf{k}}\right), \tag{5.1}$$

where the mode functions satisfy:

$$\ddot{v}_{\mathbf{k}} + \omega_k^2(t)v_{\mathbf{k}} = 0.$$

In the case of a free field in flat space, one gets:

$$v_{\mathbf{k}}(t) = \frac{1}{\sqrt{\omega_k}}\,\mathrm{e}^{i\omega_k t}, \tag{5.2}$$

with $\omega_k = \sqrt{k^2+m^2}$. In particular, the mode functions are isotropic (independent of the direction of $\mathbf{k}$).
The vacuum energy is in general infinite. Each point is an oscillator by itself and therefore contributes $\frac{1}{2}\omega_{\mathbf{k}}$. In the free field case in flat space, one can renormalize the Hamiltonian by imposing a normal-ordering condition, thus substracting the vacuum energy.
Another way to understand the appearence of mode functions goes as follows (see [6]):

1. Start with a Lagrangian, p.e.

   $$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2\right)$$

   $\xi = 0$ gives the standard Klein-Gordon equation and therefore easy equations of motion. For $\xi = 0$ the field is conformally coupled, so the action is invariant under $g_{\mu\nu} \to \mathrm{e}^\lambda\, g_{\mu\nu}$. For the special case of conformally flat metrics (like FLRW metrics), the field decouples from gravity, since its action is equivalent to an action in flat space.

2. Derive the field equations[5]:

   $$\Box\phi + m^2\phi + \xi R\phi = 0.$$

   Define an inner product on solutions. To this end, pick a Cauchy surface Σ with normal unit vector $\mathbf{n}^\mu$, and set:

   $$(v_1, v_2) = i\int v_2^*\,\overleftrightarrow{\partial_\mu}\,v_1\,d\Sigma\,\mathbf{n}^\mu = i\int\left(v_2^*\partial_\mu v_1 - v_1\partial_\mu v_2^*\right)d\Sigma\,\mathbf{n}^\mu,,$$

   where v_1 and v_2 are solutions to the equations of motion.
   By the equations of motion, the definition does not depend on the specific choice of

[5] Here $\Box = \nabla^\mu\nabla_\mu$. For scalar fields: $\Box\phi = \nabla^\mu\partial_\mu\phi$.

Cauchy surface. Therefore, we get a well-defined inner product, the Klein-Gordon product.

Note that the Klein-Gordon Product is a generalization of the Wronksian

$$\Im(v_k' v_k^*) = \frac{v_k' v_k^* - v_k'^* v_k}{2i} = \frac{W[v_k, v_k^*]}{2i}.$$

3. Now proceed with canonical quantization:

$$\left[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)\right] = i\delta(\mathbf{x}', \mathbf{x}), \text{ where}$$
$$\pi = \frac{\delta L}{\delta \dot{\phi}},$$
$$\int \delta(\mathbf{x}', \mathbf{x})\, d\Sigma = 1.$$

If (v_i) is a complete set of solutions of positive norm ("positive energy" in the flat case), then as in conventional QFT, ϕ gets operator valued, and can be expanded as:

$$\hat{\phi} = \sum_i \left(v_i^* \hat{a}_i^- + v_i \hat{a}_i^+\right),$$

where a^+, a^- are creation and annihilation operators. In the non-discrete case, we are back in (5.1).

The problem of curved space-times is the ambiguity in a choice of basis v_i. While in the flat case, one picks positive frequency solutions (5.2), there is no canonical choice in the curved case.

5.1 Fields in FLRW models

Now consider a field ϕ with Lagrangian ($\xi = 0$ above)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$$

in a FLRW universe with metric

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = a^2(\eta)\left(dt^2 - d\mathbf{x}^2\right).$$

It is equivalent to a field $\chi = a\phi$ in flat spacetime, where χ obeys the equation

$$\chi'' - \Delta\chi + m_{\text{eff}}^2 \chi = 0,$$

where $m_{\text{eff}}^2 = a^2 m^2 - \frac{a''}{a}$ and the prime denotes taking the derivative with respect to the conformal time η. For ξ non-zero, $m_{\text{eff}}^2 = a^2\left[m^2 + \left(\xi - \frac{1}{6}\right) R\right]$.

Again, the oscillator decouple after a Fourier transformation. Now the mode functions satisfy

$$v_k'' + \omega_k^2(\eta) v_k,$$

with $\omega_k(\eta) = \sqrt{k^2 + m_{\text{eff}}^2(\eta)}$. This equation has two linear independent solutions, which we interpret as a complex solution v_k. Making use of the Klein-Gordon product, we normalize:

$$(v_k, v_{k'}) = \delta_{kk'}.$$

5.2 Bogolyubov transformations

For general manifolds, two isotropic mode functions are related via

$$v_k^* = \alpha_k u_k^* + \beta_k u_k,$$

with Bogolyubov coefficients α_k, β_k. Normalization is kept, if $|\alpha_k|^2 - |\beta_k|^2 = 1$. The operators are related via the Bogolyubov transformation:

$$\begin{aligned} \hat{b}_{\mathbf{k}}^- &= \alpha_k \hat{a}_{\mathbf{k}}^- + \beta_k^* \hat{a}_{-\mathbf{k}}^+, \\ \hat{b}_{\mathbf{k}}^+ &= \alpha_k^* \hat{a}_{\mathbf{k}}^+ + \beta_k \hat{a}_{-\mathbf{k}}^-. \end{aligned}$$

The b vacuum can be expressed in terms of the a vacuum:

$$|0_b\rangle = \left[\prod_{\mathbf{k}} \frac{1}{\sqrt{|\alpha_{\mathbf{k}}|}} \exp\left(-\frac{\beta_{\mathbf{k}}^*}{2\alpha_{\mathbf{k}}} \hat{a}_{\mathbf{k}}^+ \hat{a}_{-\mathbf{k}}^+ \right) \right] |0_a\rangle .$$

The product converges, if $|\beta_{\mathbf{k}}|^2 \to 0$ faster than $\frac{1}{k^3}$. This is the mean density of b-particles in the mode $\chi_{\mathbf{k}}$ in the a-vacuum, since $\langle 0_a|\hat{N}_b|0_a\rangle = |\beta_{\mathbf{k}}|^2 \delta^{(3)}(0)$. The complete mean density of b-particles is $\int d^3\mathbf{k}\, |\beta_k|^2$ which converges under the same condition as above.

5.3 Choice of vacua

One possible vacuum is the instantaneous lowest energy vacuum: minimize the energy at a given instant of time η_0. A solution exists, if $\omega_k^2(\eta_0) > 0$. The initial conditions for the mode functions in this case are[6]:

$$\begin{aligned} v_{\mathbf{k}}(\eta_0) &= \frac{1}{\sqrt{\omega_k(\eta_0)}}, \\ v_{\mathbf{k}}'(\eta_0) &= i\sqrt{\omega_k(\eta_0)}, \end{aligned}$$

which turn out to be isotropic. The Hamiltonian is:

$$\hat{H}(\eta_0) = \int d^3\mathbf{k}\, \omega_k(\eta_0) \left(\hat{a}_{\mathbf{k}}^+ \hat{a}_{\mathbf{k}}^- + \frac{1}{2}\delta^{(3)}(0) \right).$$

$|0_{\eta_0}\rangle$ is the vacuum of instantaneous diagonalization, since the Hamiltonian is diagonal in its exited states. At later times, the vacuum usually turns into a superposition of excited states. In general, there is no unique prescription to define a vacuum. Several problems may occur:

- It makes sense to talk about a particle with momentum p, if $\Delta p \ll p$ and hence the spatial size $\lambda \gg \frac{1}{p}$. But in this region, the spatial curvature may change. For high wave numbers and small curvature, this effect might become irrelevant, even on a cosmological scale.
- In general the 4d curvature counts, not only the spatial curvature. For example for a FLRW universe, the frequency $\omega_k(\eta) = \sqrt{k^2 + m_{\mathrm{eff}(\eta)}^2}$ can become imaginary. In this case, the modes become growing and decaying. No lowest energy state exists.

[6] Compare also with the free field solution (5.2).

- Further the definition depends on the chosen reference frame.
- Even for small deviations in the curvature and on a small time interval, the particle production from $|0_{\eta_1}\rangle$ to $|0_{\eta_2}\rangle$ can be infinite.

Another possible vacuum choice which avoids the last difficulty is the adiabatic vacuum. If the variation of ω_k is only small in some range of η one can use a WKB approximation. In this adiabatic regime, at fixed η_0 one solves for the mode function of the adiabatic vacuum, s.t. $v_k(\eta_0)$ and $v'_k(\eta_0)$ coincide with the WKB approximation

$$v_k^{\mathrm{WKB}}(\eta) = \frac{1}{\sqrt{\omega_k(\eta)}} \exp\left[i \int_{\eta_0}^{\eta} d\tilde{\eta}\, \omega_k(\tilde{\eta})\right]$$

at η_0.

6 Particle creation

In the Heisenberg picture, the states are time-independant, while the operator change in time. Let us prepare our system in the state $|0_a\rangle$, the vacuum corresponding to operators $\hat{a}^{\pm}_{\mathbf{k}}$. These operators can be chosen in a natural way, if the universe is static at initial time (see section 2). After an interaction with the gravitational field, let us assume, that our universe is again in a static state. But now the natural operators are $\hat{b}^{\pm}_{\mathbf{k}}$, with vacuum $|0_b\rangle$. Since our universe is still in the state $|0_a\rangle$, but particles are counted with $\hat{N}_b$, the mean density of b-particles is

$$\int d^3\mathbf{k}\ |\beta_k|^2\,,$$

as we have seen above. The energy density is

$$\int d^3\mathbf{k}\, \omega_k\, |\beta_k|^2\,.$$

6.1 Example

Consider a massless scalar field in a FLRW universe which is static in the past and future. We want to solve

$$\begin{aligned} \chi''_k - \Delta\chi_k + m^2_{\mathrm{eff}}\, \chi_k &= 0, \\ m^2_{\mathrm{eff}} &= a^2\left(\xi - \frac{1}{6}\right) R, \end{aligned}$$

with initial condition (in-state)

$$\chi_k^{(\mathrm{in})}(\eta) = \frac{\mathrm{e}^{-i\omega\eta}}{\sqrt{2\omega}}.$$

We can reformulate the problem as an integral equation

$$\chi_k(\eta) = \chi_k^{(\mathrm{in})}(\eta) + \frac{1}{\omega}\int_{-\infty}^{\eta} d\tilde{\eta}\, \chi_k(\tilde{\eta})\, m_{\mathrm{eff}}(\tilde{\eta}) \sin\omega(\eta - \tilde{\eta}),$$

which can be solved perturbatively by iteration.
One application is particle creation due to gravitational interactions during inflation. After a

deSitter expansion ($p = -\rho$, $a \sim \mathrm{e}^{Ht}$), the universe enters a matter dominated phase ($p = 0$). The energy contribution can be estimated by

$$\rho \simeq (1-6\xi)^2 \rho_{\text{Vac}} \frac{\rho_{\text{Vac}}}{\rho_{\text{Planck}}}.$$

Here ρ_{Vac} is the density driving inflation and ρ_{Pl} is the Planckian density. For GUT scale inflation, this contribution is negligible, if reheating is efficient:

$$\begin{aligned}\rho_{\text{Vac}} &\simeq \left(10^{11}\ \text{GeV}\right)^4 \\ \rho_{\text{Planck}} &\simeq \left(10^{15}\ \text{GeV}\right)^4\end{aligned}$$

7 Fluctuations

First we calculate fluctuations in a vacuum state. There are two ways to calculate fluctuations, leading to the same result:

- Equal time correlation functions:

$$\langle 0|\hat{\chi}(\mathbf{x},\eta)\,\hat{\chi}(\mathbf{y},\eta)|0\rangle \sim \int d^3\mathbf{k}\ |v_k(\eta)|^2\,\frac{\sin kL}{kL} \sim k^3|v_k|^2, \text{ at } k \sim \frac{1}{L},$$

  where $L = |\mathbf{x}-\mathbf{y}|$ is the comoving distance (the physical distance being $a(\eta)L$).

- Fluctuations over averaged fields, with window-function (bump-function) W_L of order 1 at a scale L:

$$\langle 0|\left(\int d^3\mathbf{x}\,W_L(\mathbf{x})\,\hat{\chi}(\mathbf{x},\eta)\right)^2|0\rangle \sim \int d^3\mathbf{k}\,|v_k|^2|\widetilde{W}_1(\mathbf{k}L)|^2.$$

  Since the Fourier transformed window-function $\widetilde{W}_1(\mathbf{k}L)$ is of order 1 at scale $|\mathbf{k}| \sim \frac{1}{L}$, the result is the same as above, up to factors of order 1.

For the free field, one gets:

$$\delta\chi_L(\eta) \sim \frac{k^{3/2}}{(k^2+m^2)^{1/4}} \sim \begin{cases} k^{3/2} & \text{for large } L \\ k & \text{for small } L\end{cases}$$

In a non-vacuum state, the fluctuations become:

$$(\delta\chi_L^{(b)})^2 = (\delta\chi_L)^2\left(1+2|\beta_k|^2\right).$$

Another oscillating term occurs, which vanishes, averaging over large times, but can in general yield a factor even smaller than 1.

7.1 Example

For the effective mass:

$$m_{\text{eff}}^2(\eta) = \begin{cases} m_0^2 & \eta < 0 \text{ and } \eta > \eta_1 \\ -m_0^2 & 0 < \eta < \eta_1 \end{cases}$$

one gets particle densities

$$|\beta_k|^2 \sim \begin{cases} \left(\frac{m_0}{k}\right)^4 & \text{for large } k \\ \mathrm{e}^{2m_0\eta_1} & \text{for small } k \end{cases}$$

The fluctuations now are given by

$$\delta\chi_L(\eta) \sim \frac{k^{3/2}}{(k^2+m^2)^{1/4}} \sim \begin{cases} \mathrm{e}^{m_0\eta_1}\, k^{3/2} & \text{for large } L \\ k & \text{for small } L \end{cases}$$

hence there is an enhancement of the fluctuations of $\mathrm{e}^{m_0\eta_1}$ on large scales.

8 Fields in de Sitter spacetime

A de Sitter universe is a FLRW universe with $a = a_0\,\mathrm{e}^{Ht}$, corresponding to $\epsilon = -p$, leading to $\epsilon = \text{const}$. The usual coordinate patch

$$ds^2 = dt^2 - a^2 d\mathbf{x}^2$$

covers only a finite part of de Sitter space-time[7], which is nevertheless enough for cosmological applications. In cosmology, a de Sitter universe is used for inflation, with a large Hubble parameter H. Due to dark matter and dark energy, also our current universe seems to resemble de Sitter on large scales, but with a tiny Hubble parameter. So the expansion of our universe increases exponentially.
In a de Sitter universe, there exists a horizon with scale $a(t_0)\,r_{\max}(t_0) = \frac{1}{H}$ which can only be reached asymptotically by lightlike particles.
Switch to conformal coordinates[8]:

$$ds^2 = a^2(d\eta^2 - d\mathbf{x}^2)$$

Now consider

$$k|\eta| \sim \frac{1}{L}\frac{1}{aH} = \frac{L_{\text{horizon}}}{L_{\text{phys}}},$$

with physical wavelength $L_{\text{phys}} = a(\eta)L$. Small values give superhorizon modes, heavily affected by gravity. In de Sitter the modes go like a power of η. Large values correspond to subhorizon modes, almost unaffected by gravity, so one gets flat modes. At a time when

[7] This follows from the fact, that for a massive observer $-\infty < t < 0$ corresponds to finite eigentime.
[8] Coordinate transformation:

$$\eta = -\frac{1}{H}\,\mathrm{e}^{-Ht}, \qquad a(\eta) = -\frac{1}{H\eta}.$$

Here $-\infty < \eta < 0$ corresponds to $-\infty < t < \infty$.

$L_{\text{horizon}} = L_{\text{phys}}$, horizon crossing occurs. Superhorizon modes do not have a particle interpretation. Only correlation functions make sense.
The effective frequencies in de Sitter are imaginary for $k|\eta|$ small enough (superhorizon modes). Hence instantaneous vacuum states cannot be defined. Instead define the Bunch-Davies vacuum by applying the Minkowski vacuum prescription as $\eta \to -\infty$:

$$v_k \to \frac{1}{\omega_k} \, \mathrm{e}^{i\omega_k \eta}, \qquad v_k' \to i\omega_k v_k.$$

An explicit solution can be given in terms of Bessel functions. In inflation, de Sitter space-time is usually applied for times $\eta_i < \eta < \eta_f$. Therefore in this setup the Bunch-Davies vacuum can only be chosen for subhorizon modes, $k|\eta_i| \gg 1$.
The fluctuation look as follows:

$$\delta\Phi_{L_{\text{phys}}}(\eta) = \begin{cases} \text{unknown}, & \eta < \eta_i, \text{ or } L_{\text{phys}} \gtrsim L_{\max} = H^{-1}\frac{\eta_i}{\eta} \\ L_{\text{phys}}^{-1}, & L_{\text{phys}} < H^{-1} \\ H|L_{\text{phys}}H|^{n-3/2}, & L_{\text{phys}} > H^{-1} \end{cases}$$

with $n = 3/2 - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \simeq 1$ for small masses. Since $L_{\max} \sim \mathrm{e}^{Ht}$, the unknown regime gets shifted to high physical scales rather quickly. A de Sitter inflation therefore helps to iron out unkown fluctuations. On small scales, the fluctuations are like in Minkowski space $\sim L_{\text{phys}}^{-1}$, while on large scales they approach a constant value H. These fluctuations account for inhomogeneities in the early universe in the inflation model.

9 Some physical effects

9.1 Lamb shift

Energy splitting between $2p$ and $2s$: different geometry of electron cloud $\longrightarrow$ different interaction with vacuum fluctuations.

9.2 spontaneous radiation of hydrogen atoms

$2p \to 1s$, due to interactions of electrons with vacuum fluctuations of electromagnetic field (otherwise: $2p$ is stable).

9.3 Schwinger effect

e^+e^- pair production in an electromagnetic field. If virtual pairs get separated, they can become real, if $leE \geq 2m_e$. Probability:

$$P \sim \exp\left(-\frac{m_e^2}{eE}\right).$$

9.4 Pair production due to gravity

Works only for time-dependant (non-static) gravitational field, otherwise the virtual pair does not get apart.

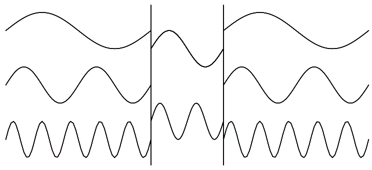


Figure 1: Casimir effect

9.5 Casimir effect

One might ask oneself the question, whether the vacuum energy has any reasonable meaning by itself or if it is just a mathematical relict. In fact, differences in vacuum energy in different space-time regions lead to effects that can be measured experimentally. An example is the Casimir effect, an attractive force between two uncharged plates in vacuum, which was indeed experimentally verified.
Consider two plates at a distance L in vacuum in $1+1$ dimensions. For simplicity, we only deal with a scalar field. In experiments, the force between the plates is due to fluctuations of the electromagnetic field. We want to solve

$$\partial_t^2 \phi - \partial_x^2 \phi = 0$$

with boundary conditions:

$$\phi(t,x)|_{x=0} = 0 = \phi(t,x)|_{x=L} .$$

The solutions are standing waves with nodes on the plates. Therefore the frequencies between the plates are no longer continuous (like for a field in a box). Rather, the classical solution between the plates consists only of a superposition of discrete frequencies:

$$\phi(t,x) = \sum_{n=1}^{\infty} \left(A_n \, \mathrm{e}^{-i\omega_n t} + B_n \, \mathrm{e}^{i\omega_n t} \right) \sin \omega_n x,$$

with $\omega_n = \frac{n\pi}{L}$. Quantization gives:

$$\phi(t,x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \frac{\sin \omega_n x}{\sqrt{2\omega_n}} \left(\hat{a}_n^- \, \mathrm{e}^{-i\omega_n t} + \hat{a}_n^+ \, \mathrm{e}^{i\omega_n t} \right) .$$

The prefactor guarantees the right commutation relations. To compute the vacuum energy between the plates, we need the Hamiltonian:

$$\hat{H} = \frac{1}{2} \int_0^L dx \left[\left(\partial_t \hat{\phi} \right)^2 + \left(\partial_x \hat{\phi} \right)^2 \right]$$

Some straightforward algebra gives the energy density per unit length:

$$\epsilon_0 = \frac{1}{L} \langle 0 | \hat{H} | 0 \rangle = \frac{1}{2L} \sum_k \omega_k = \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n,$$

where $|0\rangle$ is the Minkowski vacuum defined w.r.t. $\hat{a}_-^\pm$.
Seems, we run into trouble, since the energy turns out to be infinite. This is a standard problem

that plagues quantum field theory. Many results diverge, and to extract physical prediction, one has to introduce a regularization method. There are many different regularization schemes in QFT, for example dimensional regularization. Here, instead of d dimensions, we work in $d+\epsilon$ dimensions. The end result contains usually a term divergent in $\frac{1}{\epsilon}$. After renormalization, the terms are finite again.
We want to apply another method, which might seem even more obscure, namely zeta-function regularization. The zeta-function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$ by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Comparing to our result, we see that the vacuum energy corresponds to $s = -1$. But the sum representation above is not defined for $s = -1$. Nevertheless, the zeta-function can be analytically continued into the complex plane. In particular, the value at $s = -1$ is well-defined, and turns out to be $-\frac{1}{12}$, giving us a result for the vacuum energy[9]. The expectation value for the energy is negative, which is possible in QFT.
The outside region can also be described by our formula above, but now we shift the second plate to infinity, $L \to \infty$. The difference between the energies per unit length L inside and outside is:

$$\epsilon - \epsilon_0 = \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n - \lim_{L\to\infty} \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n = -\frac{\pi}{24L^2}.$$

The energy difference results in the Casimir force

$$F = -\frac{d}{dL}(\epsilon - \epsilon_0) = -\frac{\pi}{24L^2}.$$

In $3+1$ dimensions, one gets:

$$\frac{F}{A} = -\frac{\pi^2}{240L^4}.$$

10 The Unruh effect

10.1 Rindler space-time

We follow [4], section 4.5, with slightly different metrics and sign conventions.
The Unruh effect deals with an accelerated observer in flat Minkowski space-time. For simplicity, we only look at the 2-dimensional case. The relevant metric is Rindler space-time:

$$ds_R^2 = (a\rho)^2 d\tau^2 - d\rho^2.$$

[9] The same argument is used in bosonic string theory to calculate the central charge c of the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}.$$

In the light-cone gauge in D dimensions, the $D-2$ transverse oscillators contribute to the Hamitonian. Normal ordering gives a term $(D-2)\frac{1}{2}\sum n = -\frac{D-2}{24}$ (zero-point energy). The contribution from the Faddeev-Popov ghosts (these guys turn up in the gauge fixing procedure of the path integral) is 1. To avoid a conformal anomaly (non-zero charge), that spoils gauge invariance, the two contributions must cancel. Therefore critical bosonic string theory is formulated in $D = 26$ dimensions. In superstring theory, the Faddeev-Popov obey a different CFT with different central charge, and the critical dimension is 10.

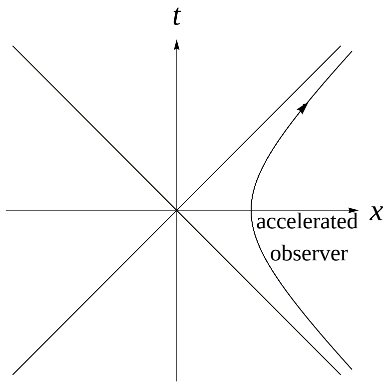


Figure 2: Rindler space-time

The metric is the analogue of polar coordinates for Minkowski space. A transformation to $ds_M^2 = dt^2 - dx^2$ can be given via

$$\begin{aligned} x &= i\rho \cosh a\tau \\ t &= \rho \sinh a\tau. \end{aligned} \tag{10.1}$$

In Euclidean space, the lines of constant radius are circles, while in the Minkowski case, lines of constant ρ are hyperbolas $x^2 - t^2 = \text{const}$. But in special relativity, an observer that is uniformally accelerated with acceleration a follows a hyperbola exactly parametrized by (10.1). Hence Rindler space-time describes uniformal acceleration, as sketched in Figure 10.1.

Null geodesics in flat Minkowski space satisfy

$$0 = ds_M^2 = dt^2 - dx^2.$$

So out-going geodesics (moving in positive x-direction) are described by constant u_M, while in-going geodesics correspond to constant v_M, where

$$\begin{aligned} u_M &= t - x, \\ v_M &= t + x \end{aligned}$$

are the null coordinates. The same works for Rindler space-time. Here geodesics satisfy

$$0 = ds_R^2 = (a\rho)^2 d\tau^2 - d\rho^2,$$

and null coordinates are

$$\begin{aligned} u_R &= a\tau - \log \rho, \\ v_R &= a\tau + \log \rho. \end{aligned}$$

In null coordinates, the metrics read:

$$\begin{aligned} ds_M^2 &= du_M \, dv_M, \\ ds_R^2 &= \mathrm{e}^{v_R - u_R} \, du_R \, dv_R. \end{aligned}$$

Written in this way, the Rindler metric can be seen to be conformally flat (as is any metric in two dimensions).

An observer following the hyperbola as sketched in Figure 10.1 can only receive signals from the lower quarter, while it can only send signals to the upper quarter. Hence there is no way to synchronize clocks in these cases and hence no notion of distance. The Rindler metric therefore only covers one quarter of Minkowski space. In particular there is a horizon, corresponding to the boundary of the Rindler wedge $x > |t|$ in Minkowski space time. The horizon consists of $u_R = \infty$ (or $u_M = 0$) and $v_R = -\infty$ (or $v_M = 0$).
Next we add a second wedge, which we get by reflecting the old one in the origin. The old one is called $\mathfrak{R}$ (right), the new one is called $\mathfrak{L}$ (left). The remaining wedges are called $\mathfrak{F}$ (future) and $\mathfrak{P}$ (past).
We want to find out the physical implications of the acceleration. Let us quantize a scalar field as usual, that obeys the Klein-Gordon equation

$$\left(\partial_t^2 - \partial_x^2\right)\phi = \partial_{u_M}\partial_{v_M}\phi = 0.$$

Since the equation of motion is conformally invariant, in Rindler space it is simply

$$\partial_{u_R}\partial_{v_R}\phi = 0.$$

The solutions are the same as given above in equation (5.2). In Minkowski space, they are:

$$v_k^M = \frac{1}{\sqrt{\omega_k}}\,\mathrm{e}^{i\omega t}\,.$$

In Rindler space, we can solve in $\mathfrak{R}$ or $\mathfrak{L}$ respectively:

$$v_k^{R(\mathfrak{R})} = \begin{cases} \frac{1}{\sqrt{\omega_k}}\,\mathrm{e}^{i\omega(v_R-u_R)} & \text{in } \mathfrak{R} \\ 0 & \text{in } \mathfrak{L} \end{cases}$$

$$v_k^{R(\mathfrak{L})} = \begin{cases} 0 & \text{in } \mathfrak{R} \\ \frac{1}{\sqrt{\omega_k}}\,\mathrm{e}^{-i\omega(v_R-u_R)} & \text{in } \mathfrak{L} \end{cases}$$

Now $v_k^{R(\mathfrak{R})}$ is a complete set of solutions in $\mathfrak{R}$, while $v_k^{R(\mathfrak{L})}$ is complete in $\mathfrak{L}$. Furthermore, one can show, that together they are complete in the whole Minkowski space.
The fields can be expanded in either set:

$$\begin{aligned}\phi &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left((v_k^M)^*\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{a}_{\mathbf{k}}^- + v_{\mathbf{k}}^M\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{a}_k^+\right) = \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left((v_k^{R(\mathfrak{R})})^*\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{b}_{\mathbf{k}}^{\mathfrak{R}-} + v_{\mathbf{k}}^{R(\mathfrak{R})}\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{b}_k^{\mathfrak{R}+} + (v_k^{R(\mathfrak{L})})^*\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{b}_{\mathbf{k}}^{\mathfrak{L}-} + v_{\mathbf{k}}^{R(\mathfrak{L})}\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{b}_k^{\mathfrak{L}+}\right)\end{aligned}$$

Now there are two vacuum states:

$$\begin{aligned}\text{Minkowski vacuum: } & \hat{a}_k^- \left|0_M\right\rangle = 0, \\ \text{Rindler vacuum: } & \hat{b}_k^{\mathfrak{R}-}\left|0_R\right\rangle = \hat{b}^{\mathfrak{L}-}\left|0_R\right\rangle = 0.\end{aligned}$$

The system must be in the Minkowski vacuum, being the lowest energy state of an inertial observer. To prepare a state in Minkowski space in the Rindler vacuum, one needs an infinite amount of energy: if the energy is uniformly distributed in Rindler space, the energy density diverges (after zero-energy subtraction) at the horizon, which corresponds to an infinite

coordinate region in Rindler space. This is not sensible, since it would lead to an infinite backreaction due to Einstein equations.
To compute the excitation number, an accelerated observer measures in the Minkowski vacuum, we have to compute generalized Bogolyubov transformations, i.e. the transformation is no longer diagonal in the momentum. Rather:

$$\hat{d}_\Omega^- = \int_{-\infty}^{\infty} d\omega \, \left(\alpha(\omega,\Omega)\, \hat{c}_\omega^- + \beta(\omega,\Omega)\, \hat{c}_\omega^+\right), \ldots$$

The Rindler mode functions must pick up some negative frequency, since they are non-smooth at the horizon, due to the sign flip in the exponent. Therefore, they cannot be superpositions of the analytic Minkowski mode functions.
To do the calculation, we construct two combinations that are analytic and bounded in the lower half planes in u^R and v^R, namely:

$$\frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} \left(u_k^{R(\mathfrak{R})} + \mathrm{e}^{-\frac{\pi\omega}{a}} (u_{-k}^{R(\mathfrak{L})})^*\right)$$
$$\frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} \left((u_{-k}^{R(\mathfrak{R})})^* + \mathrm{e}^{\frac{\pi\omega}{a}}\, u_k^{R(\mathfrak{L})}\right)$$

as one can check in a straight-forward fashion.
Expanding in these new mode functions, the corresponding operators must now annihilate the *Minkowski* vacuum. By taking inner products $(\phi, u_k^{R(\mathfrak{R})})$ and $(\phi, u_k^{R(\mathfrak{L})})$ one gets the needed Bogolyubov coefficients.
Between the accelerated frame and the flat metric, one finds the mean density of particles with momentum Ω as seen from the accelerator to be:

$$n_\Omega = \frac{\exp\left(-\frac{\pi\omega}{a}\right)}{2\sinh\frac{\pi\omega}{a}} = \frac{1}{\exp\left(\frac{2\pi\omega}{a}\right) - 1}.$$

This corresponds to a Bose-Einstein blackbody radiation of Unruh temperature

$$T = \frac{a}{2\pi}.$$

11 Black holes and thermodynamics

11.1 Basic facts on Black holes

Several solutions for black holes were found. They can be listed by increasing complexity:

1. The first solution found was the Schwarzschild black hole (1916):

   $$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + r^2 d\Omega^2,$$

   where $r_s = 2G_N M = 3\frac{M}{M_\odot}$km is the Schwarzschild radius.
   It solves the Einstein equations in vacuum $R_{ab} = 0$, and is

   - stationary: there exists a time-like Killing vector field ξ^a (or a one-parameter group of isometries ϕ_t with time-like orbits).

- spherically symmetric: the isometry group contains SO(3).
- static[10]: the orbits of ϕ_t are integrable, meaning there exist hypersurfaces orthogonal to ϕ_{t_0} for any t_0. By Frobenius' theorem:

$$\xi^\flat \wedge d\xi^\flat = 0.$$

  Here $\xi^\flat$ is the corresponding one-form, which in abstract-index notation is written $\xi_a = g_{ab}\xi^a$.

2. There exists a further generalization to a charged black hole (Reissner-Nordström metric):

$$ds^2 = \left(1 - \frac{r_s}{r} + \frac{e^2}{r^2}\right) dt^2 - \frac{dr^2}{1 - \frac{r_s}{r} + \frac{e^2}{r^2}} - r^2 d\Omega^2,$$

   with $r_2 = 2G_N M$ as above.

3. The most general metric is the Kerr metric, describing a rotating, charged black hole:

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a(r^2 + a^2 - \Delta)\sin^2\theta}{\Sigma} dt\, d\phi + \\ &+ \frac{(r^2+a^2)^2 - \Delta\, a^2 \sin^2\theta}{\Sigma} \sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2, \\ \Sigma =& r^2 + a^2\cos^2\theta, \\ \Delta =& r^2 + a^2 + e^2 - 2Mr. \end{aligned}$$

   To avoid naked singularities (which should be forbidden according to the Cosmic Censorship Conjecture), one requires:

$$e^2 + a^2 \leq M^2, \tag{11.1}$$

   so for a given mass, a black hole cannot have an arbitrarily high charge, or cannot rotate arbitrarily fast. If equality holds, the black hole is called extremal.

11.2 Hawking radiation

First, it was discovered, that rotating black holes can emit particles. If the inequality (11.1) holds strictly, Δ vanishes for two values of r:

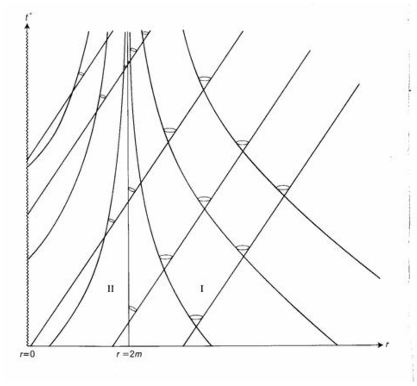
$$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}.$$

r_- is the horizon. The region between r_+ and r_- is the so called ergosphere. The Killing field ξ^a of stationarity, which gives time translations asymptotically at infinity, flips as one enters the ergosphere, and becomes space-like. Now the energy of a particle with momentum p^a, as measured by an observer at infinity is given by $E = -p^a\xi_a$. If the particle stays outside the ergosphere, the energy is strictly positive. But, since the Killing vector ξ^a becomes space-like

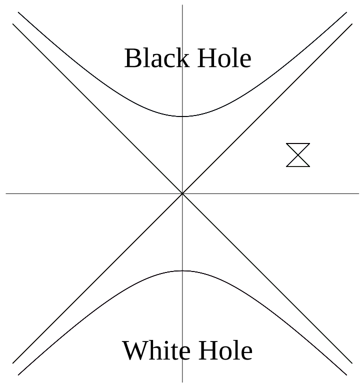
[10] Staticity follows in fact from being spherically symmetric, which is known as Birkhoff's theorem (and generalizes a theorem by Newton). In other words, even if a mass distribution oscillates in a spherically symmetric fashion, the space-time outside is static, hence there is no emission of gravitational waves. There exist no gravitational monopoles.

inside the ergosphere, real particles can carry negative energy. This allows one to invent a mechanism to extract energy from a black hole. In the Penrose process, a particle of positive energy E is sent into the ergosphere, where it splits into two particles. We can take one particle to have negative energy. The other particle escapes again out to infinity (note that we are still outside the horizon), while the particle of negative energy is swallowed by the black hole. From outside, it seems that the black hole emitted positive energy.
One can only extract about 29% from the black hole, since the absorbed particle will also have negative angular momentum, thereby slowing down the process. This situation will change for quantized fields.
While the Hawking radiation described below is pretty weak and quite possibly unobservable, the Penrose process might have big impacts. For example, the giant eruptions of quasars are commonly explained by particle emission of a rotating super-massive black hole.
In 1974, Hawking realized that also non-rotating and uncharged black holes emit particles, due to quantum effects. We want to sketch several derivations in the following:

1. First, we derive the equation for a static Schwarzschild black hole. In 1+1 dimensions, we can gain the result in analogy to the Unruh effect. First, let us transform to conformal coordinates. These are described by lightcone coordinates, so we want to solve the geodesic equation for light-like particles. A shortcut is again given by observing that, for lightlike particles:
$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 = 0.$$
The solutions are "tortoise" coordinates (Eddington-Finkelstein coordinates):
$$t \text{ and } r_* = r - 2M + 2M \log\left(\frac{r}{2M} - 1\right).$$
Light-cone tortoise coordinates are:
$$u = t - r_* \text{ and } v = t + r_*,$$
and the metric reads
$$ds^2 = \left(1 - \frac{2M}{r}\right) du\, dv.$$
$r_* \to -\infty$ corresponds to the Schwarzschild radius $r = 2M$ (explaining the "tortoise"-like behaviour of these coordinates), while for $r_* \to \infty$, the coordinates approach Schwarzschild coordinates of an observer at infinity. An observer must accelerate to stay at constant r, and these coordinates will therefore correspond to Rindler coordinates in our calculation of the Unruh effect.
The tortoise-coordinates do not cover the interior of the black hole. A completion is given by Kruskal coordinates (see section 6 in [15] for a proper motivation):
$$\bar{u} = -4M \exp\left(-\frac{u}{4M}\right) \text{ and } \bar{v} = 4M \exp\left(\frac{v}{4M}\right),$$
yielding the metric
$$ds^2 = \frac{2M}{r} \exp\left(1 - \frac{r}{2M}\right) d\bar{u}\, d\bar{v},$$
where r is understood as a function of $\bar{u}$ and $\bar{v}$. The Kruskal coordinates serve to describe a particle falling freely into the black hole.
To summarize, similar to the Unruh effect, the coordinates describe different observers:



(a) Eddington-Finkelstein coordinates



(b) Kruskal coordinates

- Kruskal coordinates: locally inertial (freely falling) observers. The corresponding vacuum is $|0_K\rangle$.
- Tortoise coordinates: observers at constant r (they must accelerate to stay at a fixed position). Vacuum: $|0_T\rangle$.

Again the Bogolyubov coefficient are calculated, as above. The proper acceleration is given by $a = \frac{1}{4M}$, leading to the Hawking temperature:

$$T_H = \frac{\hbar c^3}{Gk} \cdot \frac{1}{8\pi M} \sim \frac{M_p^2}{M}.$$

Unruh and Hawking effect are quite similiar in two dimensions:

Unruh	Hawking
inertial observer ("free fall" in flat space): $\lvert 0_M\rangle$	free-fall observer: $\lvert 0_K\rangle$
accelerated observer: $\lvert 0_R\rangle$	observer at constant r (accelerated in curved space): $\lvert 0_T\rangle$
$T = \frac{1}{2\pi}\, a$	$T = \frac{1}{8\pi}\frac{1}{M}$

With massive fields, the particle density is given by:

$$n = \frac{1}{\exp\left(\frac{E}{T_H}\right) - 1},$$

with $E = \sqrt{m^2 + k^2}$ which is only large enough for small masses. Radiation of massless particles is exponentially suppressed.
In $3+1$ dimensions, one must include a greybody factor < 1 accounting for a barrier-like potential in the Laplace equation.

2. Hawking's original derivation [9] of the black hole radiation was somewhat different. He did not consider an eternal Schwarzschild black hole, but instead he looked at matter collapsing to a black hole. Intuitively, black hole evaporation can be understood as highly energetic particles staying close to the horizon for a long time. Eventually, they

can escape and reach the detector. Though the two approaches seem to desribe different physical setups (in the first case, an eternal black hole emitting particles, in the second a black hole forming), they give the same result, namely a black body radiation with Hawking temperature.

3. Another, more heuristic interpretation goes as follows: consider a pair of particles, one with positive, the other with negative energy. The latter can tunnel through the event horizon. Inside the black hole, the time-like killing vector field turns space-like, and hence the particle becomes real (now propagating time-like). The particle of positive energy can propagate to infinity. The probability of this effect to happen is proportional to the surface gravity, the area of the black hole horizon.
   This explanation resembles the Penrose process described above. But note that the Hawking effect is a pure quantum effect. The Penrose process happens outside the Black hole horizon. For non-rotating black holes, the Killing vector flips only inside the horizon, and a splitting procedure as for the Penrose process cannot work, since there is no way that one particle can again escape to infinity.

4. Finally, there is a very elegant and short derivation, to get the temperature from the metric. We follow [3], page 562.
   A thermodynamical system can be completely described (in the canonical ensemble) by its partition function:
$$Z = \mathrm{Tr}\ \mathrm{e}^{-\beta H},$$
   where $\beta = \frac{1}{k_B T}$. In the following, we fix $k_B = 1$. Statistical mechanics can be discribed very laxly as a quantum field theory in Euclidean time. In quantum mechanics, time-evolution is given by e^{-iHt}. The partition function above can be understand as a theory with Euclidean time compactified on a circle. Taking the trace means, time is compactified on a circle of length β.
   First we have to compute the analytic continuation of the Schwarzschild metric, by replacing $t \to i\tau$. Now, at the horizon, the metric has to be regular at the horizon. Therefore expand, close to the horizon: $r = r_s(1+\rho^2)$. The metric transforms to:
$$ds^2 = 4r_s^2\left(d\rho^2 + \rho^2\left(\frac{d\tau}{2r_s}\right)^2 + \frac{1}{4}d\Omega^2\right).$$
   This is a metric of a flat plane (times a sphere), provided that τ is an angular parameter with period $2r_s = \beta = \frac{1}{T}$.

11.3 Thermodynamics

A first hint to the thermodynamical behaviour of black holes was the area theorem, which states, loosely speaking, that the area of the horizon surface of a black hole can never decrease in time. For the general formulation and a proof, see Theorem 12.2.6 of [15].
Replacing the area by an entropy, we get the second law of thermodynamics.
Indeed, having derived the Hawking temperature, we can compute the Bekenstein entropy of black holes:
$$S_{\mathrm{BH}} = \int \frac{dE}{T} = \int \frac{dM}{T} = \frac{1}{4}4\pi(2M)^2 = \frac{1}{4}A,$$

using the black hole radius $r_s = 2M$.
Stefan-Boltzmann's law[11]:

$$L = \gamma \sigma T_H^4 A,$$

leads to:

$$M = -\int dt\, L = M_0 \left(1 - \frac{t}{t_L}\right)^{1/3}, \qquad t_L = 5120\pi \frac{M_0^3}{\gamma},$$

and hence leads to a finite life-time t_L (without quantum gravity effects)[12].
Black holes have negative heat capacity

$$C_{\mathrm{BH}} = \frac{\partial E}{\partial T} = \frac{\partial M}{\partial T} = -\frac{1}{8\pi T^2},$$

so they become colder, if they absorb mass and energy. This is a very interesting property, we want to explore a bit further.

11.3.1 Detour: entropy

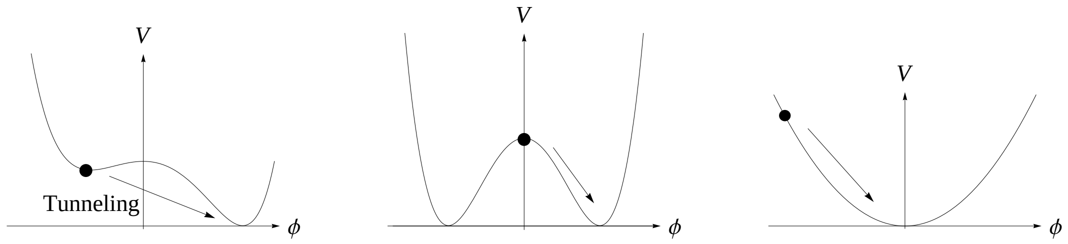
An excellent overview is given in chapter 27 of Penrose's epic book [14].
Entropy is a coarse graining phenomenon. Divide the phase space into several regions, such that all states within such a region have the same macroscopic behaviour. Define the entropy of a state by

$$S = k_B \log V,$$

where V is the volume of the region the state belongs to. Of course there is some arbitrariness in the definition, since different observers might disagree about what are macroscopically indistinguishable states. Nevertheless, thanks to the logarithm, the differences are ridiculously small. Now a state in thermal equilibrium corresponds to a state of highest entropy, contained within the largest volume in the phase space. The second law seems intuitively clear: if we are in a state in a small volume, it is highly likely that we will end up in a state of higher entropy (higher volume) after some time, until finally we are in the biggest box (maximal entropy, thermal equilibrium). In other words: entropy never decreases.
There is a subtle point underlying these hand-waving arguments: our state must be in a small volume (low entropic state) to begin with. We would not even exist in thermal equilibrium. Let's aks the question, how can we do anything at all? To tidy up my room, the entropy of my room decreases, hence my own entropy must increase, to get a gain in entropy in the end. How does it work? We owe all that to the sun, being a "hot spot in a cold universe". During the day, the sun sends us highly-energetic photons. At night, the earth emits again all the energy, but now in form of photons of low energy. There are much more of these, hence more degrees of freedom, hence higher entropy (larger volume in phase space). So only, because the energy from the sun has a very low entropy, plants can use it to decrease their own entropy, while we eat plants to decrease our own.
So how can the sun emit low entropy photons? The reason lies in the peculiar interplay of gravitation and entropy:

- A gas is in thermal equilibrium, if it is spread all over a given volume.

[11] L: flux of energy, γ: degrees of freedom, $\sigma = \frac{\pi^2}{60}$: Stefan-Boltzmann constant.
[12] Only for very light black-holes, the life-time is small w.r.t. the age of the universe.



(c) Old inflation (d) New inflation (e) Chaotic inflation

Figure 3: Several models of inflation

- A gravitational system is in thermal equilibrium, if it clumps to objects of large density.

We can understand it from above: the entropy is proportional to the area of a black hole, and therefore increases, if the black hole gets larger.
The second law is supposed to hold for the complete system, i.e.

$$\Delta S(\text{black holes}) + \Delta S(\text{matter}) \geq 0.$$

For example, Hawking radiation reduces the area of a black hole as seen above, thereby lowering the entropy. But the entropy of the radiation makes it positive again.
Now there is a problem with our second law: assume a closed universe, then at the big crunch, the entropy is ridiculously high. Even in open and flat models, the entropy is very large, due to black holes. Now the assumption underlying the second law is very sharp: at the big bang, matter was uniformaly distributed (thermal equilibrium in the matter sense, but *not* from the gravitational viewpoint). This corresponds to a very low entropy, so the universe was initially in a extremely small volume of the total phase space! Penrose motivates a factor of

$$10^{10^{123}}.$$

That's Loschmidt's paradox: there exists an arrow of time! Particle physics is essentially invariant under time inversion (ignoring some rare meson processes, while even here CPT is still conserved), but at macroscopic scales, a time direction exists. Claiming that a room gets more and more messy every day, already asserts that someone did tidy up in the first place. So someone must have been cleaning our universe at the big bang, preparing it in an extremely low-entropic state. If physics were just invariant under time translations, we could argue in the opposite direction, claiming that a messy place cleans itself in the course of time. Nice idea, but intuitively, something is wrong.

11.3.2 How to solve Loschmidt's paradox?

There are some approaches to explain the extremely special initial conditions at the big bang:

1. The "canonical explanation" is inflation. For a short review, see [8]. Inflation assumes the presence of an additional scalar field, the inflaton. Now, there are several scenarios, depicted in figure 3. In the old model, figure 3(c), the potential is initially in the "false vacuum" and afterwards is tunneling to the real vacuum. In this model the fluctuations are too large compared to experiments. In the "new inflation" model, figure 3(d), the

false vacuum is on a plateau and the inflaton rolls down the potential. In the third model, "chaotic inflation", figure 3(e), no plateau is needed, and the inflaton starts in a generic point and rolls down the potential. The inflaton serves as a force to drive an inflation of space-time, hence the name. The force is given by:

$$dW = -p\,dV = \rho dV$$

where the last expression is the energy of the expanded matter. Assuming a FLRW space-time[13]:

$$\ddot{a} = -\frac{4\pi}{3}(\rho + 3p)a = \frac{8\pi}{3}\rho a.$$

leading to an exponential expansion:

$$a \sim \mathrm{e}^{\xi t}\text{ , with } \xi = \sqrt{\frac{8\pi}{3}\rho}.$$

ξ^{-1} sets a time-scale for inflation, called an e-folding.
In the new inflation and the chaotic inflation models, so-called "internal inflation" can set in. The decay of the false vacuum is an exponential process, just as inflation itself. In fact inflation can be even stronger, so always a portion of the original space remains in the false vacuum, starting inflation again. Lots of baby universes evolve.
In principle this is a dangerous result, making it hard to make any predictions at all. Nevertheless, it might be a criterion to rule out vacua in M-theory of string theory. Vacua leading to stronger inflation will dominate, even if they have a lower *a priori* possibility.
Now the history of the universe is as follows:

- The universe is initially in a chaotic state.
- If some small patch was in the false vacuum, this patch gets inflated as the inflaton rolls down the potential. Here false vacuum denotes the minimum in the old inflation model, the plateau in the new model or a state of high expectation value (due to quantum fluctuations) in the chaotic model.
- When the inflaton is in the true minimum of the potential, inflation stops.
- The inflaton couples to other particles and therefore its energy gets converted to a "hot soup" of particles.
- Now we are at the hot big bang, and the standard model of cosmology sets in.

Inflation served to solve several problems:

(a) It explains, why the universe is large. After about 60 e-foldings, the particle content of at least 10^{90} baryons can be explained.

(b) It explains, why the universe is isotropic and homogeneous. All of the visible universe started in a small patch, which thermalized (in the radiation sense, not in the gravitational sense!). This explains the properties of the cosmic microwave backgrounds. Inflation of quantum fluctuations of the potential even serve to describe the fine-structure of the CMB (small anisotropy).

[13] this result should also hold in more general initial space-times, according to the cosmological "no-hair" conjecture.

(c) It explains the flatness of the universe. Flatness occurs, if the density of the universe is critical. In cosmological models:

$$\frac{\rho}{\rho_{\text{crit}}} = \begin{cases} 1 + \text{const} \cdot t & \text{radiation-dominated} \\ 1 + \text{const} \cdot t^{2/3} & \text{matter-dominated} \\ 1 + \text{const} \cdot \mathrm{e}^{-2Ht} & \text{inflation} \end{cases}$$

Only inflation serves to drive the value towards 1.

(d) GUT theories predict topological defects (monopoles, cosmic strings, domain walls). The energy density of monopoles would completely dominate the universe, which is unobserved. Inflation solves the monopole problem, by diluting the monopole density. Inflation must set in after monopole formation.

2. If Penrose's argumentation above is true, there must be at least some doubt on the motivation of inflation. Inflation tries to explain the homogeneity due to thermalization effects (matter can interact until it gets inflated apart). But thermalization increases the entropy, so the universe must have been even more special before.
An alternative idea, due to Penrose, is the Weyl curvature conjecture that states, that at the big bang the Weyl curvature did vanish. Loosely speaking, curvature can be split into a Ricci and a Weyl part:

   - Ricci curvature "contracts". Consider the effect of a massive object on a sphere of test mass surrounding it. It will be contracted spherically symmetric towards the object. Ricci curvature also enters Einstein's equations.
   - Weyl curvature is "tidal". Consider a little sphere outside a massive object. Different parts of the sphere get attracted with different strength, leading to tidal effects (compare moon/earth)! Weyl curvature also appears in the presence of gravitational waves.

11.4 Laws of black hole thermodynamics

1. Zeroth law: for a body in thermal equilibrium the temperature is constant. For black holes, this generalizes to the fact that for stationary black holes, the surface gravity κ is constant over the event horizon. Some notions have to be explained:

   - The future horizon h^+ is the boundary $\dot{I}^-\left(\bigcup\gamma\right)$, where the union goes over all geodesics tangent to Killing vector fields in the region outside the black hole. $I^-(\gamma)$ is the set of points that can be joint to γ by time-like future-directed curves ending on γ (for more precise definitions, see chapter 5.3 in [16]).
   - Look at a black hole with stationary Killing vector field ξ^a. If the black hole is rotating, ξ^a is in general not normal to h^+. Rather there exists a second Killing vector field χ^a, which can be written as:

   $$\chi^a = \xi^a + \Omega\phi^a,$$

   where $\xi^a\xi_a \to -1$ at infinity, and ϕ^a is a field with closed orbit, normalized to period 2π. Here, Ω is the angular velocity of the horizon.

Thermodynamics	**Black Holes**
generic situation	
E	M
V	J
S	$-2\pi \int_\sigma \frac{\delta L}{\delta R_{abcd}}\, n_{ab}\, n_{cd}$
	$Q_{\rm el}, Q_{\rm magn}$
equilibrium	
thermal equilibrium	stationary black hole
P	$-\Omega$
T	$\frac{1}{2\pi}\kappa$, Schwarzschild: $\kappa = \frac{1}{4M}$
S	$\frac{1}{4}A$
laws	
$0^{\rm th}$: T constant in thermal equilibrium	$0^{\rm th}$: κ constant for stationary black holes
$1^{\rm st}$: $\Delta E = T\Delta S - P\Delta V$	$1^{\rm st}$: $\Delta M = \frac{1}{2\pi}\kappa\Delta\left(\frac{1}{4}A\right) + \Omega\Delta J$
$2^{\rm nd}$: $\Delta S \geq 0$	$2^{\rm nd}$: $\Delta A \geq 0$

Table 1: Comparison between thermodynamics and black holes.

- On h^+, we have $\chi^b \chi_b = 0$, and therefore $\nabla^a(\chi^b\chi_b)$ is normal to h^+. χ^a is also itself normal to the Killing surface, and therefore they are proportional:

$$\nabla^a(\chi^b\chi_b) = -2\kappa\chi^a,$$

the proportionality constant is the surface gravity. χ^a is a Killing vector, satisfying $\nabla_{(a}\chi_{b)} = 0$. So we can rewrite:

$$\chi^b\nabla_b\chi^a = \kappa\chi^a, \tag{11.2}$$

a nonaffinely parametrized geodesic equation. For a Schwarzschild black hole, $\kappa = \frac{1}{4M}$. The surface gravity generalizes the well-known surface gravity of the earth $g = \frac{GM}{r^2}$.

2. First law: $\Delta E = T\Delta S - P\Delta V$. For black holes, one gets: $\Delta M = \frac{1}{2\pi}\kappa\Delta\left(\frac{1}{4}A\right) + \Omega\Delta J$. We want to outline the proof of this formula, and refer for details to [15] and [16]. Let $\mathcal{N}$ be the event horizon of a black hole, or more generally a null hypersurface. Let k^a be tangents to null geodesic generators, parametrized affinely with parameter λ, so $k^a k_a = -1$. Define the following quantities:

$$\begin{aligned}
h_{ab} &= g_{ab} + k_a k_b && \text{spatial metric}\\
B_{ab} &= \nabla_b k_a = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\theta\, h_{ab} &&\\
\theta &= h^{ab}B_{ab} && \text{expansion}\\
\sigma_{ab} &= B_{(ab)} - \frac{1}{3}\theta\, h_{ab} && \text{shear}\\
\omega_{ab} &= B_{[ab]} && \text{twist}
\end{aligned}$$

To get more intuition, look at a family of geodesics γ_a. The tensor $B^a{}_b$ measures how an observer on a geodesic γ_0 sees geodesics nearby. They get stretched and rotated by $B^a{}_b$. Expansion measures how they spread, and similiarly for shear and twist. If $\theta > 0$, the geodesic run towards each other, if $\theta < 0$, they get separated. θ fulfills the equation $\theta = \frac{1}{A}\frac{dA}{d\lambda}$.
Now, Raychaudhuri's equation hold:

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b. \tag{11.3}$$

Proof.

$$\begin{aligned}
k^c\nabla_c B_{ab} = k^c\nabla_c\nabla_b k_a &= k^c\nabla_b\nabla_c k_a + R_{cba}{}^d k^c k_d =\\
&= \nabla_b(k^c\nabla_c k_a) - (\nabla_b k^c)(\nabla_c k_a) + R_{cba}{}^d k^c k_d =\\
&= 0 - B^c{}_b B_{ac} + R_{cba}{}^d k^c k_d.
\end{aligned}$$

Tracing, the left-hand side gives $k^a\nabla_a\theta = \frac{d\theta}{d\lambda}$, while the right-hand side gives (using the orthogonal decomposition above):

$$-\frac{1}{9}\theta\, h_{ab}h^{ba} - \sigma_{ab}\sigma^{ba} - \omega_{ab}\omega^{ba} - R_{cd}k^c k^d.$$

□

Now let us take a stationary black hole and alter it by an infinitesimal physical process, such that the black hole is not destroyed, but stationary again.
To first order in ΔT_{ab}, we neglect σ_{ab}, ω_{ab}, and θ. Einsteins equation yields:

$$\frac{d\theta}{d\lambda} = -8\pi \Delta T_{ab} k^a k^b. \tag{11.4}$$

Further:

$$k^a = \left(\frac{\partial}{\partial \lambda}\right)^a = \frac{1}{\kappa\lambda}\frac{\partial}{\partial v} = \frac{1}{\kappa\lambda}\chi^a = \frac{1}{\kappa\lambda}(\xi^a + \Omega\phi^a).$$

Here v is the Killing parameter, such that $\chi^a \nabla_a v = 1$, while λ is the affine parameter. Indeed, the definition $k^a = e^{-\kappa v}\chi^a$ gives, with help of equation (11.2):

$$k^b \nabla_b k^a = \mathrm{e}^{-2\kappa v}\left(\chi^b \nabla_b \chi^a - \chi^a \chi^b \nabla_b(\kappa v)\right) = 0.$$

Therefore $\left(\frac{\partial}{\partial \lambda}\right)^a = \mathrm{e}^{-\kappa v}\left(\frac{\partial}{\partial v}\right)^a$, or $\lambda = \mathrm{e}^{\kappa v}$, justifying the intermediate step.
We multiply (11.4) with $\kappa\lambda$ and integrate over horizon. Using $\theta = \frac{1}{A}\frac{dA}{d\lambda}$, the left-hand side gives:

$$\kappa \int_0^\infty d\lambda \int d^2S \; \lambda \frac{d\theta}{d\lambda} = \kappa \int d^2S \;\; \theta\lambda|_0^\infty - \kappa \int d^2S \int_0^\infty d\lambda \; \theta = 0 - \kappa \Delta A.$$

The right-hand side is evaluated to:

$$-8\pi \int_0^\infty d\lambda \int d^2S \; \Delta T_{ab}\left(\xi^a + \Omega\phi^a\right) k^b = -8\pi(\Delta M - \Omega \Delta J),$$

proving the first law.

3. Second law: in a closed system, entropy never decreases, $\Delta S \geq 0$. For Black holes, the area of the event horizon never decreases, $\Delta A \geq 0$. Since the expansion of null-geodesic generators of the event horizon of a black hole can be written as $\theta = \frac{1}{A}\frac{dA}{d\lambda}$, it measures an increase of the area. An assumption needed for the proof, is the null energy condition:

$$T_{ab} k^a k^b \geq 0.$$

Einstein's equations lead to $R_{ab} k^a k^b \geq 0$, and Raychaudhuri's equation (11.3) gives:

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2. \tag{11.5}$$

It follows:

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{2}\lambda.$$

Now, assume the convergence is negative. If the geodesic can be extended far enough, one gets $\theta(\lambda) = -\infty$ for some $\lambda \leq \frac{2}{|\theta_0|}$. This cannot happen, as long since the horizon is achronal (see chapter 9 of [15]). A set S is called achronal, if there exist no $p, q \in S$, such that $p \in I^+(p)$.

The analogy between the laws of thermodynamics and the laws of black holes are summed up in table 1.

12 The holographic principle and String Theory

For nice papers and reviews, see [10, 5], and for a textbook covering the information paradox and the holographic principle, see [12].

12.0.1 Problems and Paradoxa

In the holographic picture, a particle falling into the black hole leaves a fingerprint on the area of the black hole, storing information about its position and mass. Information is not lost, but rather stored holographically on the black hole horizon. Also the Hawking radiation is not completely black. The principle was motivated by several problems one runs into by applying only the machinery described above, namely QFT in a classical curved background:

1. A first hint to the holographic principle can be seen from the fact that entropy is proportional to the *surface*, and not the volume. At Planckian scales, the degrees of freedom are not 3 + 1 dimensional, but rather two-dimensional Boolean variables on a two-dimensional lattice. At the Planck scale a physical state corresponds to a basis vector in a Hilbert space. While at large scales quantum mechanics is described by a superposition of states, at the Planck scale they become irrelevant.
   The entropy of black holes
   $$S = \frac{1}{4}A = 4\pi M^2$$
   describes not only the metric, but also all quantum fields in its neighbourhood.
   Let us analyze the problem in more detail: conventional quantum field theory is a local theory. In the example of a Klein-Gordon field, to every space-time point, an harmonic oscillator is assigned. Let us estimate the degrees of freedom. First, to get a finite result, let us assume that QFT is only well-defined down to the Planck-scale, where a new theory of quantum gravity must set in. Therefore, let us assume that space-time is coarse-grained, such that degrees of freedom can only be assigned to Planckian volumes. We also cut off the harmonic oscillator spectrum at the Planck mass; let us say there are n states in the spectrum. The number of independent quantum states is n^V, where V is the volume in Planckian units. The degrees of freedom are given by
   $$N \sim \log n^V = V \log n \gtrsim V.$$
   Collapsing the system into a black hole via the so-called Susskind-process, the entropy is bounded by the black hole entropy $\sim A$. Since we can make the volume V in principle larger than A it seems that we decreased the entropy in contrast to the generalized second law. Local QFT seems to overcount degrees of freedom. In fact, consider a system of size of the Planckian volume, containing energy. If we don't want it to be already a black hole, its mass must be below the Planck length:
   $$m \lesssim l_P.$$
   Now, combine N of these systems to a ball of radius R (in Planckian units). Naively, in QFT:
   $$M \lesssim N\, l_P = \frac{R^3}{l_P^3} l_P = \frac{R^2}{l_P^2} R.$$

But, again, if we do not want the system to be a black hole, we need:

$$M \lesssim R.$$

For a ball of radius 1 cm, we get a number of degrees of freedom of 10^{99} instead of 10^{33}, a clear overcount. String theory helps out here, since strings are extended objects, and therefore non-local. This makes an implemenation of the holographic principle possible, like in AdS/CFT, as we will see below.

2. In classical general relativity, unitarity seems to be violated. Due to the no-hair theorem, black holes are described by only three parameters, mass M, charge Q, and angular momentum a. If a particle falls into a black hole, its information is lost. As we have seen above, Schwarzschild radiation has a black-body spectrum, and therefore maximal entropy (no information). The opinions about the fate of black holes differ, some (like Stephen Hawking) favour a complete loss of information, others (like Leonard Susskind) prefer conservation of information (Susskind refers to these discussions as the "Black hole wars"). Recent progresses, mainly due to String-Theory-inspired AdS/CFT, seem to point into the second direction. AdS/CFT claims a duality between a certain gravitational theory and a conformal field theory. In particular, the latter is unitary, so the first must be unitary as well. It is not possible to loose information in black holes. We say more about this later on. Let us from now on assume that quantum mechanics hold, namely that information is stored, and all processes are unitary.

3. The entropy can be calculated not only via the first law of black hole thermodynamics, but also directly from the QFT and the Hawking temperature. In the calculation the entropy is cut-off dependent and diverges at the horizon. This is again an indication of an over-counting of local QFT.

4. In quantum mechanics, it is forbidden to copy information. Let a test-particle (i.e. yielding no big back-reaction) fall into a black hole. If Hawking radiation carries information about the particle, it must be possible for a second observer to store this information, then fall itself into the black hole, and meet the first particle again. It has copied information, in contrast to the Xeroxing principle.

5. Baryon number is not preserved, when a particle falls into a black hole.

12.1 Black hole complementarity

To overcome these problems, several steps forward have been taken. Black hole complementarity is the assertion that the laws of physics are valid for all observers. Let us consider again the problem of Quantum Xeroxing, as described above. The solution of the paradox is quite subtle, but shows, how a black hole "cares about" the laws of nature. First, we have to know a little bit more about information and entropy. Let us look at a system $\mathfrak{S}$, described by a density matrix ρ. It consists of several subsystems $\mathfrak{S}_i$ with respective density matrices ρ_i. There are several definitions that become relevant:

1. Von-Neumann entropy (fine-coarsed entropy):

$$S(\mathfrak{S}) = -\operatorname{Tr} \rho \log \rho.$$

Von-Neumann entropy is not additive.

2. Thermal entropy (coarse grained entropy):

$$S_{\text{thermal}}(\mathfrak{S}) = \sum S(\mathfrak{S}_i).$$

   The entropy is by definition additive.

3. Information is defined by:

$$I(\mathfrak{S}) = S_{\text{thermal}}(\mathfrak{S}) - S(\mathfrak{S}),$$

   and is usually measured in bits.

To understand the difference between the two entropies, consider a system in a pure state. The Von-Neumann entropy will vanish, since $\operatorname{Tr}\rho = 0$. The subsystems nevertheless get entangled, and in general the respective entropies will be non-vanishing. Looking only at one subsystem, and forgeting the others, more states look the same. In this way, information gets lost at large scales (via thermal coarse graining), while it is still retained microscopically (von-Neumann entropy).

Now, interestingly, systems smaller than half the complete system, carry less than one bit, i.e. for $\Sigma_i < \frac{1}{2}\Sigma$:

$$\begin{aligned} S(\mathfrak{S}_i) &\cong S_{\text{thermal}}(\mathfrak{S}_i), \\ I(\mathfrak{S}_i) &\cong 0. \end{aligned}$$

Therefore, the observer outside the black hole must wait, until the black hole evaporated to half its size until it gets any information at all! Here comes the problem: in order to provide a possiblitiy for the two observers to meet again inside the black hole, the first particle must accelerate against the attraction of the black hole for a long time in order not to fall into the singularity. But this requires an amount of energy so big, that the observer is a black hole by itself. The situation changes completely, and our assumptions, that the observers are only "test-particles" is no longer valid. The black hole does not allow Quantum Xeroxing that easily and the paradoxon no longer exists!

Baryon number violation receives a similar solution. There must be some process in nature violating baryon number in order to explain the excess of baryons over anti-baryons which must result from processes in the early universe. Possible sources are sphalerons (non-perturbative effects within the standard model), or massive particles coupling baryons to anti-baryons or leptons, as occur naturally in GUT-theories. For example, assume a pXe^+ coupling, i.e. an interaction between a proton, a positron and a field X with large mass m_X. The particle is assumed to be massive, so the process was relevant only at higher energies in an earlier stage of the universe. This interaction leads to proton decay (as predicted in GUT theories). These processes are always present, but for a observer in the rest frame of the proton, the oscillations occur on time scales $\frac{1}{m_X}$ which are extremely small, so time-averaged, the proton is stable (as observed in nature). Nevertheless, now let the proton cross the horizon of a black hole. From far away, it takes infinite time for the particle to approach the horizon. Now let there be an oscillation from proton to a positron. From far away it takes an infinite time to get back the original proton state; the proton seems to have decayed. This is also expected, since from far away, the proton is inside a heat bath (due to Hawking radiation), that provides energy for creation of X particles. The processes look completely different for the two observers, nevertheless no physical laws are violated[14]!

It is common to define the *stretched horizon*, i.e. a small deformation of the horizon away from the singularity. In AdS/CFT this is quite naturally. Here AdS_5 comes naturally with a boundary. On the boundary a CFT is defined on a stack of D-branes. The scattering amplitudes of strings off D-branes can be calculated and yield an effective thickness of the D-branes, some sort of open string halo. The stretched horizon serves to cut off some singularities.

The stretched surface has interesting properties. If a charged particles falls across the stretched surface, the surface acts as a conductor obeying Ohm's law! From the point of view of an observer outside, the charge density gets distributed quickly over the surface, while for the particle falling freely, nothing special happens. For details, compare chapter 7 in [12].

12.2 The holographic principle

Crudely speaking, the holographic principle states that all entropy contained in some volume can be stored on its boundary. Of course this definition is much to vague, but it can be sharpened to agree with all reasonable physical effects and space-times!

A first guess is to take space-like volumes and claim that the entropy is bounded by the entropy on the boundary. However this cannot work. For example consider a closed universe. Then we can take the volume arbitrarily large, such that the boundary gets arbitrarily small.

A more refined definition is needed. In fact, the bound that seems to hold is the covariant entropy bound: the entropy of any light-sheet of a given space-like surface B is bounded by the area of B:

$$S\big(L(B)\big) \leq \frac{A(B)}{4}.$$

The notion of a light-sheet has to be explained. Consider any space-like surface B. It possesses four orthogonal null directions, future-directed or past-directed as well as ingoing of outgoing. As a shorthand, we call them $(\pm, \pm)$, where the tuple corresponds to time and direction (t, r), $+$ denoting future-directed, or outgoing respectably, while $-$ corresponds to past-directed, or ingoing. We have seen in equation (11.5), that under the null-energy condition $T_{ab}k^a k^b \geq 0$ (stating that matter has an effect of focusing on light-rays, rather than anti-focusing), null geodesic run together, ending at so called caustics, given that the expansion θ is negative at some point (this is referred to the focussing theorem). Now, at least two null directions have non-positive expansion. As an example, consider a spherical room, with light-bulbs all over the wall, both inside and outside. Now switch on the light for a second. The light will run towards the center of the room, where the light-rays intersect (in-going), and similarly, light runs of to infinity (out-going). Similarly, the process can be looked at backwards in time. The null directions of negative expansions are $(+, -)$ and $(-, +)$. These directions of negative expansion span the light-sheet, ending at caustics (in the example above, the cone towards the center of the room, ending on the wall, in both time directions). The problem of the closed universe is circumvented, since null directions only converge into the smaller volume.

[14] This is analogous to our intuitive explanation of the Hawking process: while usually particle/anti-particle creation occurs on very tiny scales, if the particle runs outside the horizon, and the anti-particle inside the horizon back in time, from outside it takes an infinite amount of time until annihilation. Effectively, a particle was created.

12.3 AdS/CFT

We want to give a short glimpse at AdS/CFT and its relation to holography. From the string theory point of view, one considers a stack of N coincident D3-branes located in 10 dimensions. These branes are $3+1$ dimensional objects, where open strings can end. They can be charged under fields in the Ramond-Ramond sector of superstring theory. Branes can serve to describe black holes in string theory. If two open string excitations collide on the brane volume, they may combine into an open string that can escape away from the brane, leading to Hawking radiation. One can solve the equations of motion for the background metric in the supergravity approximation (small string coupling). The metric turns out to be

$$ds^2 = H_p^{-1/2}\big(-dx_0^2 + \ldots + dx_3^2\big) + H_p^{1/2}(dr^2 + r^2 d\Omega_5^2).$$

The first part corresponds to the metric on the D3 brane, while the second part describes the metric in the $10-4=6$ dimensions transversal to the brane, written in spherical coordinates, so $d\Omega_5^2$ is the metric of an S^5. The function relating the two metrics is:

$$1 + \left(\frac{R}{r}\right)^4.$$

Near the horizon $r = R$, the metric takes the form

$$ds^2 \simeq \left(\frac{r}{R}\right)^2 dx_{1,3}^2 + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_5^2.$$

Defining $z = \frac{R^2}{r}$, one gets:

$$ds^2 \simeq R^2 \frac{dx_{1,3}^2 + dz^2}{z^2} + R^2 d\Omega_5^2,$$

the metric of $\mathrm{AdS}_5 \times S^5$, both with radius R. AdS_{d+1} spaces can be defined as a hyperboloid in a $d+2$ dimensional space with signature $(2,d)$. Time corresponds to the radial coordinate, which gets decompactified by going to the universal covering space. This space has topology $\mathbb{R} \times B_{p+1}$ with boundary homeomorphic to $\mathbb{R} \times S^{p+1}$. The Poincaré patch with coordinate z as above covers only a part of AdS. $z = 0$ corresponds to the boundary.

The conjecture is now that type IIB superstring theory in the $\mathrm{AdS}_5 \times S^5$ background is equivalent to $\mathcal{N} = 4$, $D = 4$ super Yang-Mills theory with gauge group $\mathrm{SU}(N)$. The latter can be understood as the field theory on the stack of N D-branes, giving via the Chan-Paton method the required gauge group. The parameter N occurs on the string-theory side via the following relations:

$$\begin{aligned}\lambda &= g_{\mathrm{YM}}^2 N\\ g_{\mathrm{YM}}^2 &= 4\pi g_s\\ \lambda^4 &= \frac{R}{r_s}\end{aligned}$$

The first line defines the t'Hooft coupling λ, g_{YM}^2 is the coupling of the field theory, g_s the coupling of the string theory. The second equation is very special for D3 branes, since only in this dimension, the coupling is independent of the dilaton field which induces a running coupling. The coupling is independent of the energy, and the theory is conformal, simplifying many calculations.

A first check is a matching of symmetries:

1. AdS_5 was defined as a hyperboloid in $\mathbb{R}^{4,2}$ and therefore has symmetry SO(4, 2). Further the S^5 gives a symmetry group SO(6). These combine to the bosonic subgroup of the covering group SU(2, 2) × SU(4). Furthermore, type IIB string theory is supersymmetric, so in addition to the space-time symmetries, we have supersymmetries, giving superspace symmetries PSU(2, 2|4).

2. The Yang-Mills theory is supersymmetric. It can be obtained from the unique theory in 10 dimensions via dimensional reduction. Since it has $\mathcal{N} = 4$ supercharges, the R-symmetry group is SU(4). Since the theory is conformal, it has a conformal symmetry group in four dimensions SO(4, 2). Including supersymmetries, again gives PSU(2, 2|4).

In other words, the symmetry of the sphere corresponds to the R-symmetry of the supercharges, while the symmetry of AdS_5 gets identified with the conformal symmetry group in 4 dimensions.

The AdS/CFT conjecture relates a gravity theory (or a superstring theory) in 10 dimensions to a field theory in 9 dimensions, indicating some sort of holography. Further, the duality links strong coupling to weak coupling. Indeed, small curvature $\frac{1}{R^2}$ in the bulk theory, corresponds to large coupling λ in the field theory, and vice versa.

Several checks have been done, with the help of supersymmetry preserving some quantities while passing from weak to strong coupling (quantities saturating the BPS bound). Important limits, that simplify the calculations are:

1. First fix the t'Hooft coupling λ and consider N large, forcing the string coupling to be small. Quantum corrections (string loops come with factors of g_s) get suppressed.

2. Next, send λ to infinity, corresponding to a large radius (small curvature). Stringy corrections get suppressed in this limit.

13 Path integral methods

Effective action:

$$\mathrm{e}^{i\Gamma[J]} = \int \mathcal{D}q \; \mathrm{e}^{iS[q,J]} \, .$$

Get:

$$\begin{aligned}
\frac{\langle 0_{\text{out}}| T\, \hat{q}(t_1) \cdots \hat{q}(t_n) |0_{\text{in}}\rangle}{\langle 0_{\text{out}}|0_{\text{in}}\rangle} &= \frac{\frac{\delta^n \, \mathrm{e}^{i\Gamma[J]}}{\delta J(t_1)\cdots\delta J(t_n)}}{\mathrm{e}^{i\Gamma[J]}} \\
\frac{\langle 0_{\text{in}}| T\, \hat{q}(t_1) \cdots \hat{q}(t_n) |0_{\text{in}}\rangle}{\langle 0_{\text{in}}|0_{\text{in}}\rangle} &= \left. \frac{\frac{\delta^n \, \mathrm{e}^{i\Gamma[J]}}{\delta J(t_1)\cdots\delta J(t_n)}}{\mathrm{e}^{i\Gamma[J]}} \right|_{G_{\mathrm{F}} \to G_{\text{ret}}}
\end{aligned}$$

where we replaced the Feynman propagator by the retarded propagator in the last equation.

13.1 Backreaction

For a system classical background plus quantum system, one gets a total effective action:

$$S_{\text{eff}}[J] = S_{\text{background}}[J] + \Gamma[J].$$

The equation of motion for the background now becomes:

$$\frac{\delta S_B[J]}{\delta J(t)} + \left.\frac{\delta\Gamma[J]}{\delta J(t)}\right|_{G_{\rm F}\to G_{\rm ret}} = 0.$$

The effective potential Γ leads to source terms $\langle 0_{\rm in}|\hat{q}(t)|0_{\rm out}\rangle$.
Examples:

- Electromagnetism: J is here replaced by A_μ, the equations of motion are

$$\frac{1}{4\pi}\partial_\nu F^{\mu\nu} + \langle \hat{j}^\mu\rangle = 0,$$

with:

$$\frac{\delta\Gamma[J]}{\delta A_\mu} = \frac{\int \mathcal{D}\psi\, i\frac{\delta S}{\delta A_\mu}\, {\rm e}^{iS}}{\int \mathcal{D}\psi\ {\rm e}^{iS}} = \frac{\int \mathcal{D}\psi\, j^\mu\, {\rm e}^{iS}}{\int \mathcal{D}\psi\ {\rm e}^{iS}} = \langle j^\mu\rangle\,.$$

- Semi-classical Gravity: J gets replaced by $g^{\mu\nu}$, with equations of motion:

$$-\frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{\sqrt{-g}}{2}\langle\hat{T}_{\mu\nu}\rangle = 0.$$

with:

$$\frac{\delta\Gamma[J]}{\delta g^{\mu\nu}} = \frac{\int \mathcal{D}\phi\, \frac{i\delta S}{\delta g^{\mu\nu}}\, {\rm e}^{iS}}{\int \mathcal{D}\phi\ {\rm e}^{iS}} = \frac{\int \mathcal{D}\phi\, \frac{\sqrt{-g}}{2}T_{\mu\nu}\, {\rm e}^{iS}}{\int \mathcal{D}\phi\ {\rm e}^{iS}} = \frac{\sqrt{-g}}{2}\langle\hat{T}_{\mu\nu}\rangle\,,$$

leading to a vacuum polarization.

14 Heat kernel

For an action $S = \frac{1}{2}\int dt\, \phi\mathcal{O}\phi$, the effective action is given by[15]:

$$\Gamma = \frac{1}{2}\log\det\mathcal{O}.$$

Find an appropriate Hilbert space and an Hermitian operator $\hat{M}$ with the same eigenvalues λ_n. Define the zeta function of the operator $\hat{M}$ by

$$\zeta_{\hat{M}}(s) = \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\right)^s = {\rm Tr}\,\hat{M}^{-s}.$$

Now:

$$\Gamma = -\frac{1}{2}\left.\frac{d\zeta_{\hat{M}}(s)}{ds}\right|_{s=0}.$$

[15] Since in Euclidean notation:

$${\rm e}^{\Gamma} = \int \mathcal{D}\phi\ {\rm e}^{-\frac{1}{2}\int dt\,\phi\mathcal{O}\phi} = \sqrt{\det\mathcal{O}}.$$

Define the heat kernel:

$$\hat{K}_{\hat{M}}(\tau) = \sum_n \mathrm{e}^{-\lambda_n \tau} \, |\psi_n\rangle \, \langle\psi_n|$$

The effective potential can be calculated in terms of the trace of the heat kernel, using the identity[16]:

$$\zeta_{\hat{M}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, t^{s-1} \, \mathrm{Tr} \, \hat{K}_{\hat{M}}(\tau).$$

Solve perturbatively:

$$\hat{K}_{\hat{M}} = \hat{K}_0 + \hat{K}_1 + \ldots.$$

Writing $-\hat{M} = \Box + \hat{s}$, one gets a chain of equations:

$$\begin{aligned}
\frac{d\hat{K}_0}{d\tau} &= \Box \hat{K}_0, & \hat{K}_0(0) &= \hat{1}, \\
\frac{d\hat{K}_1}{d\tau} &= \Box M \hat{K}_1 + \hat{s}\hat{K}_0, & \hat{K}_1(0) &= \hat{0}, \\
\frac{d\hat{K}_2}{d\tau} &= \Box M \hat{K}_2 + \hat{s}\hat{K}_1, & \hat{K}_2(0) &= \hat{0}, \\
&\ldots
\end{aligned}$$

One can either expand in curvature or in τ. The first corrections are the same. Seeley-DeWitt expansion in d dimensions is given by

$$\langle x|\hat{K}(\tau)|x\rangle = \frac{\sqrt{g}}{(4\pi\tau)^{d/2}} \left(1 + a_1(x)\tau + a_2(x)\tau^2 + \ldots\right).$$

For $\mathcal{O} = -\Box + V(x)$, the first Seeley-DeWitt coefficient is given by

$$a_1(x) = \frac{1}{6} R(x) - V(x).$$

The $\zeta_{\hat{M}}$-function diverges for $\tau \to 0$, due to UV divergences[17]. The divergences have to be regularized and renormalized by defining new physical parameters in addition to the bare parameters (like the cosmological constant, Newton's constant and a R^2 coupling constant). Finite terms can be found in the expansion up to second order in the curvature (not in τ), yielding the Polyakov action (in two dimensions)

$$\begin{aligned}
\Gamma[g_{\mu\nu}] &= \frac{1}{96\pi} \int d^2x \, \sqrt{g} R \Box^{-1} R = \\
&= \frac{1}{96\pi} \int d^2x \, d^2y \, \sqrt{g(x)} \sqrt{g(y)} R(x) G(x,y) R(y).
\end{aligned}$$

[16] Use:

$$\begin{aligned}
\zeta_{\hat{M}}(s) &= \sum_n (\lambda_n)^{-s}, \\
\mathrm{Tr} \, \hat{K}_{\hat{M}}(\tau) &= \sum_n \mathrm{e}^{-\lambda_n \tau} \\
\Gamma(s) &= \lambda^s \int_0^\infty d\tau \, \mathrm{e}^{-\lambda\tau} \, \tau^{s-1}.
\end{aligned}$$

[17] Another divergence for $\tau \to \infty$ only appears in the Seeley-DeWitt expansion. This is not a serious problem, since the expansion is only valid for small τ anyway.

The variation of Γ under a conformal transformation leads to a conformal anomaly:

$$\langle 0|\hat{T}^{\mu}{}_{\mu}(x)|0\rangle = \frac{a_1(x)}{4\pi} = \frac{R(x)}{24\pi}.$$

This origins from the fact, that in path integral quantization, while S is conformally invariant, the measure $\mathcal{D}\Phi$ cannot be chosen to be generally covariant and conformally covariant at the same time.
An alternative derivation comes from looking at the Seeley-DeWitt expansion to first order.

15 C^*algebras

Definition 15.1 (C^*-algebra)**.** A C^*-algebra A is a associative $\mathbb{C}$-algebra with norm and $\star : A \to A$, $\mathbb{C}$ anti-linear, s.t.

1. $a^{**} = a$
2. $(ab)^* = b^* a^*$
3. $\|ab\| \leq \|a\| \, \|b\|$
4. $\|a^*\| = \|a\|$ (isometry)
5. $\|aa^*\| = \|a\|^2$ (C^*-property)

Definition 15.2. Let A be a C^* algebra, $a \in A$.
Resolvent set:

$$r_A(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ invertible}\}.$$

Spectrum:

$$\sigma_A(a) = \mathbb{C} \setminus r_A(a).$$

Spectral radius:

$$\rho_A(a) = \sup\{|\lambda| \mid \lambda \in \sigma_A(a)\}.$$

A self-adjoint element a is called positive, if one of the following holds[18]:

1. $a = b^2$, for some b self-adjoint,
2. $a = c^* c$, for some c,
3. $\sigma_A(a) \subset [0, \infty)$.

Lemma 15.3. *$\sigma_A(a)$ is nonempty and compact. Further:*

$$\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \|a\|.$$

The norm is uniquely determined by A and $\star$:

$$\|a\|^2 = \sqrt{\rho_A(a^* a)}.$$

Lemma 15.4. *Let $f : A \to B$ be an injective and unit-preserving $\star$-morphism. Then:*

$$\|f(a)\| = \|a\|.$$

[18] An element can be decomposed into positive elements, $a = a_+ - a_-$ with $a_+ = \frac{1}{2}(a + |a|)$ and $a_- = \frac{1}{2}(a - |a|)$.

16 States, observables, and operators

Definition 16.1. Let A be a C^* algebra, $\mathcal{H}$ a Hilbert space. A *representation* is a $\star$-map from A to $\mathcal{L}(\mathcal{H})$, the linear bounded operators. The *universal* representation is given by

$$\bigoplus_{\tau \in S(A)} \pi_\tau : A \to \mathcal{L}\left(\bigoplus_{\tau \in S(A)} H_\tau\right),$$

where $S(A)$ is the set of all states. $\mathcal{H}_\tau$ is the Hilbert space defined in the GNS representation below. A *state* is a positive linear functional $\tau : A \to \mathbb{C}$ with $\|\tau\| = 1$. An *observable* is a self-dual operator on $\mathcal{H}$.

- Pure vector state: $\mathcal{H}$ Hilbert space, $\mathcal{O} \in \mathcal{L}(\mathcal{H})$ linear bounded operator in $\mathcal{H}$, $\Psi \in \mathcal{H}$ vector. Set:
$$\tau(\mathcal{O}) = (\Psi, \mathcal{O}\Psi) = \langle \Psi | \mathcal{O} | \Psi \rangle .$$
In physics notation, $|\Psi\rangle$ is called state.
- Mixed state: density matrix (trace class operator). Let $\sum_i \lambda_i = 1$ and $\lambda_i > 0$. Define[19]:
$$\tau(\mathcal{O}) = \sum_i \lambda_i (\psi_i, \mathcal{O}\psi_i) = \sum_i \lambda_i \langle \psi_i | \mathcal{O} | \psi_i \rangle = \mathrm{Tr}\left[\left(\sum_i \lambda_i |\psi_i\rangle \langle \psi_i|\right) \mathcal{O}\right] = \mathrm{Tr}(\rho \mathcal{O}).$$
If $\mathrm{Tr}(\rho^2) = 1$, the state is pure, if it is less than 1, the state is mixed.
Note: a convex combination of states in the algebraic sense leads to mixed states (or density matrices), while a convex combination of states as vectors (in the usual QM sense) gives again a pure state.

Theorem 16.2 (GNS representation)**.** *Let τ be an arbitrary state. A scalar product is given by $\langle a, b\rangle = \tau(b^* a)$ Next we construct a Hilbert space. Therefore mod out the null space N_τ consisting of elements $a \in C^*$ with $\tau(a^* a) = 0$. Complete the pre-Hilbert coset-space A/N_τ to a Hilbert space H_τ.*
Now the Gelfand-Naimark-Segal (GNS) representation is given by:

$$\pi_\tau : A \to \mathcal{L}(H_\tau),$$
$$(\pi_\tau(a))\,[b] = [ab].$$

17 Wightman axioms

Consider the following set-up:

- States are one-dimensional subspaces of a separable complex Hilbert space $\mathcal{H}$. $\mathcal{H}$ carries a unitary representation of the Poincaré group $P = \mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$. *Leads to notions of energy, momentum, angular momentum, center of mass.*

[19] This has to be distinguished from a convex linear combination (superposition) of pure states (in physics sense), which would lead to
$$\tau(\mathcal{O}) = \langle \sum_i \lambda_i \psi_i | \mathcal{O} | \sum_j \lambda_j \psi_j \rangle ,$$
and $\sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$ cannot be written in terms of $\sum_i \sum_j \widetilde{\lambda}_i \widetilde{\lambda}_j |\psi_i\rangle \langle \psi_j|$.

- $\mathcal{F}$ is a collection of fields. A field $\Phi \in F$ is a operator-valued distribution ($\mathcal{S}$ denotes the Schwartz functions), i.e.

$$\begin{aligned}\Phi &: \mathcal{S}(\mathbb{R}^{1,3}) \to \mathcal{O}(\mathcal{H}),\\ \Phi(f) &: D_{\Phi,f} \to \mathcal{H}.\end{aligned}$$

  Here $\Phi(f)$ is defined on $D_{\Phi,f}$, a linear dense subspace of $\mathcal{H}$. Further there exists a linear dense subspace D with $D_{\Phi,f} \subset D$ for all Φ and f. *Leads to notions of locality.*

- There exists a vacuum vector $\Omega \in D$.

Now the Wightman axioms are:

1. Covariance: $\forall A \in P, \forall f \in \mathcal{S}, \forall \Phi \in \mathcal{F}, \forall x \in \mathbb{R}^{1,3}$, the following hold:

$$\begin{aligned}U(A)\Omega &= \Omega,\\ U(A)D &\subset D,\\ \Phi(f)D &\subset D,\\ U(A)\Phi(f)U(A)^* &= \Phi(Af),\\ Af(x) &= f(A^{-1}x)\end{aligned}$$

2. Spectrum condition. Let T_0 denote time-translations and $T_j \in P,\ j = 1, 2, 3$ spatial translations. Then

$$\begin{aligned}U(T_0) &= \exp(iP_0),\ P_0 \text{ Hamiltonian},\\ U(T_j) &= \exp(-iP_j).\end{aligned}$$

   The joint spectrum of T_j is the forward light cone C_+.

3. Locality, microscopic causality: if supp(f) and supp(g) are causally separated, $\Phi(f)$ and $\Psi(g)$ commute (or anti-commute).

Remark 17.1. Irreducible representations of the Poincaré group P of non-negative energy, can be described by the so-called Wigner classification. The mass $m = \sqrt{P^2}$ is a Casimir invariant of P. Now the following situations can occur

- $m > 0$: in the rest frame: $P_0 = m, P_i = 0$, allowing for a representation of Spin(3).
- $m = 0, P_0 > 0$. Go to the light-cone frame $P = (k, 0, 0, -k)$, giving us a representation of the double cover of SE(2), denoting the special Euclidean group (rotations plus translations). There occur irreducible representations characterized by the helicity (multiple of $\frac{1}{2}$) and continuous spin representations.
- $m = 0, P = 0$. This is the vacuum, leading to the trivial representation.

18 Algebraic Quantum Field Theory

Definition 18.1. We define three Categories:

1. **Loc**:

   - obj(**Loc**): globally hyperbolic Lorentzian manifolds, oriented and time-oriented. Or: bundles over space-time.
   - mor(**Loc**): isometric embeddings. Or: conformal transformations (for CFTs). The morphisms preserve orientation and time-orientation. Further, for $a, b \in M_1$ causally connected, also causally curves between $\psi(a), \psi(b)$ must be contained in $\mathrm{im}(\psi) \subseteq M_2$.

2. **Obs**

   - obj(**Obs**): C^*-algebras. Or: $\star$-algebras. Or: von Neumann algebras.
   - mor(**Obs**): injective, unital $\star$-homomorphisms.

3. **Test**

   - obj(**Test**): smooth, compactly supported test functions.
   - mor(**Test**): push-forward.

Now a locally covariant QFT is a covariant functor $\mathcal{A}$ between **Loc** and **Obs**. Let $\mathcal{D}$ be a covariant functor between **Loc** and **Test**. A locally covariant quantum field Φ is a natural transformation between $\mathcal{A}$ and $\mathcal{D}$.

$$\begin{tikzcd} \mathbf{Loc} \arrow[dd, swap, "\mathcal{A}"] & & \mathbf{Loc} \arrow[dd, "\mathcal{D}"] \\ {} & \overset{\Phi}{\Longrightarrow} & {} \\ \mathbf{Obs} & & \mathbf{Test} \end{tikzcd}$$

The following conditions must be satisfied:

1. $\mathcal{A}$ is causal: if the images of two manifolds are causally separated, the images of the corresponding algebras commute.

2. $\mathcal{A}$ obeys the time-slice-axiom: if the image of a manifold M_1 contains a Cauchy surface in M_2, the image of the algebra $\mathcal{A}(M_1)$ is the complete algebra $\mathcal{A}(M_2)$, not just a subset.

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