9. One Forms

Version 1.2

As before, W is an open subset of a manifold M and $\varphi : U \to V$ is a chart with $U \subset W$, $\varphi = (q^1, \dots q^n)$.

(9.1) Def. A 1-form* η on W is a map $\eta : W \to T^*M$, $T^*M := \bigcup \{ T_aM^* \mid a \in M \}$ with

$$\eta_a := \eta(a) \in T_aM^*, \; a \in W \quad (F^* \text{ dual of a vector space } F)$$

$$a \longmapsto \eta_a(X(a)) \text{ is smooth for all } X \in \mathcal{V}(W)$$

Notation: Let $\mathcal{V}^*(W)$ be the $\mathcal{E}(W)$-module of one forms on W.

A basic example is the differential $df(a)([\gamma]_a) := \frac{d}{dt}(f \circ \gamma)|_{t=0}$ for $[\gamma]_a \in T_aM$, $a \in W$: $df \in \mathcal{V}^*(M)$.

In particular, $dq^i \in \mathcal{V}^*(U)$.

In local coordinates $\eta \in \mathcal{V}^*(U)$ can be written as

$$\eta = \eta_j \cdot dq^j \quad \text{with } \eta_j = \eta\left(\frac{\partial}{\partial q^j}\right) \in \mathcal{E}(U)$$

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* Also called differential form of degree 1 or Pfaffian form or Pfaffian.

(9.2) Remark: $df(X) = L_X f$ for $f \in \mathcal{E}(W)$, $X \in \mathcal{V}(W)$.

(9.3) Alternative Definitions: A 1-form η on W is

1° a smooth section $\eta : W \longrightarrow T^*M$ in the cotangent bundle $T^*M \xrightarrow{\tau} M$ (with bundle structure analogous to TM, cf. 6.1).

2° a collection of components $\eta_1, \dots, \eta_n \in \mathcal{E}(U)$ for each chart transforming as $\bar{\eta}_j = \frac{\partial q^k}{\partial \bar{q}^j} \eta_k$

3° a $\mathcal{E}(W)$-linear map $\eta : \mathcal{V}(W) \to \mathcal{E}(W)$. Hence, $\mathcal{V}^*(W)$ is essentially the dual module $(\mathcal{V}(W))^*$ of $\mathcal{V}(W)$.

We will later see: $(\mathcal{V}(W))^{**} \cong \mathcal{V}(W)$, cf. section 10.

4° a smooth map $TM \to \mathbb{R}$ which is linear in the fibres $T_a M$, $a \in M$.