8. Flows

Version 1.1

Let $\gamma : I \to M$ be a smooth curve in a manifold M. $\dot{\gamma} = \frac{d\gamma}{dt} : I \to TM$ is defined to be

$$\dot{\gamma}(t_0) := \left[\gamma(t_0 + t)\right]_{\gamma(t_0)} \in T_{\gamma(t_0)} M .$$

One obtains the "lift" $\dot{\gamma} \in \mathcal{E}(I, TM)$, $\tau \circ \dot{\gamma} = \gamma$.
In local coordinates

$$\dot{\gamma}(t) = \frac{d}{dt}(q^j \circ \gamma)(t_0) \frac{\partial}{\partial q^j} .$$

(8.1) Proposition: Assume $X \in \mathcal{V}(M)$.

1° To every $a \in M$ corresponds a unique maximal solution $\gamma_a : I_a \to M$ of

$$\boxed{\dot{\gamma}(t) = X(\gamma(t)),\ \gamma(0) = a.}$$ (ordinary differential equation)

(This is an initial value problem of the ODE $\dot{\gamma} = X(\gamma)$ which has a unique maximal solution according to the theorem of Picard-Lindelöf: Look at $\dot{\gamma} = X(\gamma)$ in local coordinates and conclude that the ODE satisfies a Lipschitz condition because it is smooth.)

2° $\Sigma := \bigcup \{ I_a \times \{a\} \mid a \in M \} \subset \mathbb{R} \times M$ is open and the map $\varphi : \Sigma \to M$, $(t,a) \mapsto \gamma_a(t) = \varphi(t,a)$ is smooth. $\varphi = \varphi^X$ is called the flow of X.

3° For $t \in M$ let $M_t := \{ a \in M \mid (t,a) \in \Sigma \}$ and $\varphi_t := \varphi(t, \;)$. Then:
$M_t \subset M$ is open and $\varphi_t : M_t \to M_{-t}$ is a diffeomorphism.
Moreover, $\varphi_t^{-1} = \varphi_{-t}$, $\varphi_0 = id_M$ and

$$\varphi_s \circ \varphi_t = \varphi_{s+t}$$

for all $a \in M$ where $\varphi_{s+t}(a)$, $\varphi_t(a)$ and $\varphi_s(\varphi_t(a))$ are defined.
Altogether (φ_t) is a local 1-parameter group (of diffeomorphisms).

(8.2) Proposition: Conversely, any local 1-parameter group (λ_t) of diffeomorphisms induces a vector field $X \in \mathcal{V}(M)$ such that $t \mapsto \lambda_t(a)$ is a solution of

$$\dot{\gamma} = X \circ \gamma, \quad \gamma(0) = a.$$

We have $X(a) = \frac{d}{dt}\left(\lambda_t(a)\right)\Big|_{t=0}$.

As a result we obtain two additional characterizations of vector fields (alternative definitions, cf. 7.4) on M:

4° A vector field is the same as an autonomous ODE $\dot{\gamma} = X(\gamma)$ (or a dynamical system).

5° A vector field is the same as a local 1-parameter group of diffeomorphisms.

(8.3) Definition: $X \in \mathcal{V}(M)$ is complete if $\Sigma = \mathbb{R} \times M$.

Hence X is complete if all maximal solution curves are defined on all of $\mathbb{R}$ ("from $-\infty$ to $+\infty$").

Example: $M = \mathbb{R}$, $X(a) = (a, 1+a^2) \in T_a\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ leads to the ODE $\dot{x} = 1 + x^2$. $\varphi^X(t,a) = \tan(t + \arctan a)$, $|t - \arctan a| < 1$. X is not complete since $\gamma(t) = \tan(t + \arctan a)$ cannot be continued beyond $\arctan a - 1$ and beyond $\arctan a + 1$.

(8.4) Fact: In case of a compact M all vector fields are complete.

(8.5) Definition: For $\varphi \in \mathcal{E}(M,N)$ the pullback φ^* of φ is

$$\varphi^*(g) := g \circ \varphi \,,$$

for $g \in \mathcal{E}(N)$. The pullback

$$\varphi^* : \mathcal{E}(N) \to \mathcal{E}(M)$$

is $\mathbb{R}$-linear.

(8.6) Proposition: For the flow $\varphi = \varphi_X$ of a vector field $X \in \mathcal{V}(M)$ the following formula holds true

$$\frac{d}{dt} \varphi_t^* f \Big|_{t=0} = L_X f \,.$$

(8.7) Definition: For a diffeomorphism $\varphi \in \mathcal{E}(M,N)$ and $Y \in \mathcal{V}(N)$ we set analogously

$$\varphi^* Y(a) := T_a \varphi^{-1}(Y(\varphi(a))) \qquad \text{pullback}$$

$$\varphi^* : \mathcal{V}(N) \to \mathcal{V}(M) \qquad \mathbb{R}\text{-linear}$$

Note that $\varphi^* Y = (\varphi^{-1})_* Y$ (cf. section 7)

(8.8) Definition: For $X, Y \in \mathcal{V}(M)$ $\quad L_X Y := \frac{d}{dt}(\varphi_t^* Y)\Big|_{t=0}$.

(8.9) Fact: $L_X Y = [X, Y]$.