

7. Vector Fields

Version 1.1

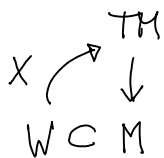
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In the following let W be an open subset of a manifold M and $\varphi: U \rightarrow V$ a chart with $U \subset W$, $\varphi = (\varphi^1, \dots, \varphi^n)$ ($n = \dim M$).

(7.1) Definition: A vector field X on W is a section in TM over W , i.e. a smooth map

$$X: W \rightarrow TM \quad \text{with} \quad \tau \circ X = \text{id}_W.$$



In local coordinates: $X(a) = X^j(a) \frac{\partial}{\partial \varphi^j}(a)$,
 $a \in U$, where $X^j \in \mathcal{E}(U)$.

Notation: $\mathcal{W}(W)$ is the $\mathcal{E}(W)$ -module of vector fields on W : $f \in \mathcal{E}(U)$, $X \in \mathcal{W}(W)$
 $\Rightarrow fX \in \mathcal{W}(W)$ with $fX(a) := f(a)X(a)$, $a \in W$.

(7.2) Fact: $\mathcal{W}(U)$ is a free $\mathcal{E}(U)$ -module of rank n .
Basis: $\frac{\partial}{\partial \varphi^1}, \dots, \frac{\partial}{\partial \varphi^n}$.

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(7.3) Definition: For $X \in \mathcal{M}(W)$ & $f \in \mathcal{E}(W)$ define

$$L_X f(a) := \left. \frac{d}{dt} (f \circ \gamma)(t) \right|_{t=0}$$

with $X(a) = [\gamma]_a$, $a \in W$. L_X is the Lie derivative in the direction of X .

Fact: $L_X f \in \mathcal{E}(W)$. And $L_X(fg) = (L_X f)g + f L_X g$ (derivation)

In local coordinates set $\frac{\partial f}{\partial q^j} := L_{\frac{\partial}{\partial q^j}} f$. This means

$$\frac{\partial f}{\partial q^j}(a) = \left. \frac{d}{dt} f \circ \bar{\varphi}^{-1}(\varphi a + t e_j) \right|_{t=0} = \frac{\partial (f \circ \bar{\varphi}^{-1})}{\partial q^j}(\varphi a), a \in U.$$

Fact: $L_X f = X^j(a) \frac{\partial f}{\partial q^j}(a)$ for $X = X^j \frac{\partial}{\partial q^j}$.

(7.4) Alternative Definition: A vector field X on W is equivalently

1° A map $X: W \rightarrow TM$ with $\tau \circ X = id_W$ such that $L_X f \in \mathcal{E}(W)$ for all $f \in \mathcal{E}(W)$,

2° A collection of $X^1, \dots, X^n \in \mathcal{E}(U)$ for each chart $\varphi: U \rightarrow V$ such that $\bar{X}^j = \frac{\partial \bar{q}^j}{\partial q^k} X^k$ for coordinate changes.

3° An \mathbb{R} -linear map $L_X: \mathcal{E}(W) \rightarrow \mathcal{E}(W)$ with

$$L_X(fg) = (L_X f)g + fL_X g \quad \text{for all } f, g \in \mathcal{E}(W).$$

Remarks on notations: $L_X f$ is often denoted by Xf .

(7.5) Proposition: Any derivation $D: \mathcal{E}(W) \rightarrow \mathcal{E}(W)$ [i.e. D \mathbb{R} -linear & $D(fg) = (Df)g + fDg$ for all $f, g \in \mathcal{E}(W)$] is of the form $D = L_X$ with a suitable $X \in \mathcal{W}(W)$.

Proof (sketch): Locally we have $X|_U := D(q^i) \frac{\partial}{\partial q^i}$

(7.6) Definition - Proposition: According to 7.5 for every pair $X, Y \in \mathcal{W}(W)$ of vector fields there exists uniquely $Z \in \mathcal{W}(W)$ with $L_Z = L_X \circ L_Y - L_Y \circ L_X$. This Z is the Lie bracket $[X, Y] \in \mathcal{W}(W)$. We can show:

1° $[\ , \]: \mathcal{W}(W) \times \mathcal{W}(W) \rightarrow \mathcal{W}(W)$ is \mathbb{R} -linear & antisym.

2° $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi-identity)

In local coordinates: $[X, Y] = \left(X^k \frac{\partial Y^j}{\partial q^k} - Y^k \frac{\partial X^j}{\partial q^k} \right) \frac{\partial}{\partial q^j}$

$\mathcal{W}(W)$ with $[\ , \]$ is a (n infinite dimensional) Lie algebra over \mathbb{R} .

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For a diffeomorphism $\varphi: M \rightarrow N$ of manifolds a vector field $X \in \mathfrak{X}(M)$ induces a vector field $\varphi_* X \in \mathfrak{X}(N)$ by

$$\varphi_* X(b) := T\varphi(X(\varphi^{-1}b)) \quad , \quad b \in N.$$

$\varphi_* X$ is called the pushforward of X by φ .

(7.7) Fact: For $g \in \mathcal{E}(N)$ and $X \in \mathfrak{X}(M)$:

$$L_{\varphi_* X} g(b) = X(g \circ \varphi)(\varphi^{-1}(b))$$

(7.8) Corollary: $\varphi_*([X, Y]) = [\varphi_* X, \varphi_* Y]$.