7. Vector Fields

Version 1.1

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In the following let W be an open subset of a manifold M and $\varphi : U \to V$ a chart with $U \subset W$, $\varphi = (q^1, \dots q^n)$ $(n = \dim M)$.

<u>(7.1) Definition:</u> A *vector field* X on W is a section in TM over W, i.e a smooth map

$$X : W \longrightarrow TM \quad \text{with} \quad \tau \circ X = id_W .$$

$$\begin{tikzcd} & TM \arrow[d] \\ W \arrow[ur, bend left, "X"] & \subset M \end{tikzcd}$$

In local coordinates: $X(a) = X^j(a) \frac{\partial}{\partial q^j}(a)$, $a \in U$, where $X^j \in \mathcal{E}(U)$.

<u>Notation:</u> $\mathcal{V}(W)$ is the $\mathcal{E}(W)$ – module of vector fields on W: $f \in \mathcal{E}(U)$, $X \in \mathcal{V}(W)$

$\Rightarrow$ $fX \in \mathcal{V}(W)$ with $fX(a) := f(a)X(a)$, $a \in W$.

<u>(7.2) Fact:</u> $\mathcal{V}(U)$ is a free $\mathcal{E}(U)$ module of rank n.

Basis: $\frac{\partial}{\partial q^1}, \dots \frac{\partial}{\partial q^n}$.

(7.3) Definition: For $X \in \mathcal{V}(W)$ & $f \in \mathcal{E}(W)$ define

$$L_X f(a) := \frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0}$$

with $X(a) = [\gamma]_a$, $a \in W$. L_X is the Lie derivative in the direction of X.

Fact: $L_X f \in \mathcal{E}(W)$. And $L_X(fg) = (L_X f)g + f L_X g$ (derivation)

In local coordinates set $\frac{\partial f}{\partial q^j} := L_{\frac{\partial}{\partial q^j}} f$. This means

$$\frac{\partial f}{\partial q^j}(a) = \frac{d}{dt} f \circ \varphi^{-1}(\varphi a + t e_j)\Big|_{t=0} = \frac{\partial (f \circ \varphi^{-1})}{\partial q^j}(\varphi a), \ a \in U.$$

Fact: $L_X f = X^j(a) \frac{\partial f}{\partial q^j}(a)$ for $X = X^j \frac{\partial}{\partial q^j}$.

(7.4) Alternative Definitions: A vector field X on W is equivalently

1° A map $X : W \to TM$ with $\tau \circ X = \mathrm{id}_W$ such that $L_X f \in \mathcal{E}(W)$ for all $f \in \mathcal{E}(W)$.

2° A collection of $X^1, \dots X^n \in \mathcal{E}(U)$ for each chart $\varphi : U \to V$ such that $\bar{X}^j = \frac{\partial \bar{q}^j}{\partial q^k} X^k$ for coordinate changes.

3° An $\mathbb{R}$-linear map $L_X : \mathcal{E}(W) \to \mathcal{E}(W)$ with
$$L_X(fg) = (L_X f)g + f L_X g \quad \text{for all } f, g \in \mathcal{E}(W).$$

Remark on notations: $L_X f$ is often denoted by Xf.

(7.5) Proposition: Any derivation $D : \mathcal{E}(W) \to \mathcal{E}(W)$ [i.e. D $\mathbb{R}$-linear & $D(fg) = (Df)g + f Dg$ for all $f, g \in \mathcal{E}(W)$] is of the form $D = L_X$ with a suitable $X \in \mathcal{V}(W)$.

Proof (sketch): Locally we have $X|_U := D(q^j) \frac{\partial}{\partial q^j}$

(7.6) Definition - Proposition: According to 7.5 for every pair $X, Y \in \mathcal{V}(W)$ of vector fields there exists uniquely $Z \in \mathcal{V}(W)$ with $L_Z = L_X \circ L_Y - L_Y \circ L_X$. This Z is the Lie bracket $[X, Y] \in \mathcal{V}(W)$. We can show:

1° $[\ ,\ ] : \mathcal{V}(W) \times \mathcal{V}(W) \to \mathcal{V}(W)$ is $\mathbb{R}$-linear & antisymm.

2° $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi-identity)

In local coordinates: $[X, Y] = \left(X^k \frac{\partial Y^j}{\partial q^k} - Y^k \frac{\partial X^j}{\partial q^k} \right) \frac{\partial}{\partial q^j}$

$\mathcal{V}(W)$ with $[\ ,\ ]$ is a (n infinite dimensional) Lie algebra over $\mathbb{R}$.

For a diffeomorphism $\varphi: M \to N$ of manifolds a vector field $X \in \mathcal{V}(M)$ induces a vector field $\varphi_* X \in \mathcal{V}(M)$ by

$$\varphi_* X(b) := T\varphi\left(X(\varphi^{-1} b)\right) \quad , \; b \in N.$$

$\varphi_* X$ is called the *pushforward* of X by φ.

<u>(7.7) Fact</u>: For $g \in \mathcal{E}(N)$ and $X \in \mathcal{V}(M)$:

$$L_{\varphi_* X}\, g\,(b) = X(g \circ \varphi)\,(\varphi^{-1}(b))$$

(7.8) <u>Corollary</u>: $\varphi_*([X,Y]) = [\varphi_* X, \varphi_* Y]$.