41. Weil Homomorphism

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In this section G is a Lie group with its Lie algebra $\mathfrak{g}$ and its adjoint representation $Ad : G \to GL(\mathfrak{g})$. And $\xi = (P, \pi, M, G)$ will be a principal fibre bundle.

(41.1) DEFINITION: A symmetric multilinear map $f : \mathfrak{g}^k \longrightarrow \mathbb{C}$ is G-invariant if

$$f(X_1, \dots X_k) = f(Ad_g X_1, \dots Ad_g X_k) \quad \forall g \in G \ \forall X_i \in \mathfrak{g}.$$

$S_G^k(\mathfrak{g})$ is the $\mathbb{C}$ vector space of k-multilinear, symmetric, G-invariant maps and

$$S_G^\bullet(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} S_G^k(\mathfrak{g})$$

is the corresponding algebra:

$$f \cdot g\,(X_1, \dots, X_{k+l}) := \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l}} f(X_{\tau_1}, \dots, X_{\tau_k})\, g(X_{\tau_{k+1}}, \dots X_{\tau_{k+l}}).$$

Whenever N is a manifold and $\omega_1, \dots \omega_k$ are $\mathfrak{g}$-valued forms every $f \in S_G^k(\mathfrak{g})$ determines $f(\omega_1 \wedge \dots \wedge \omega_k) \in \mathcal{A}^q(N, \mathfrak{g})$, $q = i_1 + \dots + i_k$, $\omega_j \in \mathcal{A}^{i_j}(N, \mathfrak{g})$ by

$$f(\omega_1 \wedge \dots \wedge \omega_k)(Y_1, \dots, Y_q) :=$$
$$= \frac{1}{i_1! \dots i_k!} \sum_{\tau \in S_q} \operatorname{sign} \tau \, f\left(\omega_1(Y_{\tau_1}, \dots Y_{\tau_{i_1}}), \dots, \omega_k(Y_{\tau_{q-i_k+1}}, \dots Y_{\tau_q})\right)$$

<u>(41.1) PROPOSITION:</u> $f \in S^k_G(\mathfrak{g})$. Then $f(\Omega^\omega \wedge \dots \wedge \Omega^\omega) \in \mathcal{A}^{2k}(P, \mathbb{C})$ is horizontal, right invariant ($\Psi_g^* \eta = \eta$) and closed. And for $\omega, \omega' \in \mathcal{A}(P)$: $f(\Omega^\omega \wedge \dots \wedge \Omega^\omega) - f(\Omega^{\omega'} \wedge \dots \wedge \Omega^{\omega'}) = d\eta$ for a suitable $\eta \in \mathcal{A}^{2k-1}(P, \mathbb{C})$.

<u>Pf</u>: $f(\Omega^\omega \wedge \dots \wedge \Omega^\omega)$ is horizontal since Ω^ω is horizontal!
$\Psi_g^* f(\Omega^\omega \wedge \dots \wedge \Omega^\omega) = f(\Psi_g^* \Omega^\omega \wedge \dots \wedge \Psi_g^* \Omega^\omega)$ (Definition!)
$= f(\mathrm{Ad}\, g^{-1} \Omega^\omega \wedge \dots \wedge \mathrm{Ad}\, g^{-1} \Omega^\omega) = f(\Omega^\omega \wedge \dots \wedge \Omega^\omega)$.
For horizontal and right invariant forms η one has $d\eta = D\eta$.
Hence,
$$d f(\Omega^\omega \wedge \dots \wedge \Omega^\omega) = Df(\Omega^\omega \wedge \dots \wedge \Omega^\omega) = f(D(\Omega^\omega \wedge \dots \wedge \Omega^\omega))$$
$$= \sum_{i=1}^{k} f(\Omega^\omega \wedge \dots \wedge \underbrace{D\Omega^\omega}_{\text{ith position}} \wedge \dots \Omega^\omega) = 0,$$
since $D\Omega^\omega = 0$ (Bianchi).

The proof of the second part of the proposition is more involved:
We set $\omega_t := \omega' + t(\omega - \omega') \in \mathcal{A}(P)$ and see with $\beta := \omega - \omega'$

$$\frac{d}{dt}\left(\Omega^{\omega_t}\right) = D^{\omega_t}\beta .$$

Then $\eta := k\int_0^1 f(\beta\wedge \underbrace{\Omega^{\omega_t}\wedge\ldots\wedge\Omega^{\omega_t}}_{(k-1)\text{ times}})\,dt$ satisfies $d\eta=\beta$:

η and $f(\beta\wedge\Omega^{\omega_t}\wedge\ldots\wedge\Omega^{\omega_t})$ are horizontal and right invariant.

$$\begin{aligned}
d\eta &= k\int_0^1 df(\beta\wedge\Omega^{\omega_t}\wedge\ldots\wedge\Omega^{\omega_t})\,dt\\
&= k\int_0^1 D^{\omega_t}f(\beta\wedge\Omega^{\omega_t}\wedge\ldots\wedge\Omega^{\omega_t})\,dt\\
&= k\int_0^1 f(D^{\omega_t}\beta\wedge\Omega^{\omega_t}\wedge\ldots\wedge\Omega^{\omega_t})\,dt \qquad (\text{Bianchi!})\\
&= k\int_0^1 f\left(\tfrac{d}{dt}(\Omega^{\omega_t})\wedge\ldots\wedge\Omega^{\omega_t}\right)dt\\
&= \int_0^1 \frac{d}{dt} f\left(\Omega^{\omega_t}\wedge\ldots\wedge\Omega^{\omega_t}\right)dt\\
&= f(\Omega^{\omega_1}\wedge\ldots\wedge\Omega^{\omega_1}) - f(\Omega^{\omega_0}\wedge\ldots\wedge\Omega^{\omega_0}) .
\end{aligned}$$

$\square$

In general, for a horizontal and right invariant form $\hat\eta\in\mathcal{A}^k(P,\mathbb{C})$ there exists $\eta\in\mathcal{A}^k(M,\mathbb{C})$ with $\hat\eta=\pi^*\eta$:

$$\eta_a(z_1,\ldots z_k) = \hat\eta_p(\check z_1,\ldots \check z_k)$$

Hence, there is a unique $2k$ form on M corresponding to $f(\Omega^\omega \wedge \ldots \wedge \Omega^\omega) \in \mathcal{A}^{2k}(P, \mathbb{C})$ which will be denoted by the same symbol $f(\Omega^\omega \wedge \ldots \wedge \Omega^\omega) \in \mathcal{A}^{2k}(M, \mathbb{C})$.

(41.3) DEFINITION: $W_P(f) := [f(\Omega^\omega \wedge \ldots \wedge \Omega^\omega)] \in H^{2k}_{dR}(M, \mathbb{C})$ for $f \in S^\bullet_G(\mathfrak{g})$ is a cohomology class which is independent of $\omega \in \mathcal{A}(P)$

$$W_P : S^\bullet_G(\mathfrak{g}) \longrightarrow H^\bullet_{dR}(M, \mathbb{C}) \;,\; f \longmapsto [f(\Omega^\omega \wedge \ldots \wedge \Omega^\omega)],$$

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is called the Weil Homomorphism.

Check: W_P is an algebra homomorphism!

(41.4) EXAMPLE: For the trivial pfb $P = M \times G$ the Weil homomorphism W_P vanishes: $W_P = 0$. The same is true if P admits a flat connection.

(41.5) PROPOSITION: Let $\varphi : M' \to M$ be a smooth map and $\varphi^* P =: M'$ the pullback pfb. Then for $f \in S^\bullet_G(\mathfrak{g})$:

$$W_{\varphi^* P}(f) = \varphi^* W(f)$$

For $P' \sim P$ we have $W_P = W_{P'}$. And W behaves well under reduction of the structure group $H \subset G$.

We describe $S^{\bullet}_G(\mathfrak{g})$ by polynomials on $\mathfrak{g}$.
A homogeneous polynomial on $\mathfrak{g}$ of degree k with respect to a basis $(b_1, \dots b_r)$ of $\mathfrak{g}$,

$$Q : \mathfrak{g} \to \mathbb{C}, \quad p(X) = \sum_{\nu_1=1}^{r} \cdots \sum_{\nu_k=1}^{r} a_\nu X^{\nu_1} \dots X^{\nu_k} \quad , \; a_\nu \in \mathbb{C},$$

where $X = X^s b_s$, $X^s \in \mathbb{K}$, $a_\nu = a_{\nu_1 \dots \nu_k} = a_{(\nu_1 \dots \nu_k)} \in \mathbb{C}$.

$P^k(\mathfrak{g})$ $\mathbb{C}$-vector space of k homog. polyn. on $\mathfrak{g}$

$P^{\bullet}(\mathfrak{g}) = \bigoplus_{k \geq 0} P^k(\mathfrak{g})$ algebra of polynomials on $\mathfrak{g}$

$Q \in P^{\bullet}(\mathfrak{g})$ is G-invariant $:\iff$

$$Q(\mathrm{Ad}_g X) = Q(X) \quad \text{for all } g \in G, \; X \in \mathfrak{g}.$$

$P^{\bullet}_G(\mathfrak{g})$ denotes the subalgebra of G-invariant polynomials.

Fact: $P^{\bullet}_G(\mathfrak{g}) \cong S^{\bullet}_G(\mathfrak{g})$.

Each $f \in S^k_G(\mathfrak{g})$ yields the homogeneous polynomial

$$Q_f(X) := f(X, X, \dots X) = \sum_{\nu_i=1}^{r} f(b_{\nu_1}, \dots, b_{\nu_k}) X^{\nu_1} \dots X^{\nu_k} \text{ the}$$

with $Q_f \in P_G^k(\mathfrak{g})$, and $f \mapsto Q_f$ is a bijectiv algebra homomorphism.

Now, let $\Omega^\omega = \Omega^\varrho b_\varrho$ with horizontal 2 forms $\Omega^\varrho, \varrho = 1, \dots r$.
When $f \in S_G^k(\mathfrak{g})$ is given by $Q = \sum a_\nu X^{\nu_1} \dots X^{\nu_k}$ we see

$$W_P(f) = [(f(\Omega \wedge \dots \wedge \Omega)] = [\sum a_\nu \, \Omega^{\nu_1} \wedge \dots \wedge \Omega^{\nu_k}]$$

and by definition for $Q = \sum a_\nu X^{\nu_1} \dots X^{\nu_k}$

$$W_P(Q) = [\sum a_\nu \Omega^{\nu_1} \wedge \dots \wedge \Omega^{\nu_k}] \in H^{2k}_{dR}(M, \mathbb{C})$$

So we arrive at an alternative description of the Weil homomorphism as

$$W_P : P_G^\bullet(\mathfrak{g}) \longrightarrow H^{2\bullet}_{dR}(M, \mathbb{C}) .$$