40. Gauge Field Theories

So far, we have described the kinematics of gauge field theories - if at all.
We come to the dynamics, - mainly in the special case of Yang Mills theory.

Fixed data are:

M a semiRiemannian oriented mfd of dim. n.
G a Lie group with its Lie algebra $\mathfrak{g}$.
β a symmetric, $\mathbb{R}$-bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $\det \beta \neq 0$, and bi-invariant $(\beta(\alpha_g X, \alpha_g Y) = \beta(X, Y)$, $X, Y \in \mathfrak{g})$.

Moreover, we study principal fiber bundles $\xi = (P, \pi, M, G)$ with M and G given.

(40.1) DEFINITION: The configuration space of pure YM theory is the space

$$\mathcal{A} = \mathcal{A}(P) = \{\omega \in \mathcal{A}^1(P, \mathfrak{g}) : \omega \text{ connection form}\}$$
$$\cong \omega_0 + \mathcal{A}^1(M, \mathrm{Ad}\, P) \qquad (\text{cf. } 38.6)$$

The YM-density is $\mathcal{L}(\omega) := -\frac{1}{2} \|\Omega^\omega\|^2$, $\omega \in \mathcal{A}$.

Here, $\|\Omega\|^2|_U = (\Omega,\Omega)|_U := \beta(F_{\mu\nu}, g^{\mu\lambda}g^{\nu\kappa}F_{\lambda\kappa})$ where $F = s^*\Omega$ for a section $s: U \to P$ and $F = F_{\mu\nu}\, dq^\mu \wedge dq^\nu$ in local coordinates. The expression is independent of the choice of the section since β is bi-invariant. Therefore, it extends to all of M.

(40.2) DEFINITION: The *motions* of the system are the stationary points of the action functional

$$S(\omega) := \int_M \mathcal{L}(\omega)\, \mathrm{vol}_M ,$$

i.e.

$$\omega \text{ motion} \iff \forall \eta \in \mathcal{A}^1(M, \mathrm{Ad}P): \frac{d}{d\varepsilon} S(\omega+\varepsilon\eta)\Big|_{\varepsilon=0} = 0$$

(40.3) PROPOSITION: The equations of motion are

$$D^*\Omega = 0 .$$

Here $D = D^\omega$ is the covariant differential, $\Omega = \Omega^\omega$ is the curvature and D^* is the adjoint of D

Pf. $$\Omega^{\omega+\varepsilon\eta} = D^{\omega+\varepsilon\eta}(\omega+\varepsilon\eta) = d(\omega+\varepsilon\eta) + \tfrac{1}{2}[(\omega+\varepsilon\eta),(\omega+\varepsilon\eta)]$$
$$= \left(d\omega + \tfrac{1}{2}[\omega,\omega]\right) + \varepsilon\left(d\eta + \tfrac{1}{2}([\omega,\eta]+[\eta,\omega])\right) + \tfrac{1}{2}\varepsilon^2[\eta,\eta]$$

By 39.8 we have $D\eta = d\eta + [\omega, \eta]$ for $\eta \in \mathcal{A}^1(M, \mathrm{Ad}\,P)$, hence

$$\Omega^{\omega+\varepsilon\eta} = \Omega^\omega + \varepsilon\, D^\omega \eta + \frac{1}{2}\varepsilon^2[\eta, \eta].$$

It follows: ω motion

$$\begin{aligned}
&\iff \int_M \frac{d}{d\varepsilon} \| \Omega^{\omega+\varepsilon\eta} \| \Big|_{\varepsilon=0} \mathrm{vol} = 0 && \forall \eta \\
&\iff \int_M (\Omega^\omega, D^\omega \eta)\, \mathrm{vol} = 0 && \forall \eta \\
&\iff \int_M (D^*\Omega, \eta)\, \mathrm{vol} = 0 && \forall \eta \\
&\iff D^*\Omega = 0
\end{aligned}$$

□

(40.4) COROLLARY: The motions satisfy

$$D^*\Omega = 0 \quad \text{and} \quad D\Omega = 0.$$

(The Bianchi $D\Omega = 0$ is always satisfied.)

The operator D^* can also be described bei the Hodge star product $*$. $*$ is induced by the metric and gives a map

$$* : \mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M)$$

such that $D^{\omega *} = \pm * D^{\omega} *$.

In the case of d and d^* on M with a Riemannian metric the Laplacian is

$$\Delta = dd^* + d^*d : \mathcal{A}^k \to \mathcal{A}^k,$$

and the forms η with $\Delta\eta = 0$ are the harmonic forms. Regarding 40.4 there is a close analogy between the YM-motions and the theory of harmonic forms (Hodge theory). Insofar, YM-theory is considered as a kind of nonabelian Hodge theory.

We want to describe how connections and curvatures change under general automorphisms

$$\Phi : \xi \to \xi$$

of the pfb ξ (global gauge transformation).

(40.5) DEFINITION: A gauge transformation (automorphism) of ξ is given by a smooth $\Phi : P \to P$ with $\pi \circ \Phi = \pi$ and $\Phi \circ \Psi_g = \Psi_g \circ \Phi$ for all $g \in G$.

Let $\mathcal{G}(P)$ denote the group of gauge transformations. $\mathcal{G}(P)$ is in a natural bijection to

$$\mathcal{E}(P,G)^G := \{ f : P \to G \mid f(pg) = g^{-1} f(p) g \}.$$

The bijection $f \longmapsto \Phi_f$ is given by

$$\Phi_f(p) := pf(p) \ , \quad p \in P.$$

Two connections ω and ω' which are related by a gauge transformation Φ in the manner

$$\omega' = \Phi^* \omega$$

describe the same physics and should be identified if they are related by $\omega' = \Phi^*\omega$. As a result instead of $\mathcal{A}(P)$ the quotient

$$\mathcal{A}(P)/\mathcal{G}(P) = \mathcal{A}(P)/\sim \qquad (\omega' \sim \omega :\Longleftrightarrow \exists \Phi \in \mathcal{G}(P) : \omega' = \Phi^*\omega)$$

is the true configuration space. The space $\mathcal{A}(P)/\mathcal{G}(P)$ of gauge orbits is not easy to handle since $\mathcal{A}(P)$ and $\mathcal{G}(P)$ are infinite dimensional (manifolds).

Comparing ω and $\Phi^*\omega$ we have the following [31.01.11]

(40.6) PROPOSITION: Let Φ be a gauge transformation on ξ and let ω be a connection. Then $\Phi^*\omega$ is a connection as well and we have

1° $\Phi^*\omega = Ad_{f^{-1}}\omega + f^*\kappa$ $\qquad (\Phi = \Phi_f)$

2° $\Phi \circ P_\gamma^{\Phi^*\omega} = P_\gamma^{\omega} \circ \Phi$ $\qquad \kappa([g e^{tX}]) = X$

3° $D^{\Phi^*\omega} \circ \Phi^* = \Phi^* \circ D^\omega$

4° $\Omega^{\Phi^*\omega} = \Phi^*\Omega^\omega = Ad_{f^{-1}}\Omega^\omega$

Pf. We only show that $\Phi^*\omega$ is a connection form: For $X \in \mathfrak{g}$ and $p \in P$

$$T_p\Phi(X^*(p)) = T_p\Phi([pe^{tX}]_p) = [\Phi(pe^{tX})]_{\Phi(p)} = [\Phi(p)e^{tX}]_{\Phi(p)} = X^*(\Phi(p))$$

Hence, $\Phi^*\omega$ satisfies (ω1):

$$\Phi^*\omega(X^*(p)) = \omega_{\Phi(p)}(T_p\Phi(X^*(p))) = \omega_{\Phi(p)}(X^*(\Phi(p))) = X,$$

But (ω2) is satisfied, too:

$$\Psi_g^*\Phi^*\omega = \Phi^*\Psi_g^*\omega = \Phi^*(Ad_{g^{-1}}\omega) = Ad_g(\Phi^*\omega).$$

The rest of the proof is standard. Notice, that some of the formulas resemble the formulas for local gauge transformations, in particular 1° and 4°. In fact, looking at the definition of Φ_f it could be denoted as Ψ_f as well, and we would have $\Psi_f^* \omega = Ad_{f^{-1}}\omega + f^*\kappa$ (1°) and $\Psi_f^*\Omega = Ad_{f^{-1}}\Omega$ (4°).

It is in general difficult to describe solutions of the YM equations $D^*\Omega = 0$. But in the case of a 4 dim. Riemannian manifold M we can look for special solutions. In that case the Hodge operator on 2 forms maps into 2 forms

$$* : \mathcal{A}^2(M) \to \mathcal{A}^2(M)$$

To describe $*$ on $\mathcal{A}^2(M)$ we use the fact that the Riemannian metric induces a Riemannian metric on T^*M and for an orthonormal basis $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in T_a^*M

$$*(\sigma_1 \wedge \sigma_2) = \sigma_3 \wedge \sigma_4, \quad *(\sigma_1 \wedge \sigma_3) = -\sigma_2 \wedge \sigma_4 \quad \text{etc.}$$

$$*(\sigma_{i_1} \wedge \sigma_{i_2}) = \mathrm{sgn}(i_1, i_2, i_3, i_4)\, \sigma_{i_3} \wedge \sigma_{i_4}, \quad \{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}$$

<u>(40.7) DEFINITION:</u> A two-form $\eta \in \mathcal{A}^2(M)$ is self-dual (resp. antiself-dual) if $*\eta = \eta$ (resp. $*\eta = -\eta$).

A connection is <u>self-dual</u> (<u>antiself-dual</u>) if Ω^ω is selfdual (antiself-dual), as section $\Omega^\omega \in \mathcal{A}^2(M, \mathrm{Ad}P)$.

<u>(40.8) PROPOSITION</u>: Every self-dual connection is a YM-connection and the same is true for the antiself-dual connections.

<u>Pf.</u> We use the presentation $D^* = -*D*$ and can show for $*\Omega = \pm\Omega$:

$$D^*\Omega = -*D*\Omega = -D(\pm\Omega) = 0 \qquad (\text{Bianchi}). \qquad \square$$

Self-dual connections are also called <u>instantons</u>. On $\mathbb{R}^4$ with the euclidean metric they have been studied in great detail.

General gauge field theories: Only $\xi = (P, \pi, M, G)$ is fixed and a representation $\rho : G \to GL(r, \mathbb{C})$.

- $\mathcal{M}$ space of metrics (fixed signature p, q) on M
- $\mathcal{F} = \Gamma(M, E_\rho)$

more dyn. variables

$$\mathcal{L} : \mathcal{D} \to \mathcal{A}^n(M), \qquad \mathcal{D} \subset \mathcal{M} \times \mathcal{F} \times \mathcal{A} \times \dots$$

$$S = \int_M \mathcal{L}$$

$\delta S = 0 \iff$ Equations of motion.

Conditions on $\mathcal{L}$:

1. Gauge invariance in case of $\mathcal{D} = \mathcal{A}$

$\mathcal{L}(\Phi^* \omega) = \mathcal{L}(\omega) \quad \forall \Phi \in \mathcal{G}(P)$. Example: $\mathcal{L}_{YM}$

Result. $\omega \sim \omega'$: ω motion $\iff$ ω' motion

2. Gauge invariance in case of $\mathcal{D} = \mathcal{A} \times \mathcal{F}$

$\mathcal{L}(\Phi^* \omega, \Phi^* s) = \mathcal{L}(\omega, s) \quad \forall \Phi \in \mathcal{G}(P)$

$\left(s(a) = [p, f_s(p)] \Rightarrow \Phi^* s(a) = [p, f_s \circ \Phi(p)] \right)$

"Same" result!

3. Covariance $\quad \mathcal{D} \subset \mathcal{M} \times \mathcal{A} \times \mathcal{F}$

$\mathcal{L}(f_M^* g, f^* \omega, f^* s) = \mathcal{L}(g, \omega, s)$, $f \in \mathrm{Diff}_\pi(P)$

Other: Naturality, Conformal invariance, ...

$\rightsquigarrow$ Principle of gauge invariance

$\rightsquigarrow$ Examples.