39. Curvature and Structure equations

Version 1.1

Throughout this section $\xi = (P, \pi, M, G)$ will be a principal fibre bundle (pfb) with a connection mostly given by a connection form $\omega \in \mathcal{A}^1(P, \mathfrak{g})$, $\mathfrak{g} = \mathrm{Lie}\, G$.

(39.1) DEFINITION: The *curvature* (*form*) $\Omega = \Omega^\omega$ of the connection is

$$\Omega := D^\omega \omega \quad (= h^* \circ d(\omega)) \in \mathcal{A}^2(P, \mathfrak{g}).$$

(39.2) PROPOSITION: The curvature Ω satisfies

1° $\Omega(Y, Y') = 0$ if Y or Y' is vertical,

2° $\Psi_g^* \Omega = \mathrm{Ad}_{g^{-1}}(\Omega)$.

Hence, $\Omega \in \mathcal{A}^2_{hor}(P, \mathfrak{g})^{G, \mathrm{Ad}}$.

Pf. 1° Let Y be vertical, i.e. $v(Y) = Y$ and $h(Y) = 0$. Then $\Omega(Y, Y') = d\omega(h(Y), h(Y')) = 0$.

2° $\Psi_g^* \Omega = \Psi_g^* h^* d\omega = h^* \Psi_g^* d\omega = h^* d(\Psi_g^* \omega) = h^* d(\mathrm{Ad}_{g^{-1}} \omega)$ according to $\omega(2)$. Finally,

$\Psi_g^* \Omega = h^* d(\mathrm{Ad}_{g^{-1}} \omega) = \mathrm{Ad}_{g^{-1}}(h^* d\omega) = \mathrm{Ad}_{g^{-1}}(\Omega)$. □

Recall the product $[\ ,\ ]$ on $\mathfrak{g}$-valued forms (cf. section 27):

(39.3) DEFINITION: 1° For $\eta, \vartheta \in \mathcal{A}^1(P, \mathfrak{g})$:

$$[\eta, \vartheta](Y, Y') := [\eta(Y), \vartheta(Y')] - [\eta(Y'), \vartheta(Y)], \quad Y, Y' \in \mathcal{V}(P).$$

2° For $\beta \in \mathcal{A}^2(P, \mathfrak{g})$, $\eta \in \mathcal{A}^1(P, \mathfrak{g})$:

$$[\beta, \eta](X, Y, Z) := [\beta(X, Y), \eta(Z)] + [\beta(Y, Z), \eta(X)] + [\beta(Z, X), \eta(Y)]$$

$$[\eta, \beta] := -[\beta, \eta].$$

The following formulas hold true:

$$[[\eta, \eta], \eta] = 0 \qquad \text{and}$$

$$d([\eta, \eta]) = [d\eta, \eta] - [\eta, d\eta] = 2[d\eta, \eta].$$

(39.4) PROPOSITION: The curvature fulfills the structure equations

$$\boxed{\Omega = d\omega + \tfrac{1}{2}[\omega, \omega].}$$

Proof. We have to show $\Omega_p(Y, Y') = d\omega_p(Y, Y') + \frac{1}{2}[\omega_p, \omega_p](Y, Y')$ for all $Y, Y' \in T_pP$. Since every tangent vector has a unique decomposition into the sum of a horizontal and a vertical vector field it is enough to consider the following two cases:

1. Y and Y' are both horizontal. Then $\omega_p(Y) = \omega_p(Y') = 0$, hence $[\omega_p, \omega_p](Y, Y') = 0$. Moreover, $\Omega_p(Y, Y') = d\omega_p(hY, hY') = d\omega_p(Y, Y')$ which implies that the structure equation holds.

2° Y or Y' is vertical. Then $\Omega_p(Y, Y') = 0$ by 39.2. Let Y be vertical and define $X := \omega_p(Y) \in \mathfrak{g}$. Then $Y = X^*(p)$. Extend Y' to a vector field with the same notation Y'. We use the formula $L_Z = d\, i_Z + i_Z d$ for $Z = X^*$, $L_{X^*}\omega = d i_{X^*}\omega + i_{X^*} d\omega$. Because $i_{X^*}\omega = \omega(X^*) = X$ is constant by (ω1) the formula reduces to $L_{X^*}\omega = i_{X^*} d\omega$. By definition

$$L_{X^*}\omega(Y') = \frac{d}{dt}\, \varphi_t^*\omega(Y')\Big|_{t=0}$$

with φ_t being the flow of X^* : $\varphi_t(p) = pe^{tX} = \Psi_{e^{tX}}(p)$. Hence,

$$L_{X^*}\omega(Y') = \frac{d}{dt}\, \omega(T\Psi_{e^{tX}}(Y'))\Big|_{t=0} = \frac{d}{dt}\, \mathrm{Ad}_{e^{tX}}\, \omega(Y')\Big|_{t=0}$$

(ω2!), and therefore

$$L_{X^*}\omega(Y') = [\omega(Y'), X] = -[\omega(Y), \omega(Y')]$$

Now,

$$\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right)(Y, Y') = d\omega(X^*, Y') + \tfrac{1}{2}[\omega,\omega](Y, Y')$$

$$= i_{X^*} d\omega(Y') + [\omega(Y), \omega(Y')] = 0. \qquad \square$$

(39.5) COROLLARY: 1° The structure equations yield a decomposition of $d\omega$ into horizontal and vertical forms.

[25.01.11]

$$\boxed{d\omega = \Omega - \tfrac{1}{2}[\omega,\omega].}$$

2° $\boxed{D\Omega = 0.}$ (Bianchi-identity)

Pf. 1° holds since ω is vertical and $[\omega,\omega]$ as well.

2° $d[\omega,\omega] = 2[d\omega,\omega]$ (cf. 39:3) applied to $d\Omega$ yields

$$d\Omega = d\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right) = [d\omega, \omega] = \left[d\omega + \tfrac{1}{2}[\omega,\omega], \omega\right] = [\Omega, \omega],$$

and we conclude

$$D\Omega = h^* d\Omega = h^*([\Omega,\omega]) = 0$$

since ω is vertical.

Es gibt viele weitere wichtige Formeln. Zum Beispiel:

(39.6) PROPOSITION: For horizontal k-forms $\eta \in \mathcal{A}^k_{hor}(P, \mathbb{K}^r)^{G,\rho}$ of type ρ:

$$DD\eta = \rho_*(\Omega) \wedge \eta .$$

Here, $\rho : G \to GL(r, \mathbb{K})$ is a representation and $D = D^\omega$ is the covariant differential $D = D^\omega$.

Pf. In order to show this formula for the covariant differential $D = D^\omega$ we need the following:

(39.7) Proposition: $D : \mathcal{A}^k(P, \mathbb{K}^r) \to \mathcal{A}^{k+1}(P, \mathbb{K}^r)$ satisfies $D(\mathcal{A}^k_{hor}(P, \mathbb{K}^r)^{G,\rho}) \subset \mathcal{A}^{k+1}_{hor}(P, \mathbb{K}^r)^{G,\rho}$ and for $\eta \in \mathcal{A}^k_{hor}(P, \mathbb{K}^r)^{G,\rho}$ we have

$$D\eta = d\eta + \rho_*(\omega) \wedge \eta .$$

To show 39.6 we use the fact that η and $D\eta$ are horizontal of type ρ such that

$$DD\eta = d\left(d\eta + \rho_*(\omega)\wedge\eta\right) + \rho_*(\omega)\wedge\left(d\eta + \rho_*(\omega)\wedge\eta\right)$$

$$= d\rho_*(\omega)\wedge\eta - \rho_*(\omega)\wedge d\eta + \rho_*\omega\wedge d\eta + \rho_*(\omega)\wedge\rho_*(\omega)\wedge\eta$$

$$= \rho_*(d\omega)\wedge\eta + \rho_*(\omega)\wedge\rho_*(\omega)\wedge\eta$$

Moreover,

$$\rho_*(\omega)\wedge\rho_*(\omega)\ (X,Y) = \left[\rho_*(\omega)(X), \rho_*(\omega)(Y)\right]_{gl(r,\mathbb{K})}$$

$$= \rho_*\left(\left[\omega(X), \omega(Y)\right]_{\mathfrak{g}}\right) = \rho_*\left(\tfrac{1}{2}[\omega,\omega]\ (X,Y)\right)$$

Hence,

$$DD\eta = \rho_*\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right)\wedge\eta = \rho_*(\Omega)\wedge\eta$$

□

39.7 can be shown by directly checking the statements, as well.

<u>(39.8) COROLLARY:</u> Applied to the adjoint action $\rho = Ad$ the formulas read as

$$D\eta = \omega\wedge\eta \qquad \text{for } \eta\in\mathcal{A}^k_{hor}(P,\mathfrak{g})^{G,Ad}$$

$$DD\eta = \Omega\wedge\eta \qquad \text{for } \eta\in\mathcal{A}^k_{hor}(P,\mathfrak{g})^{G,Ad}$$

with the same formula for $D: \mathcal{A}^k(M, AdP) \to \mathcal{A}^{k+1}(M, AdP)$.

<u>(39.10) PROPOSITION</u> (Local Formulas): With respect to s_j and g_{ij} we obtain for $A_j := s_j^* \omega \in \mathcal{A}^1(U_j, \mathfrak{g})$

- $A_j = \mathrm{Ad}_{g_{ij}^{-1}} A_i + g_{ij}^* \kappa$
- $F_j = s_j^* \Omega \in \mathcal{A}^2(U_j, \mathfrak{g})$ local curvature
- $F_j = \mathrm{Ad}_{g_{ij}^{-1}} F_i$
- $F_j = dA_j + \frac{1}{2}[A_j, A_j]$
- $dF_j = [F_j, A_j]$

Here $\kappa \in \mathcal{A}^1(G, \mathfrak{g})$ is defined by $\kappa_g(\tilde{X}_g) = X$ for $X \in \mathfrak{g}$ and $\tilde{X}$ the corresponding right invariant vector field.

Important for the parallel transport is

<u>(39.11) PROPOSITION</u>: For horizontal vector fields $Y, Y' \in \mathcal{V}(P)$ we have

$$\Omega(Y, Y') = -\omega([Y, Y']).$$

As a consequence,

$$\omega([Y, Y']) = -(\omega(Y, Y'))^* = -\Omega(Y, Y')^*$$

<u>Pf</u>. $\Omega(Y, Y') = d\omega(Y, Y')$ since $\omega(Y) = \omega(Y') = 0$

$d\omega(Y, Y') = L_Y \omega(Y') - L_{Y'} \omega(Y) - \omega([Y, Y']) = -\omega([Y, Y'])$.

According to the definition of ω we conclude
$v([Y, Y']) = -\Omega(Y, Y')^*$ $\square$

This result yields a condition for $H \subset TP$ being an involutive distribution. By definition a distribution $F \subset TN$ on a manifold N (i.e. a sub vector field $N \subset TN$ of the tangent bundle) is <u>involutive</u> if for all vector fields X, Y in F (i.e. $X(a), Y(a) \in F_a$, $\forall a \in N$) the bracket $[X, Y]$ is in F as well (i.e. $[X, Y](a) \in F_a$, $\forall a \in N$). The theorem of Frobenius states that all involutive distributions are <u>integrable</u>, i.e. at each $a \in N$ there exists a submanifold $S \subset N$ with $a \in S$ and $T_{a'}S = F_{a'}$ for all $a' \in S$.

<u>(39.12) PROPOSITIN:</u> 1° The vertical bundle $V \subset TP$ is involutive.

2° $H \subset TP$ is involutive if and only if $d\Omega = 0$.

<u>Pf.</u> 1° $[X, Y]^* = [X, Y]^*$ for $X, Y \in \mathfrak{g}$. And also: For $p \in P$ the fibre $\pi^{-1}(\pi p) = P_p$ is a submfd with $T_q P_p = V_q$ for all $q \in P_p$.

2° The vertical component of two horizontal vector fields Y, Y' is $-\Omega(Y,Y')$ according to 39.11. Hence $[Y,Y']$ alwas in $H \iff \Omega = 0$.

$\square$

In particular, in the case of the canonical flat connection on the product pfb $P \cong M \times G$ we conclude that the curvature vanishes.
We have explained in 37.11 and proven in the exercises that in a pfb ξ with connection, the parallel transport is locally independent of the curves if ξ with connection is locally isomorphic to the product pfb with the canonical flat connection. In particular, the curvature on such a pfb is vanishing. This yields the following characterization of a connection with $\Omega = 0$:

<u>(39.13) THEOREM:</u> Let ω be a connection on a pfb $\xi = (P, \pi, M, G)$. The following properties are equivalent:

1° ω is flat, i.e. $\Omega = D^\omega \omega = 0$.
2° The horizontal distribution $H \subset TP$ is integrable

3° There exists an open cover $(U_i)_{i\in I}$ of M such that: The p/b P_{U_j} with $\omega|_{P_{U_j}}$ is isomorphic to $U_j \times G$ with the canonical flat connection.

4° There exists an open cover $(U_i)_{i\in I}$ of M such that: Parallel transport in P_{U_j} with $\omega|_{P_{U_j}}$ is independent of the curves in U_j.

Sketch of proof. 1° $\Longleftrightarrow$ 2° by Frobenius and 39.12. 3° $\Longleftrightarrow$ 4° and 3° $\Rightarrow$ 2° as we have just explained. It remains to show 2° $\Rightarrow$ 3°. Let $p \in P$ and S a submanifold of P with $T_q S = H_q$ for $q \in S$. Let U be a normal neighbourhood of $a = \pi(p)$ with respect to any Riemannian metric on M. To define a section $s : U \to P$ let γ be the radial geodesic connecting a with $b \in U$, $\gamma(0) = a$, $\gamma(1) = b$. Let $\check{\gamma}$ be the horizontal lift of γ with $\check{\gamma}(0) = p$. Then $\check{\gamma}(1) \in S$, if U is chosen small enough, and moreover, $\{\check{\gamma}(1)\} = S \cap P_b$. We set $s(b) := \check{\gamma}(1)$ and observe that $s : U \to P$ is a section with $s(U) \subset S$. In addition, s is horizontal in the sense of

$$T_b s(T_b M) = H_{s(b)}, \quad b \in U,$$

by construction ($s(U) \subset S$!). Now the map

$$\psi : U \times G \to P_U, \quad (b,g) \mapsto s(b)g$$

is an isomorphism of pfbs and for $y=(b,g)$

$$T_y\psi(T_bM \oplus 0) = T_y\mathcal{R}_g \circ T_bs\,(T_bM) = T_y\mathcal{R}_g\left(H_{s(b)}\right) = H_{s(b)g} = H_{\psi(y)}.$$

$\square$