38. Associated Connections

Version 1.1

Let $\xi = (P, \pi, B, G)$ be a principal fibre bundle with a connection, and let $\varrho : G \to GL(r, \mathbb{K})$ be a representation.

$E_\varrho := P \times_\varrho \mathbb{K}^r$ is a vector bundle of rank r, the associated bundle (cf. section 34).

A smooth map $f \in \mathcal{E}(P, \mathbb{K}^r)$ is called *G-invariant* if

$$f(pg) = g^{-1} f(p) = \varrho(g^{-1}) f(p) \quad \text{for all} \quad (p, g) \in P \times G.$$

The vector space of G-invariant maps will be denoted by

$$\mathcal{E}(P, \mathbb{K}^r)^{G,\varrho} := \{ f \in \mathcal{E}(P, \mathbb{K}^r) : f \text{ G-invariant} \}.$$

$\mathcal{E}(P, \mathbb{K}^r)^{G,\varrho}$ is an $\mathcal{E}(M)$-module via π^*:

$$\lambda \cdot f := (\lambda \circ \pi) f = \pi^*(\lambda) f \quad , \lambda \in \mathcal{E}(M), f \in \mathcal{E}(P, \mathbb{K}^r)$$

Given $f \in \mathcal{E}(P, \mathbb{K}^r)^{G,\varrho}$ we set

$$s(a) := [(p, f(p))] , \quad p \in \pi^{-1}(a) = P_a ,$$

and obtain a smooth section $s \in \Gamma(M, E_\varrho)$: s is well-defined since for $p' = pg$:

$$(p', f(p')) = (pg, g^{-1}f(p)) \sim (p, f(p)), \text{ hence } [p', f(p')] = [p, f(p)].$$

This s will be denoted by $\Phi(f) := s$.

<u>(38.1) LEMMA:</u> $\Phi : \mathcal{E}(P, \mathbb{K}^r)^{G, \varrho} \longrightarrow \Gamma(M, E_\varrho)$ is an $\mathcal{E}(M)$-module isomorphism.

[24.01.11]

Pf. Φ is in fact $\mathcal{E}(M)$-linear:

$$\Phi(\lambda \cdot f + g)(p) = [(p, \lambda(\pi(p)) f(p) + g(p))] =$$

$$= \lambda(a) [(p, f(p))] + [(p, g(p))] = \lambda(a)\, \Phi(f)(a) + \Phi(g)(a).$$

Moreover, Φ is bijective: Clearly it is injective.
Let $s \in \Gamma(U, E_\varrho)$ and $\psi : P_U \to U \times G$ a local trivialization with $s_\psi(a) := \psi^{-1}(a, e)$. There is a $v \in \mathbb{K}^r$ with $s(a) = [(s_\psi(a), v)]$, uniquely, and we denote this v by $f_\psi(s_\psi(a)) := v$. For $p = s_\psi(a) g$ we set $f_\psi(p) = g^{-1} f_\psi(s_\psi(a))$. Similarly, for another local trivialization $\bar\psi : P_{\bar U} \to \bar U \times G$. In case of $\bar U \cap U \neq \emptyset$ we check

$$f_\psi |_{\pi^{-1}(U \cap \bar U)} = f_{\bar\psi} |_{\pi^{-1}(U \cap \bar U)} :$$

For $a \in U \cap \bar U$ there is $h \in G$ with $s_{\bar\psi}(a) = s_\psi(a) h$. From

$$[(s_\psi(a), f_\psi(\psi(a))] = [s_{\bar\psi}(a), h^{-1} f_\psi(s_\psi(a))] = [s_{\bar\psi}(a), f_\psi(s_{\bar\psi}(a))]$$

we get $s(a) = [s_{\bar\psi}(a), f_\psi(s_{\bar\psi}(a))]$, hence $f_{\bar\psi}(s_{\bar\psi}(a)) = f_\psi(s_{\bar\psi}(a))$.
Therefore, $f|_U := f_\psi$ defines a smooth function $f \in \mathcal{E}_\varrho(P, \mathbb{K}^r)$ with $\Phi(f) = s$. Hence, Φ is surjective. $\square$

<u>(38.2) DEFINITION - PROPOSITION:</u> Every connection ω on ξ induces a uniquely determined connection $\nabla = \nabla^\omega$ on $E = E_\varrho$ by

$$\nabla_X := \Phi \circ L_{\check{X}} \circ \Phi^{-1} : \Gamma(M, E_\varrho) \to \Gamma(M, \varrho), \quad X \in \mathcal{V}(M).$$

∇ is called the <u>associated connection</u>.
(The converse is true for $G = GL(r, K)$, $P = R(E)$.)

Pf. ∇_X is a connection on E:

- $\nabla_X s$ is $\mathcal{E}(M)$-linear in $X \in \mathcal{V}(M)$
- $\nabla_X s$ is $\mathbb{R}$ - " " $s \in \Gamma(M, E)$
- $\nabla_X s$ satisfies the Leibnizrule:

For $s = \Phi(f)$, $\lambda \in \mathcal{E}(M)$, we get $\Phi^{-1}(\lambda s) = \lambda f$ and

$$L_{\check{X}}(\lambda \cdot f) = L_{\check{X}}(\lambda \circ \pi) f + (\lambda \circ \pi) L_{\check{X}} f = (L_X \lambda) \circ \pi \, f + (\lambda \circ \pi) L_{\check{X}} f,$$

hence

$$\nabla_X(\lambda s) = \Phi(L_X^\vee(\lambda\cdot f)) = \Phi((L_X\lambda)\circ\pi f + (\lambda\circ\pi)L_X^\vee f)$$
$$= L_X\lambda\cdot\Phi(f) + \lambda\,\Phi(L_X^\vee f) = (L_X\lambda)s + \lambda\nabla_X s \, . \qquad \square$$

<u>(38-3) PROPOSITION:</u> Let the connection on ξ be given by local gauge connections α_j on U_j. Then the associated connection ∇ on $E = E_\varrho$ can also be described by the local connection forms $\varrho_*(\alpha_j) \in \mathcal{A}^1(U_j, \mathrm{End}(E_\xi))$:

$$\nabla s|_{U_j} = \text{"}\, d + \varrho_*(\alpha_j)\,\text{"}|_{U_j} \qquad , s \in \Gamma(M, E)$$

in the sense of

$$pr_2\circ\psi_j\,(\nabla_X s|_{U_j}) = d\,(pr_2\circ\psi_j\circ s|_{U_j}) + \varrho_*(\alpha_j)(X)\,(pr_2\circ\psi_j\circ s|_{U_j})$$

for $\psi_j : E_{U_j} \to U_j \times \mathbb{K}^r$ the trivialization (cf. 23.8).

<u>(38.4) DEFINITION:</u> A k-form $\eta \in \mathcal{A}^k(P, \mathbb{K}^r)$ is called

1° horizontal when $\eta(X_1, \dots X_k) = 0$ if one of the X_j is vertical.

2° of type ϱ when $\Psi_g^*\eta = \varrho(g^{-1})\circ\eta$ for $g \in G$.

$\mathcal{A}^k_{hor}(P, \mathbb{K}^r)^{G,\varrho}$ is the space of horizontal k-forms of type ϱ.

(38.5) PROPOSITION: In generalization to 38.1 the spaces

$$\mathcal{A}^k_{hor}(P, \mathbb{K}^r)^{G,\varrho} \quad \text{and} \quad \mathcal{A}^k(M, E_\varrho)$$

are isomorphic as $\mathcal{E}(M)$-modules.

Pf (sketch): Let $\beta \in \mathcal{A}^k_{hor}(P, \mathbb{K}^r)^{G,\varrho}$. For $X_1, \dots X_k \in \mathcal{V}(M)$ and $a \in M$ we set

$$\sigma_\beta(a)\,(X_1(a), \dots, X_k(a)) := [(p, \beta_p(\check{X}_1(p), \dots X_k(p))] \in E_a .$$

To show that σ_β is well defined let $p' = pg$. Then

$$(p', \beta_{p'}(\check{X}_1(p'), \dots, \check{X}_k(p')) = (p', \beta_{pg}(T_p\Psi_g\, \check{X}_1(p), \dots T_p\Psi_g\, \check{X}_k(p)))$$

$$= (pg, \varrho^{-1}(g)\,\beta_p(\check{X}_1(p), \dots \check{X}_k(p)) \sim (p, \beta_p(\check{X}_1(p), \dots \check{X}_k(p)).$$

Hence, $\sigma_\beta \in \mathcal{A}^k(M, E)$. It is easy to check that $\beta \longmapsto \sigma_\beta$ is $\mathcal{E}(M)$-linear an bijective. $\square$

If ω and ω' are connection forms on P the difference $\eta = \omega - \omega' \in \mathcal{A}^1(P, \mathfrak{g})$ is horizontal (for $Y_p = X^*_p$, $X \in \mathfrak{g}$: $\omega(Y_p) = X = \omega'(Y_p)$) and of type $Ad : G \to GL(\mathfrak{g}) = GL(k, \mathbb{R})$, $k = \dim G$:

$$\Psi_g^* \eta = \Psi_g^* \omega - \Psi_g^* \omega' = Ad_{g^{-1}} \omega - Ad_{g^{-1}} \omega' = Ad_{g^{-1}}(\eta),$$

because of (ω2) (cf 37.13). By 38.5 η can be represented by a 1-form on M with values in the adjoint vector bundle $\mathrm{Ad}\,P := P \times_{\mathrm{Ad}} \mathfrak{g}$. In this way we obtain

(38.6) COROLLARY: The set of all connections on a pfb $\xi = (P, \pi, M, G)$ is an affine space with vector space $\mathcal{A}^1(M, \mathrm{Ad}\,P) \cong \mathcal{A}^1_{hor}(P, \mathfrak{g})^{G, \mathrm{Ad}}$.

(38.7) REMARK: Reversing the viewpoint it is natural to ask whether a given connection ∇^E on the vector bundle $E = E_\rho$ determines a connection on ξ. This is certainly the case if for a collection of local connection forms $A_j^E \in \mathcal{A}^1(U_j, \mathrm{End}\,E)$ defining ∇^E there are suitable $\alpha_j \in \mathcal{A}^1(U_j, \mathfrak{g})$ with $\rho_*(\alpha_j) = \alpha_j^E$.

This condition is satisfied for an arbitrary vector bundle E with connection ∇^E with respect to the frame bundle $R(E)$ for P and the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ for ρ: $E \cong R(E) \times_{\mathrm{Ad}} \mathbb{K}^r$.

(38.8) COROLLARY: As a consequence the theory of connections on vector bundles is part of the theory of connections on principal fibre bundles.