37. Connections on Principal Fibre Bundles

Version 1.3

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The central geometric object in the geometry of principal fibre bundles is the concept of a connection.

On the basis of our experience with the concept of a connection in a vector bundle we present in this section five versions of a connection which are all equivalent to each other but essentially different in its formulation.

The five versions are:

Throughout this section let $\xi = (P, \pi, M, G)$ be a principal fibre bundle. The right action of G on P is denoted by $\Psi : P \times G \to P$, in particular for $g \in G$

$$\Psi_g : P \to P, \; p \mapsto \Psi_g(p) := pg = \Psi(p, g),$$

is the right action of a fixed group element $g \in G$.

<u>Version 1.</u> We start with the version of a connection as a <u>horizontal distribution</u> on P, i.e. a vector subbundle $H \subset TP$ of the tangent bundle $TP \xrightarrow{\tau_P} P$ of P.

<u>(37.1) DEFINITION:</u> The <u>vertical bundle</u> $V_P = V$ is the subbundle

$$V_P := \ker T\pi \subset TP.$$

V_P is a k-dimensional ($k = \dim_{\mathbb{R}} G$) subbundle of TP (note that $T_p\pi : T_pP \to T_{\pi(p)}M$ is surjective, hence $T\pi$ has constant rank). Similar to 24.2 we have:

<u>(37.2) LEMMA:</u> For $p \in P$ the fibre V_p of V at p has the form

$$V_p = \{ [pe^{Xt}]_p : X \in \mathrm{Lie}\, G = \mathfrak{g} \}.$$

Pf: Let $a = \pi(p) \in M$. $\pi(pe^{Xt}) = \pi(p) = a$, hence $T_p\pi([pe^{Xt}]_p) = 0$, i.e. $[pe^{tX}]_p \in V_p$. Moreover, the map

$$R_p : \mathfrak{g} \to V_p \, , \quad X \mapsto [pe^{tX}]_p ,$$

is linear and injective since $\exp : \mathfrak{g} \to G$ is a local diffeomorphism. Hence, R_p is surjective ($k = \dim \mathfrak{g} = \dim V_p$), which implies $V_p = \{ [pe^{tX}]_p : X \in \mathfrak{g} \}$. $\square$

<u>(37.3) DEFINITION:</u> A <u>connection on</u> ξ is a subbundle $H \subset TP$ of the tangent bundle TP such that

(H1) $TP = V \oplus H$

(H2) For all $(p,g) \in P \times G$: $T_p\Psi_g(H_p) = H_{pg}$.

The action of G on P carries over to an action $\tilde{\Psi}$ on TP (and on V):

$$\tilde{\Psi} : TP \times G \to TP \, , \, (Y,g) \mapsto T\Psi_g(Y),$$

with $\tau_P \circ \tilde{\Psi}_g(Y) = \Psi_g \circ \tau_P(Y)$. Condition (H2) asserts that the decomposition $H \oplus V = TP$ is invariant with respect to $\tilde{\Psi}$.

A slightly different formulation of 37.3:

<u>(37.4) PROPOSITION:</u> A connection on ξ is given by a vector bundle homomorphism $v: TP \to TP$ with

(v1) $v \circ v = v$ and $V_P = \operatorname{im} v$,

(v2) For all $g \in G$: $T\Psi_g \circ v = v \circ T\Psi_g$,

and vice versa.

Pf. It is easy to check that for such an invariant projection v onto V_P the kernel $H := \ker v$ is a connection in the sense of 37.3. Conversely a horizontal connection H as in 37.3 induces linear projections $v_p : T_pP \to T_pP$, $\operatorname{im} v_p = V_{P,p}$, $\ker v_p = H_p$ and $v = (v_p)_{p \in P}$ satisfies (v1) & (v2). □

<u>(37.5) EXAMPLE:</u> (The canonical flat connection) Let $P = M \times G$ be the product pfb: $\xi = (M \times G, pr_1, M, G)$. The vertical bundle $V = V_P$ at $p = (a,g) \in P$ is

$$V_p = T_p(\{a\} \times G) = \{0\} \oplus T_gG \subset T_pP$$

in the natural decomposition $T_pP = T_aM \oplus T_gG$. With

$$H_p := T_aM \oplus \{0\} \subset T_pP$$

a horizontal distribution $H \subset TP$ with (H1) and (H2) is given. H is called the canonical flat connection on P.

Version 2: We now come to the second version:

As in section 24 it is useful to consider the fibre product (or pullback) $\pi^*TM \cong P \times_M TM$ defined as

$$\pi^*TM = \{ (p, z) \in P \times TM \mid \pi(p) = \tau_M(z) \}.$$

π^*TM is an n-dimensional vector bundle over P which "transports TM to P". With the definition of the natural vector bundle homomorphism

$$B : TP \to \pi^*TM$$

given by $B(Y) := (p, T_p\pi(Y))$, $Y \in T_pP$, we obtain (as in section 24), when $I : V \hookrightarrow TP$ denotes the injection:

(37.6) PROPOSITION: The sequence

$$0 \to V_P \xrightarrow{I} TP \xrightarrow{B} \pi^*TM \to 0 \qquad \text{[17.1.11]}$$

is an exact sequence of vector bundle homomorphisms. Moreover, I and B are G-invariant in the sense of:

$$I \circ T\Psi_g = T\Psi_g \circ I$$
$$B \circ T\Psi_g = \Psi_g' \circ B \qquad \text{for } g \in G. \ ^*$$

Here, Ψ_g' on the right hand side of the equation is the map

$$(p, z) \longmapsto (pg, z) =: \Psi_g'(p, z) \in (\pi^*TM)_{pg}$$

with the property: $\Psi_g \circ p_1 = p_1 \circ \Psi_g'$ with $p_1 : \pi^*TM \to P$, the projection onto the first component: $(p, z) \longmapsto p = p_1(p, z)$.
We have to following commutative diagrams:

$$\begin{array}{ccc} TP & \xrightarrow{T\Psi_g} & TP \\ B\downarrow & & \downarrow B \\ \pi^*TM & \xrightarrow[\Psi_g']{} & \pi^*TM \end{array} \qquad \begin{array}{ccc} \pi^*TM & \xrightarrow{\Psi_g'} & \pi^*TM \\ p_1\downarrow & & \downarrow p_1 \\ P & \xrightarrow[\Psi_g]{} & P \end{array}$$

(37.7) PROPOSITION: A connection on ξ is given by a splitting $C : \pi^*TM \to TP$ of the exact sequence 37.6, i.e. a vector bundle homomorphism with $B \circ C = id$, and vice versa.

$$\boxed{0 \longrightarrow V \xrightarrow{I} TP \underset{C}{\overset{B}{\rightleftarrows}} \pi^*TM \longrightarrow 0}$$

$^*T\Psi_g$ defines a right action of G on TP and on V, and Ψ_g' on π^*TM.

Pf. Such a splitting is injective because of $B \circ C = \mathrm{id}$ and moreover invariant in the sense of

$$T\Psi_g \circ C = C \circ \Psi_g' \quad , g \in G,$$

since B is invariant. Now, $H := \operatorname{im} C \subset TP$ is a complement to V since $\ker C = V$ hence H satisfies (H1) of 37.3, and from the invariance one concludes (H2) (cf. 24.4 and its proof).
Conversely, let $H \subset TP$ be a connection. The restriction $(T_p\pi)|_{H_p} : H_p \to T_{\pi p}M$ is an isomorphism of vector spaces and defines, given $Z \in T_aM$, $a = \pi p$, a tangent vector

$$\check{Z}_p := \left((T_p\pi)_{|H_p}\right)^{-1}(Z) \in H_p,$$

the horizontal lift of $Z \in T_aM$ at $p \in P_a$.
With the aid of the horizontal lift we obtain the splitting $C : \pi^*TM \to TP$ by

$$C(p, Z) := \check{Z}_p \in H_p \subset T_pP.$$

Clearly, $C : \pi^*TM \to TP$ is a vector bundle homomorphism satisfying $B \circ C(p, Z) = B(\check{Z}_p) = (p, Z)$. Hence, C is a splitting of the sequence 37.6. $\square$

We repeat one essential step of the construction of C in the form of a definition for later purposes (cf. parallel transport):

(37.8) DEFINITION: For a connection $H \subset TP$ and $z \in T_aM$ the tangent vector $\check{z}_p := (T_p\pi)_{H_p}^{-1}(z) \in H_p$ is called the horizontal lift of z at $p \in P$.

(37.9) EXAMPLE: In our example of the canonical flat connection on $P = M \times G$ (cf. 37.5) we have

$$\pi^*TM = \{(p,z) \in P \times TM \mid p = (a,g) \ \& \ z \in T_aM\}$$

Hence, for $p = (a,g) \in P$: $(\pi^*TM)_p = \{((a,g),z) : g \in G, z \in T_aM\}$.
On the level of the fibres, the exact sequence 37.6 is

$$0 \longrightarrow V_p = T_gG \xrightarrow{I_p} T_pP = T_aM \oplus T_gG \xrightarrow{B_p} \{p\} \times T_aM \rightarrow 0,$$

where $I_p(X) = X$ and $B_p(Z \oplus X) = (p, Z)$.
Now, the splitting $C_p : \{p\} \times T_aM \longrightarrow T_pP$ is simply

$$C_p(p,z) := z \oplus 0 = z \ , \quad z \in T_aM$$

in this situation (recall $H_p = T_aM \subset T_pP$, cf. 37.5).

(37.10) EXAMPLE: Any other connection on the product pfb $P = M \times G$ is given by a splitting $C : \pi^*TM \to TP$ of the exact sequence 37.6 which has the following form in the fibres $C_p : (\pi^*TM)_p = \{p\} \times T_aM \to T_pP = T_aM \oplus T_gG$, $p = (a, g)$, with

$$C_p(p, Z) = Z \oplus \eta_p(p, Z) \in T_aM \times T_gG,$$

where $\eta : \pi^*TM \to TG$, $\eta(p, Z) \in T_gG$, is a vector bundle homomorphism which is equivariant:

$$\eta \circ \Psi_h^! = TR_h \circ \eta \quad \text{for } h \in G.$$

Such an η is close to an invariant $\mathfrak{g}$-valued 1-form which leads to the third version of the concept of a connection on a pfb, cf. below.

(37.11) EXAMPLE: Let M be a homogeneous space, i.e. M is a quotient $M = P/G$ of a Lie group P with respect to a closed subgroup $G \subset P$. Assume the homogeneous space to be reductive, i.e. there exists a vector space decomposition $\mathrm{Lie}\,P = \mathfrak{g} \oplus \mathfrak{h}$ where $\mathfrak{g} = \mathrm{Lie}\,G$ such that $\mathrm{Ad}_g(\mathfrak{h}) \subset \mathfrak{h}$ for each $g \in G$. Regarding $\mathrm{Lie}\,P$ as T_eP the left multiplication

$$\mathcal{L}_p : P \to P\,,\ q \longmapsto qp,$$

induces a decomposition

$$T_pP = T_e\mathcal{L}_p(\mathfrak{h}) \oplus T_e\mathcal{L}_p(\mathfrak{g})$$

where $T_e\mathcal{L}_p(\mathfrak{g}) = V_p$ is the fibre of the vertical bundle $V \subset TP$. With $H_p := T_e\mathcal{L}_p(\mathfrak{h})$ we obtain a horizontal distribution $H \subset TP$ with $H \oplus V = TP$, i.e. (H1), which is invariant (i.e. (H2)) due to the reductivity condition $Ad_g(\mathfrak{h}) \subset \mathfrak{h}$: The right action Ψ_g on P is the right multiplication R_g, $g \in G$: R_g

$$T_p\Psi_g(H_p) = T_p R_g T_e L_p(\mathfrak{h}) = T_e L_p T_p R_g(\mathfrak{h})$$
$$= T_e\mathcal{L}_p T_p\mathcal{L}_g Ad_{g^{-1}}(\mathfrak{h}) \subset T_e\mathcal{L}_{pg}(\mathfrak{h}) = H_{pg}$$

Hence $T_p\Psi_g(H_p) \subset H_{pg}$, and since $T_p\Psi_g$ is injective, we have (H2).
As splitting $\pi^*TM \xrightarrow{C} TP$ the connection is given in the following way: Each $Z \in T_aM$, $\pi(p) = a$ (i.e. $p \in a$) has the form $Z = [e^{tZ'}a]$, with a unique $Z' \in \mathfrak{h}$, and

$$C_p : (\pi^*TM)_p = \{p\} \times T_aM \to T_pP\,,\ (p,Z) \longmapsto [e^{tZ'}p].$$

<u>Version 3.</u> We now come the the third version of the concept of a connection, the connection as an G-<u>invariant $\mathfrak{g}$-valued 1-form on P</u>. Let $\mathfrak{g} = \mathrm{Lie}\, G$ be the Lie algebra of G in form of its right invariant vector fields. For each $X \in \mathfrak{g}$ let φ^X the flow of X. In particular, the curve

$$\exp(t) = e^{tX} = \varphi^X(t,e), \quad t \in \mathbb{R},$$

represents X at $e \in G$: $X(e) = [e^{tX}]_e$.

<u>(37.12) DEFINITION:</u> For every $p \in P$ the curve pe^{tX} defines a tangent vector $X^*(p) := [pe^{tX}]_p$. The vector field $X^* : P \to TP$ is called the *fundamental field* of X.

From 37.2 we know $V_p = \{X^*(p) : X \in \mathfrak{g}\}$. Let $\sigma_p : V_p \to \mathfrak{g}$ the inverse map to $X \longmapsto X^*(p)$. σ_p is an isomorphism of vector spaces. Starting with a connection on ξ given by the vertical projection $v : TP \to TP$ (cf. 37.4) we obtain the $\mathfrak{g}$-valued 1-form ω by setting

$$\omega_p : T_pP \longrightarrow \mathfrak{g}, \quad Y \longmapsto \sigma_p(v_p(Y)).$$

Evidently, $H_p = \ker \omega_p$. ω is called the <u>connection form</u>.

<u>(37.13) PROPOSITION:</u> A connection on ξ is given by a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying

$$(\omega 1) \quad \omega_p(X_p^*) = X \quad \text{for all } (p, X) \in P \times \mathfrak{g},$$

$$(\omega 2) \quad \omega_{pg}(T_p \Psi_g(Y)) = \mathrm{Ad}_{g^{-1}} \omega_p(Y) \quad \text{for } Y \in T_p P,$$

and vice versa. Short version of $(\omega 2)$: $\Psi_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$.

Recall that Ad is the <u>adjoint representation</u>, in particular,

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

is the tangent map $T_e \alpha_g : T_e G \to T_e G$ of the inner automorphism $\alpha_g : G \to G$, $h \longmapsto g h g^{-1}$.

<u>Pf.</u> Let v be a connection in the sense of 37.4. Clearly, $\omega := \sigma \circ v : TP \to \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form on P. For $(p, X) \in P \times \mathfrak{g}$ one has

$$\omega_p(X_p^*) = \sigma_a \circ v_a(X_a^*) = \sigma_a(X_a^*) = X,$$

hence $(\omega 1)$. To show $(\omega 2)$ we first prove

$$(*) \quad T_p \Psi_g(X_p^*) = (\mathrm{Ad}_{g^{-1}} X)_{pg}^*, \quad X \in \mathfrak{g}:$$

$$T_p \Psi_g(X_p^*) = [p e^{tX} g]_{pg} = [pg g^{-1} e^{tX} g]_{pg} = [pg \, \alpha_{g^{-1}}(e^{tX})]_{pg} =$$
$$= [pg \exp(\mathrm{Ad}_{g^{-1}}(X) t)]_{pg} = (\mathrm{Ad}_{g^{-1}}(X))_{pg}^* .$$

Here we have used $[\alpha_{g^{-1}} e^{tX}]_e = T_e \alpha_{g^{-1}}([e^{Xt}]_e) = Ad_{g^{-1}}(X)$.

Now let $Y \in H_p := \ker v_p \subset T_pP$. Then $v_p(Y) = 0$, hence $\omega_p(Y) = 0$. Moreover, $T_p\tau_g(Y) \in H_{pg}$ and $\omega_{pg}(T_p\tau_g(Y)) = 0$ as well. It remains to show (ω2) for $Y \in V_p$. We have $Y = X_p^*$ with a unique $X \in \mathfrak{g}$: $X = \omega_p(Y)$. From (*) we get

$$\omega_{pg}(T_p\tau_g(Y)) = \omega_{pg}((Ad_{g^{-1}}X)^*_{pg}) = Ad_{g^{-1}}X = Ad_{g^{-1}}\,\omega_p(Y).$$

So ω satisfies (ω1) and (ω2).

Conversely, let $\omega \in \Omega^1(P,\mathfrak{g})$ satisfy (ω1) and (ω2). Define

$$v_p(Y) := \omega_p(Y)^*_p \quad \text{for } Y \in T_pP,\ p \in P.$$

This is a vb-homomorphism with $v \circ v = v$ and $\operatorname{im} v = V$, i.e. v satisfies (v1).
Furthermore, for $Y \in T_pP$ and $X := \omega(Y)$, i.e. $X_p^* = v_p(Y)$, we get

$$T_p\tau_g(v_p(Y)) = T_p\tau_g(X_p^*) = (Ad_{g^{-1}}X)^*_{pg} \quad \text{(after (*))}$$

and

$$v_{pg}(T_p\Psi_g(Y)) = \omega_{pg}(T_p\Psi_g(Y))^*_{pg} \underset{\omega 2}{=} (Ad_{g^{-1}}\omega_p Y)^*_{pg} = (Ad_{g^{-1}}X)^*_{pg}.$$

This implies (V2). $\square$

[18.01.11]

(37.14) REMARK: For a matrix group G we know that $Ad_g(X) = gXg^{-1}$ for $X \in \mathfrak{g}$ and $g \in G$. Hence, (ω2) has the form

$$(\omega 2)' \quad \omega_{pg}(T_p\Psi_g(Y)) = g^{-1}\omega_p(Y)g$$

or $\Psi_g^*\omega = g^{-1}\omega g$ using $\Psi_g^*\omega(Y)_{pg} = \omega_{pg}(T_p\Psi_g(Y))$.

The local variant of a connection form leads to a slightly different version of the concept of a connection.

(37.15) PROPOSITION: Let the pfb ξ be given by a cocycle (g_{ij}), $g_{ij} \in \mathcal{E}(U_{ij}, G)$, with respect to an open cover $(U_i)_{i\in I}$ of M. A connection on P is given by a family $(\alpha_j)_{j\in I}$ of $\mathfrak{g}$-valued 1-forms $\alpha_j \in \mathcal{A}^1(U_j, \mathfrak{g})$ (the gauge potentials) satisfying

$$(Z) \quad \alpha_j = Ad_{g_{ij}^{-1}}(\alpha_i) + T(L_{g_{ij}^{-1}} \circ g_{ij}) \qquad \text{on } U_{ij} = U_i \cap U_j,$$

and vice versa. In the case of a matrix group G the condition (Z) is equivalent to

$$(Z') \quad \alpha_j = g_{ij}^{-1}\alpha_i g_{ij} + g_{ij}^{-1} dg_{ij} \quad \text{on } U_{ij}$$

(cf. 36.5: M-connection)

<u>Proof</u>. The cocycle (g_{ij}) is given by local trivializations

$$\psi_j : P_{U_j} \longrightarrow U_j \times G \,, \; j \in I,$$

such that

$$\psi_i \circ \psi_j^{-1}(a,h) = (a, g_{ij}(a).h), \; (a,h) \in U_{ij} \times G, \; i,j \in I.$$

The local trivializations determine local sections

$$s_j : U_j \to P_{U_j} \,, \quad s_j(a) := \psi_j^{-1}(a,e),$$

so that $\psi_j^{-1}(a,g) = s_j(a)g$ and $s_j = s_i g_{ij}$ on U_{ij}.

Now let $\omega \in \mathcal{A}^1(P,\mathfrak{g})$ be a connection form.
In the following we show that the family of local forms $(\alpha_j)_{j\in I}$,

$$\alpha_j := s_j^* \omega \in \mathcal{A}^1(U_j, \mathfrak{g}), \; j \in I,$$

$\left(\alpha_{j,a}(X) = \omega_{s_j(a)} T_a s_j (X) \text{ if } X \in T_a M\right)$ satisfies (Z):

Let $Z \in T_a M$ be represented by the curve $\gamma(t)$: $Z = [\gamma]_a$.
Set $p' := s_j(a)$, $p'' := s_i(a)$.

$$T_a s_j(Z) = [s_j \circ \gamma]_{p'} = [s_i(\gamma) g_{ij}(\gamma)]_{p'} = [s_i(a) g_{ij}(\gamma)]_{p'} + [s_i(\gamma) g_{ij}(a)] =$$

$$= [s_j(a) g_{ij}^{-1}(a) g_{ij}(\gamma)]_{p'} + T_{p''} \Psi_{g_{ij}(a)}([s_i(\gamma)]_{p''})$$

$$= \left(T_a\left(\mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}\right)(Z)\right)^*_{p'} + T_{p''} \Psi_{g_{ij}(a)}([s_i(\gamma)]_{p''})$$

where the equality

$$[s_j(a) g_{ij}^{-1}(a) g_{ij}(\gamma)]_{p'} = \left(T_a\left(\mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}(a)\right)(Z)\right)^*_{p'}$$

follows from the fact that $X := T_a\left(\mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}\right)(Z)$
has the presentation $X = [g_{ij}^{-1}(a) g_{ij}(\gamma)]_e \in T_e G = \mathfrak{g}$, hence
$X^*_{p'} = [p\, g_{ij}^{-1}(a) g_{ij}(\gamma)]_{p'}$.
As a result,

$$\alpha_{j,a}(Z) = \omega_{s_j(a)}(T_a s_j(Z))$$

$$= \omega_{p'}\left(\left(T_a \mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}\right) Z\right)^*_{p'} + T_{p''} \Psi_{g_{ij}(a)}([s_i(\gamma)]_{p''}))$$

$$\overset{(\omega 1)}{=} T_a\left(\mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}\right)(Z) + \omega_{p'' g_{ij}(a)}\left(T_{p''} \Psi_{g_{ij}(a)} T_a s_i(Z)\right)$$

$$\overset{(\omega 2)}{=} T_a\left(\mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}\right)(Z) + Ad_{g_{ij}(a)^{-1}}\Big(\underbrace{\omega_{p''}(T_a s_i(Z))}_{\alpha_{i,a}(Z)}\Big).$$

Thus we have shown (Z).

Conversely, let (α_j) with (Z) be given. Denote $p' = s_j(a)$, $a \in \mathcal{U}_j$, where $j \in I$ is fixed. For $z \in T_aM$ and $X \in \mathfrak{g}$ set

$$\omega^{(j)}_{p'}\left(T_a s_j(z) + X^*_{p'}\right) := \alpha_{j,a}(z) + X .$$

This yields $\omega^{(j)} \in \Gamma\left(s_j(\mathcal{U}_j), \Omega^1(TP, \mathfrak{g})\right)$. Let us extend $\omega^{(j)}$ to all of $P_{\mathcal{U}_j}$ by

$$\omega^{(j)}_{p'g}(Y) := Ad_{g^{-1}}\left(\omega^{(j)}_{p'}\left(T_{p'g}\Psi_{g^{-1}}(Y)\right)\right), \quad Y \in T_{p'g}P.$$

We have to check that $\omega^{(j)}$ satisfies $(\omega 1)$ and $(\omega 2)$: For $p \in P_{\mathcal{U}_j}$, $p = p'g$, one has for $X \in \mathfrak{g}$

$$\omega^{(j)}_p(X^*_p) = \omega^{(j)}_{p'g}(X^*_{p'g}) = Ad_{g^{-1}}\left(\omega^{(j)}_{p'}\, T_{p'g}\Psi_{g^{-1}}(X^*_{p'g})\right)$$

$$\overset{(*)}{=} Ad_{g^{-1}}\left(\omega^{(j)}_{p'}\left(Ad_g(X)^*_p\right)\right) \overset{def}{=} Ad_{g^{-1}} \circ Ad_g(X) = X,$$

hence $(\omega 1)$.

Let, moreover $h \in G$. Then $ph = p'gh$.

$$\omega^{(j)}_{ph}\left(T_p\Psi_h(Y)\right) \overset{def}{=} Ad_{(gh)^{-1}}\, \omega^{(j)}_{p'}\left(T_{p'gh}\Psi_{(gh)^{-1}}\left(T_p\Psi_h(Y)\right)\right)$$

$$= Ad_{h^{-1}\circ g^{-1}}\, \omega^{(j)}_{p'}\left(T_{p'g}\Psi_{g^{-1}} \circ T_{ph}\Psi_{h^{-1}} \circ T_p\Psi_h(Y)\right)$$

$$= Ad_{g^{-1}\circ h^{-1}}\, \omega^{(j)}_{p'}\left(T_{p'g}\Psi_{g^{-1}}(Y)\right) \overset{def}{=} Ad_{h^{-1}}\left(\omega^{(j)}_p(Y)\right), \quad \text{i.e. } (\omega 2)$$

It remains to show $\omega^{(j)} = \omega^{(i)}$ on $P_{U_{ij}}$ in order to yield $\omega \in \mathcal{A}^1(P, \mathfrak{g})$ with $\omega|_{P_{U_j}} = \omega^{(j)}$. Only here we need the condition (Z)!

It is enough to show $\omega^{(i)}|_{s_j(U_{ij})} = \omega^{(j)}|_{s_j(U_{ij})}$:

For $X \in \mathfrak{g}$ and $p' = s_j(a)$, $a \in U_{ij}$, one has $\omega^{(j)}_{p'}(X^*_{p'}) = X$ $= \omega^{(i)}_{p'}(X^*_p)$ directly by definition.

Let $Z \in T_aM$. To show: $\omega^{(i)}_{p'}(T_a s_j(Z)) = \alpha_{j,a}(Z)$.

$$\omega^{(i)}_{p'}(T_a s_j(Z)) \overset{\text{def}}{=} Ad_{g_{ij}^{-1}(a)}\left(\omega^{(i)}_{p''}\left(T_{p'}\Psi_{g_{ij}^{-1}(a)}\, T_a s_j(Z)\right)\right) \quad \left(p' = p'' g_{ij}(a)^{-1}\right)$$

We use the decomposition (see above)

$$T_a s_j(Z) = \left(T_a\left(\mathcal{L}^{-1}_{g_{ij}(a)} \circ g_{ij}\right)(Z)\right)^*_{p'} + T_{p''}\Psi_{g_{ij}(a)} \circ T_a s_i(Z)$$

and obtain (using $Ad_g(X)^*_p = T_p\Psi_{g^{-1}}(X^*_{pg})$ among others)

$$\begin{aligned}
\omega^{(i)}_{p'}(T_a s_j(Z)) &= Ad_{g_{ij}^{-1}(a)}\Big[\omega^{(i)}_{p''}\left(T_{p'}\Psi_{g_{ij}^{-1}(a)}\left(\left[T_a\mathcal{L}_{g_{ij}^{-1}(a)}\circ g_{ij})(Z)\right]^*_{p'}\right)\right) \\
&\quad + \Big(\omega^{(i)}_{p''}\big(\underbrace{T_{p'}\Psi_{g_{ij}^{-1}(a)} \circ T_{p''}\Psi_{g_{ij}(a)} \circ T_a s_i(Z)}_{T_a s_i(Z)}\big)\Big] \\
&= Ad_{g_{ij}^{-1}(a)}\left[\omega^{(i)}_{p''}\left(\left[Ad_{g_{ij}(a)}(T_a \mathcal{L}_{g_{ij}^{-1}(a)}\circ g_{ij}(Z))\right]^*_{p''} + \alpha_{i,a}(Z)\right]\right. \\
&= Ad_{g_{ij}^{-1}(a)}\, Ad_{g_{ij}(a)}\left(T_a\mathcal{L}_{g_{ij}^{-1}(a)}\circ g_{ij}(Z)\right) + Ad_{g_{ij}^{-1}(a)}\left(\alpha_{i,a}(Z)\right) \\
&= T_a\mathcal{L}_{g_{ij}^{-1}(a)}\circ g_{ij}(Z) + Ad_{g_{ij}^{-1}(a)}\left(\alpha_{i,a}(Z)\right) \underset{(Z)}{=} \alpha_{j,a}(Z)
\end{aligned}$$

$\square$

<u>Version 4</u> is now the description of connections using covariant differentiation.

Let the connection on the pfb ξ given by the vertical projection $v : TP \to TP$ and set $h := id - v$, the <u>horizontal</u> projection. Let W a finite dimensional vector space over $\mathbb{R}$. For a W-valued k-form $\beta \in \mathcal{A}^k(P, W)$ set

$$h^*\beta(Y_1, \dots Y_k) := \beta(hY_1, \dots hY_k), \quad Y_j \in \mathfrak{X}(P, W).$$

The <u>covariant differential</u> $D = D^\omega$ corresponding to ω (or v, or h, or $H \subset TP$) is defined by

$$D\beta := h^* \circ d : \mathcal{A}^k(Q, W) \to \mathcal{A}^{k+1}(Q, W), \quad Q \subset P \text{ open}.$$

<u>(37.16) LEMMA:</u> $1°$ $h^* \circ h^* = h^*$. $2°$ $h^* \circ \Psi_g^* = \Psi_g^* \circ h^*$, $g \in G$.
$3°$ $h^*(\omega) = 0$ for the connection form ω.
$4°$ $h^*(\alpha \wedge \beta) = h^*(\alpha) \wedge h^*(\beta)$ (in case of $W = \mathbb{R}$).
$5°$ $D(\alpha \wedge \beta) = D\alpha \wedge h^*\beta + (-1)^{\operatorname{grad}\alpha} h^*\alpha \wedge D\beta$ ($W = \mathbb{R}$).
$6°$ $i_{X^*} D = 0$, $X \in \mathfrak{g}$. $7°$ $D \circ \Psi_g^* = \Psi_g^* \circ D$ $8°$ $D \circ \pi^* = \pi^* \circ D$

Easy to prove.

<u>(37.17) PROPOSITION:</u> A connection on ξ is given by a <u>covariant differential</u> D, i.e. $D : \mathcal{E}(P) \to \mathcal{A}^1(P)^*$ $\mathbb{R}$-linear with

(D1) $D(fg) = Df \cdot g + f\, Dg \qquad \forall f, g \in \mathcal{E}(P)$

(D2) $Df(X^*) = 0 \qquad \forall X \in \mathfrak{g}$

(D3) $\Psi_g^* \circ D = D \circ \Psi_g^* \qquad \forall g \in G$

(D4) $Df = df \qquad \forall f \in \pi^*(\mathcal{E}(M))$

<u>Proof</u>: If ω is a connection, then the differentia D^ω satisfies (D1)-(D4) according to 37.15.
Conversely, let D be given with (D1) - (D4). For $Y \in \mathcal{V}(P)$ set

$$Q_Y : \mathcal{E}(P) \to \mathcal{E}(P) \;, \quad Q_Y f := (df - Df)(Y) .$$

Q_Y is a derivation according to (D1), and determines a vectorfield $v(Y) = V_Y$ with $Q_Y = L_{v(Y)}$. This defines a vector bundle homomorphism $v : TP \to TP$ with (v1), (v2):
For $f = g \circ \pi = \pi^* g \in \pi^* \mathcal{E}(M)$ we have $Q_Y(f) = 0$ (by D4), hence $df(v(Y)) = 0$. This implies $v(Y)_p \in V_p$ for all $Y \in \mathcal{V}(P)$, i.e. $v(TP) \subset V$.
For a vertical $Y_p \in V_p$ we know $Y_p = X_p^*$ for $X \in \mathfrak{g}$, so $Q_Y(f) = df(Y) = df(v(Y))$ by D2. Hence $v_p(Y_p) = Y_p$, i.e. (v1).

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* D can be extended to a differential $D : \mathcal{A}^k(P,W) \to \mathcal{A}^{k+1}(P,W)$ for all k and all vector spaces W, with the properties of 37.15.

(V2) is essentially (D3):

$$df(v \circ T\Psi_g(Y))_{pg} = df_{pg}(T_p\Psi_g(Y)) - Df_{pg}(T_p\Psi_g(Y))$$
$$\overset{D3}{=} d(f\circ\Psi_g)_p(Y) - D(f\circ\Psi_g)_p(Y)$$
$$\overset{def}{=} d(f\circ\Psi_g)_p(v_p(Y))$$
$$= df_{pg}(T_p\Psi_g(v_p(Y))) = df(T\Psi_g \circ v(Y))_{pg}.$$

Now, $df(T\Psi_g \circ v) = df(v \circ T\Psi_g)$ for all $f \in \mathcal{E}(P)$ implies (V2) □

<u>Version 5.</u> Parallel transport.

Version 2 of defining a connection immediately yields:

<u>(37.18)</u> PROPOSITION: Let ξ be a pfb with a connection.

1° Every vector field $X \in \mathcal{V}(M)$ on M has a horizontal lift $\check{X} \in \mathcal{V}(P)$, i.e. (cf. 37.8)

$$\check{X}(p) \in H_p,\ T_p\pi(\check{X}(p)) = X(\pi(p)),\quad p \in P.$$

2° If $Y \in \mathcal{V}(P)$ is a horizontal and right invariant vector field ($T\Psi_g \circ Y = Y \circ \Psi_g$ for all $g \in G$) then there exists a unique $X \in \mathcal{V}(M)$ with $Y = \check{X}$.

3° For $X, Z \in \mathcal{V}(M)$ and $f \in \mathcal{E}(M)$ we have
$$(X+Z)^\vee = \check{X} + \check{Z}, \quad (fX)^\vee = (f\circ\pi)X^\vee, \quad [X,Z]^\vee = h([\check{X},\check{Z}]).$$

Pf. 1° $X^\vee(p) = C(p,X)$ yields the horizontal lift and the properties 2° and 3° are easy to check. □

(37.19) LEMMA: Let H be a connection on ξ. For every curve γ in M and every $p_0 \in \pi^{-1}(\gamma(t_0)) = P_{\gamma(t_0)}$ there exists a unique horizontal lift α of γ through p_0, i.e. α is a curve in P with $\pi\circ\alpha = \gamma$, $\alpha(t_0) = p_0$ and $\dot{\alpha}(t) = \check{\dot{\gamma}}(t)$ (cf. 37.8).

Pf. We can extend $\dot{\gamma}$ locally to vector fields which are lifted horizontally and obtain α as the solution of the corresponding ordinary differential equation.
Another proof starts with an arbitrary lift β of γ, i.e. $\pi(\beta) = \gamma$ which can be described using the local trivializations of $\pi : P \to M$ and change β to a horizontal curve by $\alpha(t) = \beta(t)g(t)$ for suitable $g(t) \in G$. Now, $\dot{\alpha}(t)$ is horizontal if and only if $\omega(\dot{\alpha}(t)) = 0$. This is equivalent to
$$Ad_{g(t)^{-1}}\,\omega(\dot{\beta}(t)) + T\lambda_{g(t)^{-1}}\dot{g}(t) = 0$$
or

$$\dot{g}(t) = -TR_{g(t)}\,\omega(\dot{\beta}(t)).$$

This equation has a unique solution $g(t) \in G$, $g(t_0) = e$. $\square$

The horizontal lift of curves defines – as in the vector case – a parallel transport along curves γ in M by

$$\mathbb{P}^{\gamma}_{t_0,t_1} : P_{\gamma(t_0)} \to P_{\gamma(t_1)}, \quad p_0 \longmapsto \check{\gamma}_{p_0}(t_1),$$

where $\check{\gamma}_{p_0}$ is the unique horizontal lift of γ through p_0.
$\mathbb{P}^{\gamma}_{t_0,t_1}$ is G-invariant: $\mathbb{P}^{\gamma}_{t_0,t_1}(pg) = \mathbb{P}^{\gamma}_{t_0,t_1}(p)g$ for $(p,g) \in P \times G$.

(37.20) PROPOSITION: A connection on ξ induces a G-invariant parallel transport. And an abstract G-invariant parallel transport gives back a connection.

(37.21) EXAMPLE: In case of the canonical flat connection on $P = M \times G$ the parallel transport $\check{\gamma}$ of a curve γ in M through $p_0 = (\gamma(t_0), g)$ is $\check{\gamma}(t) = (\gamma(t), g)$:

$$(\check{\gamma})^{\cdot}(t) = (\dot{\gamma}(t), g) \in H_{\check{\gamma}(t)} = T_{\gamma(t)}M \times \{0\}.$$

As a result, the parallel transport from P_a to P_b along any curve γ from $a = \gamma(t_0)$ to $b = \gamma(t_1)$ is

$$\mathbb{P}^{\gamma}_{t_0,t_1} : P_{\gamma(t_0)} \to P_{\gamma(t_1)} \ , \ (\gamma(t_0), g) \longmapsto (\gamma(t_1), g).$$

Evidently, the parallel transport is independent of the curve connecting $\gamma(t_0)$ and $\gamma(t_1)$ (therefore „flat"). This is the exceptional situation. In fact, one can show that a connection on ξ, such that the induced parallel transport is independent of the curves, is isomorphic to the canonical flat connection on the product $P \times G$, i.e. there is an isomorphism φ from ξ to $P \times G$ such that the horizontal distribution $H \subset TP$ is mapped to the flat horizontal distribution $TM \subset T(M \times G)$.

We conclude this section with the following existence result:

<u>(37.22) PROPOSITION:</u> Every pfb ξ has a connection.

<u>Pf.</u> Let (U_j) be an open cover with local trivializations $\varphi_j : P_{U_j} \to U_j \times G$ and let $(\varepsilon_j)_{j \in I}$ be a smooth partition of unity. On P_{U_j} we have the canonical flat connection, given by its connection form ω_j. Then $\omega := \sum \varepsilon_j \omega_j$ defines a connection form on P. $\square$