34. The Concept of a Principal Fibre Bundle

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In the following G will denote a Lie group. A manifold M on which G acts from the right will be called a *G-manifold*.

(34.1) DEFINITION: Let P, P' be two G-manifolds. A smooth map $f : P \to P'$ is called *equivariant** (or a *morphism of G-manifolds*) if

$$f(pg) = f(p)g$$

for all $(p,g) \in P \times G$. With this notion we have defined the *category* $({}^G\mathrm{mfd})$ *of G-manifolds.**

A special example of a G-manifold is given by the standard action on the product $P = M \times G$:

$$P \times G \to P, \quad ((a,h),g) \longmapsto (a, hg).$$

In this situation an equivariant map $f : P \to P$

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* In a more general context f is called *equivariant* if $f(pg) = f(p)\sigma(g)$ with respect to a Lie group homomorphism $\sigma : G \to G$.

is of the form

$$f(a,g) = \big(p(a),\, t(a)g\big), \quad (a,g) \in P = M \times G,$$

where $p: M \to M$ and $t: M \to G$ are arbitrary smooth maps.

<u>(34.2) DEFINITION:</u>* A <u>principal fibre bundle</u> (<u>pfb</u>) is a quadruple $\xi = (P, \pi, M, G)$ with

1° P, M are manifolds, $\pi: P \to G$ is smooth and G is a Lie group which acts on P from the right.

2° For each $a \in M$ there exists an open neighbourhood $U \subset M$ of a and an equivariant diffeomorphism $\psi: \pi^{-1}(U) =: P_U \longrightarrow U \times G$ such that $pr_1 \circ \psi = \pi|_U$:

$$\begin{tikzcd} \psi: P_U \arrow[r, "\sim"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

G is the <u>structure group</u> of the pfb (and the typical

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* We have formulated the definition in a manner that parallels our def. of a fibration (cf. 15.1). A rather different definition uses the notions of section 33: A pfb is a G-manifold P where the G-action is free and the orbit space $P/_G \cong M$ exists, cf. 34.6 below.

fibre) and, as before, P is the total space, M is the base and π is the projection. ψ with all the properties in 2° is called a local trivialization.

(34.3) REMARKS: 1° The fibre $P_a := \pi^{-1}(a)$ is diffeomorphic to G : $\psi_a := \psi|_{P_a} : P_a \to \{a\} \times G$ is a diffeomorphism.

2° A pfb is, in particular, a fibration in the sense of 15.1. The projection $\pi : P \to M$ is a submersion and, hence, an open map. π is surjective, as well.

3° $P = M \times G$ with the standard action and the projection $\pi = pr_1$ is a pfb (in fact, a trivial pfb). A pfb is called trivial if it is isomorphic as a pfb (i.e. in $({}^G\text{pfb})$) to a product fiber bundle $P = M \times G$. We need 34.4 for the definition of isomorphism.

4° The fibre P_a is not a group, in general, although $P_a \cong G$ and we have the transitive group action

$$P_a \times G \to P_a .$$

G acts transitively and freely on P_a.
The relation between G and P_a is similar to the relation between an affine space A and its abelian

group of translations V: Once a point $a \in A$ is chosen, A can be endowed with the vector space structure induced by the bijection

$$V \to A \quad , \; v \longmapsto a+v \, .$$

If $p \in P_a$ is chosen, the bijection $G \to P_a$, $g \mapsto pg$, induces a group structure on P_a such that this map is a group isomorphism.

5^0 The action of G on P is free whenever we have a pfb (P, π, M, G). So we have a free action on P such that the orbit space $P/_G$ exists ($\cong M$). The converse will be discussed below: 34.6

(34.4) DEFINITION: A morphism of principal fibre bundles (P, π, M, G), (P', π', M', G) is a morphism

$$\varphi : P \to P'$$

of fibrations ($\pi' \circ \varphi = \varphi$ or $\pi' \circ \varphi = \varphi_M \circ \pi$) such that φ is equivariant. In this way, we define the category $({}^G\mathrm{pfb})$ with its subcategory $({}^G\mathrm{pfb}_M)$ of all pfbs over a fixed mfd M with $\varphi_M = id_M$.

(34.5) REMARK: Starting with an open cover $(U_j)_{j\in I}$ of open $U_j \subset M$ and a local trivialization $\psi_j : P_{U_j} \to U_j \times G$ for each $j \in I$ we obtain transition functions

$$\psi_{ij} : U_{ij} \to \mathrm{Diff}(G) \quad , \quad \psi_{ij}(a) = \psi_{i,a} \circ \psi_{j,a}^{-1} : G \to G$$

as in section 16:

$$\begin{aligned} \psi_i \circ \psi_j^{-1} : U_{ij} \times G &\to U_{ij} \times G \\ (a, g) &\longmapsto (a, \psi_{ij}(a).g) \end{aligned}$$

where $\psi_{ij}(a) \in \mathrm{Diff}(G)$. Since $\psi_i \circ \psi_j^{-1}$ is equivariant, we know $\psi_{ij}(a).(gh) = (\psi_{ij}(a).g)h$ for all $h, g \in G$, in particular, $\psi_{ij}(a).h = (\psi_{ij}(a).e)h = L_{\psi_{ij}(a)e}\, h$. As a consequence, with

$$g_{ij}(a) := \psi_{ij}(a).e \in G$$

we end up with a cocycle (g_{ij}) in G: $g_{ij} \in \mathcal{E}(U_{ij}, G)$

$$g_{ii} = e \quad \& \quad g_{ij}\, g_{jk}\, g_{ki} = e .$$

Conversely, such a cocycle induces a pfb uniquely up to isomorphism. And the morphisms can be described via transition functions g_{ij}.

[22.12.10]

(34.6) PROPOSITION: Let the Lie group G act on the manifold freely (from the right). If the orbit space $P/_G$ exists (cf. sections 33 and 33B) then $\pi: P \to P/_G$ is a principal fibre bundle $(P, \pi, P/_G, G)$.

Proof. The projection $\pi: P \longrightarrow P/_G =: M$ is a submersion. Hence, for every $b_0 = \pi p_0 \in M$ there exists a local section $s: V \to P$, $\pi \circ s = id_{\mathcal{U}}$ ($V \subset M$ open neighbourhood of b_0; cf. 33.13.4°). Let $\psi: V \times G \to \pi^{-1}(V) \subset P$ be defined by

$$(b,g) \longmapsto s(b)g = \psi(b,g), \ (b,g) \in V \times G.$$

Then ψ is bijective: Whenever $s(b)g = s(b')g'$, this element is in the fibre P_b, hence $b = b'$. And from $s(b)g = s(b)g'$ we deduce $g = g'$, since the action is free. ψ is smooth since s is smooth and the action is smooth. The tangent map $T_{(b,g)}\psi$ is surjective: Set $p := s(b)g$. and $\mathcal{U} := \pi^{-1}(V)$. $S := s(V) \subset \mathcal{U}$ is a submanifold and for $g \in G$, $Sg = \{cg : c \in S\} \subset \mathcal{U}$ is a submanifold as well. $T_pP = T_pSg \oplus T_pB$, where $B := pG = \pi^{-1}(b)$ is the orbit through p. For a tangent vector $X \in TB$ given by a curve β in B,

$\beta(0) = p$, we have $\beta(t) = s(b)g\gamma(t)$ with γ a curve in G, $\gamma(0) = e$. Let $\hat{X} := [(b, g\gamma(t))]_{(b,g)} \in T_{(b,g)}(U \times G)$. Then $T_{(b,g)}\psi(\hat{X}) = [s(b)g\gamma(t)]_p = [\beta(t)]_p = X$. For a tangent vector $Y \in T_p S g$, given by a curve α in Sg, $\alpha(0) = p$, we have $\alpha(t) = \sigma(t)g$ with σ a curve in S. We set $\hat{Y} = [(\pi\circ\sigma, g)]_{(b,g)} \in T_{(b,g)}(U\times G)$, and obtain $T_{(b,g)}\psi(\hat{Y}) = [s(\pi\circ\sigma)g]_p = [\sigma g]_p = [\alpha]_p = Y$ (since $s\circ\pi(q) = q$ for $q \in S$). Now, $T_{(p,g)}\psi : T_{(p,g)}(U\times G) \longrightarrow T_pP$ being surjective it is also injective (since ψ is bijective) and, as a result, ψ is a diffeomorphism.

Now, $\varphi := \psi^{-1} : \pi^{-1}(V) \to V\times G$ is a diffeomorphism as well and a local trivialization: $\pi|_U = pr_1\circ\varphi$ by construction: For $\pi p = b \in V$, $p = s(b)g$, we have $\varphi(p) = (b,g)$, hence $pr_1\varphi(p) = \pi(p)$. Moreover, for $g\in G$ and $p\in\pi^{-1}(V)$ there exists a smooth map $h : U \to G$ such that $\varphi(p) = (\pi(p), h(p))$: $h(p)$ is the unique $h(p)\in G$ with $s\circ\pi(p)h(p) = p$. It follows: $s\circ\pi(pg)h(pg) = pg$, i.e. $h(pg) = h(p)g$ and therefore,

$$\varphi(pg) = (\pi(pg), h(pg)) = (\pi(p), h(p)g) = (\pi(p), h(p))g = \varphi(p)g.$$

We have shown that φ is equivariant and, hence, $\pi : P \to M$ is a pfb. $\square$

The proof also yields:

(34.7) PROPOSITION: For a principal fibre bundle $\xi = (P, \pi, M, G)$:

ξ is trivial $\Longleftrightarrow$ There exists a global section $s: M \to P$.

The trivializing map is $M \times G \to P$, $(b,g) \mapsto s(b)g$. $\square$

(34.8) EXAMPLES: The examples of homogeneous manifolds (cf. 33) are pfbs (cf. 33.11). In particular:

1° $\mathbb{R} \to S^1$, $\mathbb{C} \to \mathbb{C}^*$ with $G = \mathbb{Z}$ (discrete group)

2° $S^{n+1} \to P_n(\mathbb{R})$, $G = \{+1, -1\}$.

3° $S^{2n+1} \to P_n(\mathbb{C})$, $G = S^1$ (Hopf-fibration)

4° Grassmann mfds $GL(n, \mathbb{K}) \to Gr_{n,r}(\mathbb{K})$, Stiefel mfds

5° If one restricts to $M' \subset M$ one gets new pfbs (M' a submanifold.

6° In general, if P is a Lie group and $G \subset P$ a subgroup the right action of G on P $(q,g) \mapsto qg$ is free. If G is a closed subgroup it is a Lie subgroup and the orbit space exists (29.8 or in the words of 33: $R = R_G \subset P \times P$ is a closed submanifold, hence 33.8 applies). Hence, $P \to P/_G$ is a pfb.

The following example gives a strong link between vector bundles and principal fibre bundles.

(34.9) EXAMPLE: The frame bundle („Reperbündel").
Let E be a vector bundle on E of rk r. We want to equip the set of

$$R(E) = \bigcup_{a\in M} \{ v \in (E_a)^r \mid v = (v_1, \dots v_r) \text{ is basis of } E_a \}$$

all ordered bases of E_a, $a \in M$, with the structure of a principal fibre bundle with structure group $G = GL(r, \mathbb{K})$.
Recall from linear algebra that for a $\mathbb{K}$-vector space V of dimension r the set R_V of ordered vector space bases is in 1-to-1 correspondence to the group $GL(r, \mathbb{K})$ of invertible $r\times r$ matrices. If one chooses $\mathring{v} = (\mathring{v}_1, \dots \mathring{v}_r) \in R_V$, then every other base $v \in R_V$ can be written as

$$v_\varrho = g_\varrho^\sigma \mathring{v}_\sigma = \mathring{v}_\sigma g_\varrho^\sigma \ , \ \varrho = 1, 2 \dots r,$$

with $(g_\varrho^\sigma) = g \in GL(r, \mathbb{K})$ uniquely determined.
This correspondence $GL(r, \mathbb{K}) \cong R_V$ yields a right action of $GL(r, \mathbb{K})$ on R_V

[10.01.11]

$$R_V \times G \to R_V, \qquad (v,g) = vg = (v_\sigma g_1^\sigma, \dots, v_\sigma g_r^\sigma),$$

since $v(gh) = (vg)h$ for $g, h \in GL(r, \mathbb{K})$:

$$v_\sigma (gh)^\sigma_\varrho = v_\sigma g^\sigma_\tau h^\tau_\varrho = (v_\sigma g^\sigma_\tau) h^\tau_\varrho = (vg)_\tau h^\tau_\varrho = (vg)h.$$

This right action extends to $R(E) = \bigcup_{a\in M} R_{E_a}$: With $R_a := R_{E_a}$

$$R_a \times G \to R_a, \quad (v,g) \longmapsto vg = (v_\sigma g_1^\sigma, \dots v_\sigma g_r^\sigma).$$

The manifold structure on $R = R(E)$ will be defined by bundle charts. Let $s_1, \dots s_r$ be linearly independent sections of E over an open subset $U \subset M$. Such sections exists always locally: If $\psi : E_U \to U \times \mathbb{K}^r$ is a local trivialization over U, the sections $s_\sigma(a) := \psi^{-1}(a, e_\sigma)$, $e_\sigma \in \mathbb{K}^r$ standard unit vectors, provides such sections. Now

$$\varphi : R_U = \pi^{-1}(U) \longrightarrow U \times GL(r, \mathbb{K}) \qquad (\pi : R(E) \to M, \ \pi(R_a) = \{a\})$$

will be defined by $\varphi(v) := (a, g)$ whenever $v \in R_a$ and $g \in GL(r, \mathbb{K})$ with $v = s(a)g$, i.e $s_\sigma(a) g^\sigma_\varrho = v_\varrho$, and $\varphi^{-1}(a,g) = s(a)g$ Clearly, the maps φ (induced by the r-tuple $s = (s_1, s_2, \dots s_r)$ of sections) are bijective.
If $\bar{s} = (\bar{s}_1, \dots \bar{s}_r)$ is another r-tuple of linear independent

sections over an open subset $\bar{U}$ in E, we obtain $\bar{\varphi}: R_{\bar{U}} \to \bar{U} \times \mathbb{K}^r$ by $\bar{\varphi}^{-1}(a,g) := \bar{s}(a)g$, $(a,g) \in \bar{U} \times G$. We have to check that the transition functions $\bar{\varphi} \circ \varphi^{-1} : (U \cap \bar{U}) \times G \to (U \cap \bar{U}) \times G$ are smooth. But $\bar{\varphi} \circ \varphi^{-1}(a,g) = (a, h(a)g)$, where $h(a) \in GL(r,\mathbb{K})$ is the matrix with $\bar{s}_\sigma(a) h^\sigma_\varrho(a) = s_\varrho(a)$ and $a \mapsto h(a)$ is smooth, since $\bar{s}_\sigma, s_\varrho$ are.

(34.10) REMARK: Let (U_j) be an open cover of M such that there are local trivializations $\varphi_j : E_{U_j} \to U_j \times \mathbb{K}^r$ of the vector bundle E. The (φ_j) define transition functions $g_{ij} : U_{ij} \to GL(r,\mathbb{K})$ (the cocycle) which in turn determine the vector bundle E up to isomorphism. (cf. section 16, 17.2). The construction of $R(E)$ shows that $R(E)$ is given by the same cocycle (g_{ij}) but now as a pfb (cf. 34.5).

(34.11) FURTHER EXAMPLES: 1° Assume that the real vector bundle admits an orientation (cf. 28.1). If we choose an orientation, and require in the above construction

(34.9) all ordered bases $v \in (E_a)^r$ to be positive, then we get the submanifold $R_+(E) \subset R(E)$ of positive bases which is a pfb with structure group $GL_+(r, \mathbb{R}) = \{g \in GL(r, \mathbb{R}) : \det g > 0\}$.

2° Similarly, let E a real vector bundle with a Riemannian metric. We restrict in (34.9) to the orthonormal bases and obtain a submanifold $O(E) \subset R(E)$ which is a pfb with respect to the group $O(r)$. If E is oriented and metric we arrive at the corresponding pfb $SO(E)$ with structure group $SO(r)$.

3° If E is equipped with other structures, e.g. a Lorentz metric or a hermitian metric we obtain corresponding "frame" bundles. E.g. in the latter case with a hermitian (complex) vector bundle of rank r we obtain the frame bundle $SU(E)$ with structure group $SU(r)$.

<u>(34.12) REMARK:</u> The examples in 34.11 give examples of "restriction" of the structure group to subgroups.