# 33. Homogeneous Spaces

Version 1.1

Notiztitel

This section should belong to the theory of Lie groups, hence it should be in the last chapter VIII. However, homogeneous spaces are very close – if not the same – as principal fibre bundles.

Therefore we start the main subject of the whole course with homogeneous spaces.

In this section, $G$ is a Lie group and $M$ is a manifold.

(33.1) DEFINITION: A *left action of $G$ on $M$* is a smooth map

$$\Phi : G \times M \to M \,, \quad (g,a) \longmapsto \Phi(g,a) = g \cdot a = ga,$$

satisfying $ea = a$ and $g(ha) = (gh)a$ for all $a \in M$ and $g, h \in G$.*

For a left action $\Phi$ the maps $\Phi_g : M \to M$, $a \mapsto ga$, are diffeomorphisms since $\Phi_g \circ \Phi_{g^{-1}} = \Phi_e$. Moreover, $\Phi$ induces a homomorphism

$$\Phi : G \longrightarrow \mathrm{Diff}(M), \quad g \longmapsto \Phi_g \,.$$

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*The definition generalizes to actions of topological groups $G$ on topological spaces.

Here Diff(M) denotes the group of automorphisms.

A right action of G on M is defined analogously. It is a smooth map

$$\Psi : M \times G \to M, \ (a,g) \mapsto \Psi(a,g) = a\cdot g = ag,$$

with $ae = a$ and $(ag)h = a(gh)$.

If $\Phi$ is a left action then $\Psi(a,g) := \Phi(g^{-1},a)$ defines a right action, and vice versa.

Whenever a left (or right) action is given the Lie group G is called a transformation group on M (or of M).

A transformation group G on M is called

- transitive, if to $a, b \in M$ there exists $g \in G$ such that $a = gb$ (resp. $a = bg$).
- simply-transitive, if the $g$ is always unique.
- free, if $\Phi_g$ (resp. $\Psi_g$) has no fix point for each $g$.
- effective, if $\Phi_g = id_M$ implies $g = e$.

(33.2) DEFINITION: A manifold $M$ with a transitive action of a Lie group $G$ is called a homogeneous space.

(33.3) EXAMPLES: 1° The Galilei group is a transformation group on Newtonian spacetime. The Lorentz group and the Poincaré group act as transformation groups on Minkowski space $\mathbb{R}^4$.

2° More generally, $G = GL(n,\mathbb{R})$ acts from the left on $\mathbb{R}^n$ through $\Phi(g,a) := ga$ (matrix multiplication). This action is transitive on $M = \mathbb{R}^n \setminus \{0\}$ but not on all of $\mathbb{R}^n$ ($\Phi(g,0) = 0$ for all $g \in G$).

3° $G = SO(n,\mathbb{R})$ acts transitively on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ by matrix multiplication but not transitively on $\mathbb{R}^n \setminus \{0\}$.

4° For a Lie subgroup $G$ of a Lie group $H$, $G$ acts on $H$ by left translation and right translation:

$$G \times H \to H, \ (g,h) \longmapsto gh = \mathcal{L}_g(h),$$
$$H \times G \to H, \ (h,g) \longmapsto hg = \mathcal{R}_g(h).$$

5° For a representation $\rho : G \to GL(V)$ on a finite dimensional real vector space $V \cong \mathbb{R}^n$ the map

$$G \times V \to V, \quad (g,v) \longmapsto \varrho(g).v,$$

is a left action on V. This generalizes 2° and 1°. Conversely, any left action of G on a finite dimensional vector space V induces a representation. As a result the concept of a left action of a Lie group G generalizes the concept of a representation by allowing the representation spaces not only to be vector spaces but general manifolds.

6° For the Lie algebra $su(3) = \{x \in \mathbb{C}^{3\times 3} : \bar{x}^T + x = 0\}$ we use coordinates $x_1, x_2, x_3$ with

$$su(2) = \left\{ \begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\} \cong \mathbb{R}^3.$$

Let $\chi(x) := \begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}$ and $\kappa : su(2) \to \mathbb{R}^3$ giving back the coordinates $x$ : $\kappa(\chi(x)) = x$. The adjoint action induces the action $\Phi$

$$SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (A, x) \longmapsto \kappa\left(A\chi(x)A^{-1}\right).$$

It is easy to see that $\Phi_A(x) := \kappa\left(A\chi(x)A^{-1}\right)$ is a

special orthogonal transformation: $\Phi_A \in SO(3)$.
One can show: $\varrho : SU(2) \to SO(3)$ is a homomorphism with $\ker \varrho = \{+1,-1\}$ and a covering. Hence, $\varrho$ is a 2-to-1 covering. As a consequence, $\varrho$ is the universal covering and $SU(2) = Spin(3)$.

7° In a similar way, $SL(2,\mathbb{C})$ acts on $\mathbb{R}^4$ as Lorentz transformations : $\mathbb{R}^4 \to \mathbb{R}^4$, and we obtain a 2-to-1 covering homomorphism $\varrho : SL(2,\mathbb{C}) \to L_+^{\uparrow}$

8° Generalizing 1° one can show the the group of isometries $\mathrm{Isom}(M,g)$ of a semi-Riemannian manifold $(M,g)$ is a Lie group, and that the natural action on $M$.

9° $\mathbb{R}^4$ as translation group $(t,x) \mapsto t+x = x+t$, acts transitively and freely.

<u>(33.4) DEFINITION:</u> For a given left action of $G$ on $M$ we call for $a \in M$:

$S_a := \{g \in G : ga = a\}$ the *stabilizer group* (or the *isotropy group*) at $a$, and

$B_a := Ga = \{ga : g \in G\}$ the *orbit* through $a \in M$.

(33.5) EXAMPLE: $G = O(n+1,\mathbb{R})$ acts transitively on $S^n \subset \mathbb{R}^{n+1}$ as we have stated in 33.3.3°. Let $e_1, \dots e_{n+1} \in \mathbb{R}^{n+1}$ the standard orthonormal base and let us determine the stabilizer group $S_a$, $a = e_{n+1}$. For a matrix $A = (A^j_k) \in O(n+1,\mathbb{R})$ the condition $e_{n+1} = Ae_{n+1} = A^j_{n+1} e_j$ means $A^j_{n+1} = 0$ for $j \leq n$ and $A^{n+1}_{n+1} = 1$. Hence, $A$ has the form

$$A = \begin{pmatrix} & R & & 0 \\ & & & \vdots \\ & & & 0 \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{with} \quad R \in O(n,\mathbb{R}).$$

We conclude $S_{e_{n+1}} \cong O(n,\mathbb{R})$ and

$$S^n \cong O(n+1,\mathbb{R}) / O(n,\mathbb{R})$$

(33.6) Lemma: Let $\Phi : G \times M \to M$ a left action and let $\Phi_a : G \to M$ be the map $g \mapsto ga = \Phi_a(g) = \Phi(g,a)$.

1° $\Phi_a$ has constant rank.

2° $S_a := \Phi_a^{-1}(a)$ is Lie subgroup.

3° $\Phi$ transitive $\Rightarrow$ $S_a \cong S_b$ for all $a, b \in M$.

Pf: 1° $\Phi_a(hg) = \Phi_h \circ \Phi_a(g)$, hence $T_{hg}\Phi_a = T_{ga}\Phi_h \circ T_g\Phi_a$ & $\Phi_h$ diffeo.
2° Submfd. by 1°. 3° $a = gb$: $S_a \to S_b$, $ha \mapsto ghg^{-1}a = ghb$ is diffeo.

Fundamental questions:

- $Ga = B_a \subset M$ submanifold?
- $M/S_a$ manifold?
- $M/G$ manifold („space of orbits") ?

The "points" of $M/G$ are the orbits $M/G = \{Ga : a \in M\}$ described by the equivalence relation

$$a \sim b \iff \exists g \in G : a = gb \iff a \in Gb \iff Ga = Gb$$

The set theoretic quotient will be equipped with the quotient topology:

$$V \subset M/G \text{ open } :\iff p^{-1}(V) = \bigcup \{Ga : pa \in V\} \text{ is open}$$

In general, the quotient $M/\sim$ of a topological space need not be Hausdorff even if $M$ is. The relation „$\sim$" has to be closed, i.e.

$$R = \{(a,b) \mid a \sim b\} \subset M \times M$$

is closed. This is the case, for example, if $M$ is a Lie group and $G \subset M$ a Lie subgroup, i.e. in particular $G \subset M$ is closed.

Even if $M/_\sim$ is a Hausdorff space there might be no manifold structure on $M/_\sim$ such that it is a quotient manifold. The quotient manifold structure is the unique manifold structure on $M/_\sim$ such that the projection $p: M \to M/_\sim$ is smooth and such that for every map

$$f: M/_\sim \to Y$$

into an arbitrary manifold $Y$ the following (universal) property is satisfied: $f$ is smooth $\Leftrightarrow$ $f \circ p: M \to Y$ is smooth.

<u>DEFINITION:</u> Use of language: "The <u>orbit space exists</u>" is (in this course*) equivalent to "$M/G$ exists as a quotient manifold" (see above) and "the projection map $p: M \to M/G$ is a submersion."

<u>(33.7) OBSERVATION:</u> If the orbit space exists then

1° $Ga = p^{-1}(pa)$ is a submanifold, in particular closed.

2° $p$ is open (any submersion is open as a map).

3° $R$ is a closed submanifold of $M \times M$: $\bar{p}^{-1}(\Delta) = R$.

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* In other texts the orbit space is just the topological quotient. Or, the existence of the orbit space means that $M/G$ exists as a quotient manifold.

(33.8) THEOREM: The orbit space exists $\iff R \subset M \times M$ is a closed submanifold.*

We know already from 29.12 that the orbit space exists if $M$ is a Lie group and $G \subset M$ a Lie subgroup. In this case, the orbits $Ga$, $a \in M$, are the cosets, and 29.12 follows from 33.8, cf. 33.10.1°.

(33.9) EXAMPLE: Let $G = \{+1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{R}$ by multiplication: Then $R = \{\{a,a\} : a \in \mathbb{R}\} \cup \{(a,-a) : a \in \mathbb{R}\}$ is not a submanifold at $0$ (although closed). The orbits are $[0,\infty[$ and this is not a manifold.
If we delete $0 \in \mathbb{R}$, i.e. consider $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ everything works: $R \subset \mathbb{R}^\times \times \mathbb{R}^\times$ is a submfd & $\mathbb{R}^\times/G \cong ]0,\infty[$.
Similarly, $\mathbb{R}^n/O(n,\mathbb{R})$ "$\cong$" $[0,\infty[$.

Comment: $\mathbb{R}/G$ can be viewed as a stratified space with manifolds as strata. In our case: $\mathbb{R}/G = \{0\} \cup ]0,\infty[$.
Many quotients exist at least as stratified spaces.

Immediate consequences of the theorem:

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*We plan to present a proof in a section 33.B.

<u>(33.10) COROLLARY:</u> Let $H \subset G$ be a Lie subgroup of $G$ with its left action $\mathcal{L}_h : G \to G$, $h \in H$.

1° $G/H$ exists a orbit space (cf. 29.12) and $G$ acts on $G/H$ transitively. $G/H$ is homogeneous.

2° $H$ normal subgroup $\Longrightarrow$ $G/H$ Lie group.

<u>Pf.</u> $R = \{(g,f) \in G \times G \mid \exists h \in H : g = hf\} = \{(hf,f) : h \in H, f \in G\}$. The map $\varphi : G \times G \to G \times G$, $(h,f) \longmapsto (hf,f)$, is a diffeom., $\varphi^{-1}(u,v) = (uv^{-1}, v)$. Hence, $\varphi(H \times G) = R$ is a submanifold the orbit space by the theorem. Transitivity is obvious. Hence 1°, and 2° is an immediate consequence.

[20.12.10]

<u>(33.11) COROLLARY:</u> If $M$ is a homogeneous manifold with left action $\Phi : G \times M \to M$. Then $G/S_a$ exists as an orbit space and is diffeomorphic to $M$. In particular:

$$\boxed{M \text{ homog. } G\text{-manifold} \iff M \cong G/H,\ H \subset G \text{ Lie subgroup.}}$$

<u>Pf.</u> $S_a \subset G$ is a Lie subgroup by 33.6.2°. We know by 33.10.1° that the orbit space $G/S_a$ exists. We set

$$\psi : M = Ga \to G/S_a,\quad ga \longmapsto gH \in G/H,\quad H := S_a.$$

$\psi$ is well-defined: $ga = ha$ implies $gh^{-1} \in S_a$, i.e. $gH = hH$. $\psi$ is a diffeomorphism since $\psi^{-1}: G/S_a \to M$ is induced by $\psi^{-1} \circ p : G \to M$, $g \mapsto ga$.

$\square$

(33.12) EXAMPLES:

1° $S^n \cong O(n+1,\mathbb{R})/O(n,\mathbb{R})$, cf. 33.5.

2° Let $\Gamma \subset \mathbb{R}^n$ be a lattice, i.e. a discrete subgroup $\Gamma \subset \mathbb{R}^n$: $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \dots + \mathbb{Z}\omega_k$ and $\omega_1, \dots \omega_k$ linear independent. Then: $\mathbb{R}^n/\Gamma$ exists as orbit space and abelian Lie group: $\mathbb{R}^n/\Gamma \cong (S^1)^k \times \mathbb{R}^{n-k}$.

3° Similarly in the complex case, but different: $\Gamma \subset \mathbb{C}^n$, $\Gamma = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_k$, $\omega_j$ lin. independent over $\mathbb{R}$, $k \leq 2n$.
$\mathbb{C}^n/\Gamma$ is a complex abelian Lie group, diffeomorphic to $(S^1)^k \times \mathbb{R}^{2n-k}$. For different $\Gamma, \Gamma'$ of the same rank $k$ the complex Lie groups $\mathbb{C}^n/\Gamma$ and $\mathbb{C}^n/\Gamma'$ are in general <u>not</u> isomorphic as complex mfds. In particular, for $n=1$, $k=2$, we obtain the elliptic curves $\mathbb{C}/\Gamma = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (i.e. the compact Riemann

surfaces of genus 1). $\mathbb{C}/_{\Gamma} \cong \mathbb{C}/_{\Gamma'}$ bihol. group homom.
$\Longleftrightarrow \exists \alpha \in \mathbb{C} : \Gamma = \alpha \Gamma' \Longleftrightarrow \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \frac{\omega_1'}{\omega_2'} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}$.

4° The projective space in a new description:

$$\mathbb{P}_n(\mathbb{K}) \cong GL(n+1,\mathbb{K}) / GL(1,n;\mathbb{K}) .$$

In fact, $GL(n+1,\mathbb{K})$ acts transitively on $\mathbb{P}_n(\mathbb{K})$:

$$A(z_0 : \ldots : z_n) := (Az_0 : \ldots : Az_n) , \quad A \in GL , \ (z_0 : \ldots : z_n) = [z] \in \mathbb{P}_n(\mathbb{K}).$$

The isotropy group at $(0 : \ldots : 1)$ is

$$GL(1,n;\mathbb{K}) = \left\{ \begin{pmatrix} B & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ * \cdots * & \lambda \end{pmatrix} : B \in GL(n,\mathbb{K}), * \in \mathbb{K}, \lambda \in \mathbb{K}^\times \right\}$$

The action is not effective. However, the subgroup $SL(n+1,\mathbb{K})$ acts effectively, and

$$\mathbb{P}_n(\mathbb{K}) \cong SL(n+1,\mathbb{K}) / SL(1,n;\mathbb{K})$$

Similarly,

$$\mathbb{P}_n(\mathbb{R}) \cong SO(n+1) / SO(1,n)$$

$$\mathbb{P}_n(\mathbb{C}) \cong U(n+1) / U(1,n)$$

5° Grassmann manifolds: $1 \leq r \leq n$.

Let $G_{r,n}(\mathbb{K})$ be the (Grassmann) manifold of $r$-dimensional $\mathbb{K}$-subvectorspaces of $\mathbb{K}^n$. Then $GL(n,\mathbb{R})$ operates transitively on $G_{r,n}(\mathbb{K})$ $\quad (G_{1,n}(\mathbb{K}) = P_{n-1}(\mathbb{K}))$. To determine the isotropy group let $p \in G_{r,n}(\mathbb{K})$ be given by $e_{n-r+1}, \dots, e_n$. For $A \in GL(n,\mathbb{K})$ with $Ap = p$ we have

$$Ae_j = \sum_{\nu > n-r} A_j^\nu e_\nu , \quad n-r < j \leq n .$$

Hence, $A_j^\nu = 0$ for $1 \leq \nu \leq n-r$, $n-r < j \leq n$. $A$ has the form

$$A = \begin{array}{cl} \left(\begin{array}{c|c} B & 0 \\ \hline C & D \end{array}\right) & \left.\vphantom{B}\right\} 1 \leq \nu \leq n-r \\ \hphantom{B\ \,} \underbrace{\hphantom{D}}_{n-r<j\leq n} & \end{array}$$

with $B \in GL(n-r, \mathbb{K})$, $D \in GL(r, \mathbb{K})$ and $C \in \mathbb{K}^{(n-r)\times r}$ arbitrary. Denote $GL(r,n;\mathbb{K})$ the subgroup of $GL(n,\mathbb{K})$ of all such matrices: $GL(r,n;\mathbb{K})$ is a Lie subgroup (in particular closed) and we have

$$G_{r,n}(\mathbb{K}) \cong GL(n,\mathbb{K}) / GL(r,n;\mathbb{K}) .$$

In the case of $\mathbb{K} = \mathbb{R}$:

$$G_{r,n}(\mathbb{R}) \cong O(n,\mathbb{R}) / GL(r,n;\mathbb{R}) \cap O(n,\mathbb{R})$$

$$G_{r,n}(\mathbb{R}) \cong O(n,\mathbb{R}) / O(n-r) \times O(r)$$

$$G_{r,n}(\mathbb{R}) \cong SO(n) / H_r$$

where $H_r = \{ (A,B) \in O(n-r) \times O(n) \mid \det A \cdot \det B = 1 \}$.

As a consequence, $G_{r,n}(\mathbb{R})$ can be viewed to be the space of $(r-1)$ spheres in $S^{n-1}$.

In the case of $\mathbb{K} = \mathbb{C}$:

$$G_{r,n}(\mathbb{C}) \cong \mathcal{U}(n) / \mathcal{U}(n-r) \times \mathcal{U}(r)$$

6° Stiefel mfds: $1 \leq r \leq n$. $O(n-r,\mathbb{R})$ is subgroup of $O(n,\mathbb{R})$ by

$$B \longmapsto \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & B \end{pmatrix} \in O(n),$$

$$S_{r,n}(\mathbb{R}) := O(n,\mathbb{R}) / O(n-r,\mathbb{R}) \quad \text{Stiefel mfd}$$

$O(n,\mathbb{R})$ acts on the mfd $(S^{n-1})^r \subset (\mathbb{R}^n)^r$. Orbit of $(e_1, \dots e_r)$ is submfd $B \subset (S^{n-1})^r$. $O(n)$ acts transitively on B. Isotropy group is $O(n-r,\mathbb{R})$.

Interpretation:

$$S_{r,n}(\mathbb{R}) = \{ (v_1, \dots v_r) \in (\mathbb{R}^n)^r \mid v_1, \dots v_r \text{ orthonormal} \}$$

There is a natural map

$$S_{r,n}(\mathbb{R}) \longrightarrow G_{r,n}(\mathbb{R}) \;,\; (v_1, \dots v_r) \mapsto \operatorname{span}\{v_1, \dots v_r\} \,,$$

which yields

$$G_{r,n}(\mathbb{R}) \cong S_{r,n}(\mathbb{R}) / O(r)$$

$r = 1$: $\quad S_{1,n} \cong S^{n-1} \longrightarrow \mathbb{P}_{n-1}(\mathbb{R})$

$r = n$: $\quad S_{n,n} = O(n,\mathbb{R})$ set of ON bases.

7° Flag manifolds: $\quad F_n = GL(n,\mathbb{K}) / \Delta_n$

where $\Delta_n$ group of matrices of the form

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL(n,\mathbb{K})$$

$e_1, \dots e_n \in \mathbb{R}^n$ base: $S_0 = \{0\} \subset S_1 = \mathbb{K}e_1 \subset S_2 = \mathbb{K}e_1 \oplus \mathbb{K}e_2 \subset \dots \subset S_n = \mathbb{K}^n$ is the standard flag. $S = (S_0, \dots S_n) \in \prod_{r=0}^{n} G_{r,n}(\mathbb{K}) =: M$. $GL(n,\mathbb{K})$ acts on $M$. Orbit: All flags $V_0 = \{0\} \subsetneq V_1 \subsetneq \dots \subsetneq V_n = \mathbb{K}^n$. $AS = S$ $\iff A \in \Delta_n$.