32. Classical Lie Groups

Version 1.1

Notiztitel

(32.1) Real Lie Groups.

Lie group	dim.	Lie algebra
1° $GL(n,\mathbb{R})$	n^2	$\mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n}$
2° $SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$	n^2-1	$\mathfrak{sl}(n,\mathbb{R}) = \{a : \operatorname{tr} a = 0\}$
3° $O(n) = O(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A^T A = 1\!\!1_n\}$	$\frac{n}{2}(n-1)$	$\mathfrak{o}(n) = \{a : a^T + a = 0\}$
4° $SO(n) = SO(n,\mathbb{R}) = O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$	"	$\mathfrak{o}(n)$
5° $Sp(n,\mathbb{R}) = \{A \in \mathbb{R}^{2n\times 2n} : A^T J A = J\}$ where $J = \begin{pmatrix} 0 & 1\!\!1_n \\ -1\!\!1_n & 0 \end{pmatrix}$	$2n^2+n$	$\mathfrak{sp}(n,\mathbb{R}) = \{a \in \mathbb{R}^{2n\times 2n} : a^T J + Ja = 0\}$
6° $O(3,1;\mathbb{R}) = \{A \in \mathbb{R}^{4\times 4} : A^T \eta A = \eta\}$ where $\eta = \begin{pmatrix} 1\!\!1_n & 0 \\ 0 & -1 \end{pmatrix}$	6	$\mathfrak{o}(3,1;\mathbb{R}) = \{a^T\eta + \eta a = 0\}$
7° $O(l,k;\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A^T \Phi A = \Phi\}$ where $\Phi = \begin{pmatrix} 1\!\!1_l & 0 \\ 0 & 1\!\!1_k \end{pmatrix}$	$\frac{n}{2}(n-1)$	$\mathfrak{o}(l,k;\mathbb{R}) = \{a^T\Phi + \Phi a = 0\}$
8° $Spin(n)$ universal cover of $SO(n)$ with two sheets	$\frac{n}{2}(n-1)$	$\mathfrak{o}(n,\mathbb{R})$
9° $U(m) = \{A \in \mathbb{C}^{m\times m} : \bar{A}^T A = 1\!\!1_m\}$	m^2	$\mathfrak{u}(m) = \{a \in \mathbb{C}^{m\times m} : \bar{a}^T + a = 0\}$
10° $SU(m) = \{A \in U(m) : \det A = 1\}$	m^2-1	$\mathfrak{su}(m) = \{a \in \mathfrak{u}(m) : \operatorname{tr} a = 0\}$
11° $U(k,\mathbb{H}) = \{A \in \mathbb{H}^{k\times k} : \bar{A}^T A = 1\!\!1_k\}$ $\mathbb{H} \cong \mathbb{R}^4$ $x = x_0 + ix_1 + jx_2 + kx_3$ $\bar{x} = x_0 - ix_1 - jx_2 - kx_3$		

Comments:

Ad 1°: $GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ open submanifold of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$.
In particular, $GL(n,\mathbb{R})$ is a real analytic manifold.

$$(A,B) \longmapsto A\circ B^{-1}$$

is real analytic, since the components of $A\circ B^{-1}$ are rational functions in the components of A and B. Hence, $GL(n,\mathbb{R})$ is a real analytic Lie group.
For $G = GL(n,\mathbb{R})$ as an open subset of $\mathbb{R}^{n\times n}$ the tangent space T_eG has a natural identification with $\mathbb{R}^{n\times n}$. And this identification induces a Lie algebra homomorphism

$$\mathfrak{g} = \mathrm{Lie}\, GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^{n\times n} = \mathrm{End}(\mathbb{R}^n)$$

with the commutator of matrix multiplacation as the Lie bracket of $\mathbb{R}^{n\times n}$. Hence, one writes

$$\mathrm{Lie}\, GL(n,\mathbb{R}) = \mathbb{R}^{n\times n} = \mathfrak{gl}(n,\mathbb{R})$$

Ad 2° $\det : GL(n,\mathbb{R}) \to \mathbb{R}$ is smooth as a polynomial, and even analytic. Because of $\operatorname{grad} \det A \neq 0$ for all $A \in SL(n,\mathbb{R}) = \det^{-1}(+1)$, $SL(n,\mathbb{R})$ is a submanifold. (This follows by applying 29.12 since $SL(n,\mathbb{R})$ is closed.) Using the fact that $SL(n,\mathbb{R})$ is a subgroup as well, we conclude that

that $SL(n,\mathbb{R})$ is a Lie group.

Let us deduce $\quad sl(n,\mathbb{R}) = \{X \in \mathbb{R}^{n^2} : \operatorname{tr} X = 0\}$ in detail:

For a curve $A : J \to SL(n,\mathbb{R})$ with $A(0) = id_n = 1\!\!1_n$ we may assume $\|A(t) - 1\!\!1_n\| < 1$ for $t \in J$ by choosing J to be a small open interval containing 0. Hence, there exists a curve $B : J \to \mathbb{R}^{n\times n}$ with $A(t) = \exp B(t)$. Because of $\det A = \det \exp B = \exp \operatorname{tr} B$ (and the continuity of B) we have $\operatorname{tr} B(t) = 0$. As a consequence, $\operatorname{tr} \dot{B}(t) = 0$, since $0 = \frac{d}{dt}(\operatorname{tr} B) = \operatorname{tr} \dot{B}(t)$. In particular, $\operatorname{tr} \dot{B}(0) = 0$. Finally, $\dot{A}(t) = \dot{B}(t) \exp B(t) = \dot{B}(t) A(t)$ and $\dot{A}(0) = \dot{B}(0) A(0) = \dot{B}(0)$. Thus $\dot{A}(0) \in \{X : \operatorname{tr} X = 0\} = sl(n,\mathbb{R})$.

Conversely, let $X \in \mathbb{R}^{n^2}$ satisfy $\operatorname{tr} X = 0$. $A(t) := \exp tX = \sum_{n=0}^{\infty} \frac{1}{n!} t^n X^n$ is a well-defined smooth curve with $\det A(t) = \exp \operatorname{tr} X = 1$, i.e. $A(t) \in SL(n,\mathbb{R})$.

Ad $3° - 7°$ similarly, with different β defining bilinear maps $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \longrightarrow A^T \beta B \quad (\beta = 1\!\!1_n, \beta = J, \beta = \eta, \beta = \Phi)$

<u>(32.2) Complex Lie Groups:</u> $1° - 7°$ as in (32.1) but with $\mathbb{C}$ instead of $\mathbb{R}$. They carry a natural complex-analytic

structure as a complex manifold.

(32.3) Compactness: $O(n,\mathbb{R})$, $SO(n,\mathbb{R})$, $U(m)$, $SU(m)$, $U(k,\mathbb{H})$ are compact Lie groups since they are bounded closed subsets of $\mathbb{K}^{n^2 \times n^2}$. The groups $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$ $(n \neq 1)$, $SL(n,\mathbb{C})$, $O(3,1;\mathbb{R})$, $O(n,\mathbb{C}) \cong O(n,\mathbb{R}) \times \mathbb{R}^n$ are not compact. Spin n is compact.

(32.4) Connectedness: $SO(n,\mathbb{R})$, $U(m)$, $SU(m)$, $Spin(n)$, $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(k,\mathbb{H})$, $SL(n,\mathbb{R})$ are connected. $SU(m)$, $U(k,\mathbb{H})$ and $Spin(n)$ are simply connected. $GL(n,\mathbb{R})$ has 2 connected components, $GL^+(n,\mathbb{R}) = \{A \in \mathbb{R}^{n^2} : \det A > 0\}$ and $GL^-(n,\mathbb{R}) = \{A \in \mathbb{R}^{n^2} : \det A < 0\}$. The Lorentz group has the following 4 connected components

$$L_+^\uparrow := \{A : \det A = 1 \ \& \ A_4^4 \geq 1\} \quad \text{(the proper Lorentz group ; } A = (A_k^j))$$
$$L_-^\uparrow := \{A : \det A = -1 \ \& \ A_4^4 \geq 1\}$$
$$L_+^\downarrow := \{A : \det A = 1 \ \& \ A_4^4 \leq -1\}$$
$$L_-^\downarrow := \{A : \det A = -1 \ \& \ A_4^4 \leq -1\}$$

Some useful examples and isomorphisms:

(32.5) $U(1) \cong SU(1) \cong SO(2,\mathbb{R}) \cong \mathbb{S}^1$, where $\mathbb{S}^1$ is the Lie group with underlying manifold $\{e^{i\varphi} \mid \varphi \in \mathbb{R}\} \subset \mathbb{C} = \mathbb{R}^2$ and $e^{i\varphi} e^{i\psi} := e^{i(\varphi+\psi)}$.

(32.6) $U(2) = \left\{ \begin{pmatrix} z & -\bar{w} e^{i\varphi} \\ w & \bar{z} e^{i\varphi} \end{pmatrix} : (z,w) \in \mathbb{C}^2, \varphi \in \mathbb{R} \,\&\, z\bar{z} + w\bar{w} = 1 \right\} \cong S^3 \times S^1$, where „$\cong$" denotes a manifold isomorphism..

(32.7) $SU(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : z\bar{z} + w\bar{w} = 1 \right\} \cong S^3$ as mflds.

$SU(2) \cong U(1, \mathbb{H})$ as Lie groups.

(32.8) $Spin(3) \cong SU(2)$ as Lie groups.
$Spin(4) \cong SU(2) \times SU(2)$ as Lie group.

(32.10) $SL(2,\mathbb{C})$ is connected and isomorphic to $S^3 \times \mathbb{R}^3$ as a manifold. And $SL(2,\mathbb{C})$ is the universal cover of $L_+^\uparrow$, the proper Lorentz group.

(32.11) Further Lie group isomorphisms:

$Sp(1,\mathbb{C}) \cong SL(2,\mathbb{C})$

$Sp(1,\mathbb{R}) \cong SL(2,\mathbb{R})$

$Spin(5) \cong U(2,\mathbb{H})$

$Spin(6) \cong SU(4)$