31. The Adjoint Representation

Version 1.1

Notiztitel

We need the adjoint representation for the description of transformation properties of connection forms and curvature on general principal fibre bundles. In the case of matrix groups this transformation behaviour simplifies because adjoint representation has a simple description.

We begin with some statements on general Lie group representations and Lie algebra representations.

(31.1) DEFINITION: Let G be a Lie group, let $\mathfrak{a}$ be a Lie algebra (with Lie bracket $[\,,\,]_{\mathfrak{a}}$) and let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

1° A representation of G in V (in case of $\dim V < \infty$) is a Lie group homomorphism*

$$\varrho : G \to GL(V)\,.$$

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*According to 30.7 it is sufficient for φ to be a representation if φ is a group homomorphism and continuous. In this way the concept of a Lie group representation has a generalization to infinite dimensional V with a topology inducing a topology on $GL(V)$.

2° A representation of $\mathfrak{a}$ in V is a Lie algebra homomorphism

$$\varphi : \mathfrak{a} \longrightarrow \mathrm{End}_{\mathbb{K}} V,$$

i.e. a $\mathbb{K}$-linear map with $[\varphi(X), \varphi(Y)] = \varphi([X,Y]_{\mathfrak{a}})$, $X, Y \in \mathfrak{a}$.

If $\varrho : G \to GL(V)$ is a representation of the Lie group G then the tangent map $T_e\varrho = \mathrm{Lie}\,\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G$. (cf. 29.5)

From given representations one can construct various new representations. E.g. for $\varrho : G \to GL(V)$, $\sigma : G \longrightarrow GL(W)$ one obtains

$$\varrho \oplus \sigma : G \to GL(V \oplus W),$$
$$g \longmapsto (v \oplus w \longmapsto \varrho(g)v \oplus \sigma(g)w.$$

$$\varrho \otimes \sigma : G \longrightarrow GL(V \otimes W),$$
$$g \longmapsto (v \otimes w \longmapsto \varrho(g)v \otimes \sigma(g)w),$$

$$\varrho^{\vee} : G \longrightarrow GL(V^*), \quad \varrho^{\vee}(g)(\mu) = \mu(\varrho(g^{-1})v),$$

$$\varrho_k : G \to GL(\Lambda^k V^*),$$

$$g \longmapsto (\mu_1 \wedge \ldots \wedge \mu_k \longmapsto \varrho^\vee(\mu_1) \wedge \ldots \wedge \varrho^\vee(\mu_k)).$$

These constructions are similar to the operations on vector bundles (cf. section 18). Analogous constructions are used for Lie algebra representations.

(31.2) DEFINITION: A representation $\varrho : G \to GL(V)$ of the Lie group G in a euclidian (resp. hermitian if $\mathbb{K} = \mathbb{C}$) vector space V with scale product $\langle\, , \rangle$ is called orthogonal (resp. unitary) if

$$\langle \varrho(g)v, \varrho(g)w \rangle = \langle v, w \rangle \quad \text{for all } g \in G,\ v, w \in V,$$

i.e. if $\langle\, , \rangle$ is G-invariant.

(31.3) PROPOSITION: Let $\varrho : G \to GL(V)$ be a representation of the compact Lie group G in V. Then there exists a G-invariant euclidean (resp. hermitian) scale product on V.

Proof. Let $\langle\ \rangle$ be an arbitrary euclidean (resp.

hermitean) scalar product on V. Choose a basis $X_1, \dots X_n$ of T_eG and let $\tilde{X}_1 \dots \tilde{X}_n \in \mathrm{Lie}\, G \subset \mathcal{V}(G)$ be the corresponding right invariant vectorfields. Then the dual forms $\mu_j \in \mathcal{V}(G)^*$ defined by $\mu_j(\tilde{X}_k) = \delta_{jk}$ give a basis of T_g^*G at each g. Hence, $\omega := \mu_1 \wedge \dots \wedge \mu_n$ give a volume form which is right invariant. We know define a new scalar product on V by

$$\langle v, w \rangle_G := \int_{g \in G} \langle \rho(g)v, \rho(g)w \rangle \, \omega \ , \quad v, w \in V.$$

$\langle \, , \rangle_G$ is well-defined since G is cpt and invariant by definition. For every $g \in G$:

$$\begin{aligned} \langle \rho(g)v, \rho(g)w \rangle_G &= \int_{h \in G} \langle \rho(h)\rho(g)v, \rho(h)\rho(g)w \rangle \, \omega \\ &= \int_{h \in G} \langle \rho(hg)v, \rho(hg)w \rangle \, \omega \\ &= \int_{h \in G} T_h R_g \langle \rho(h)v, \rho(h)w \rangle \, \omega \\ &= \int_{R_g G} \langle \rho(h)v, \rho(h)w \rangle \, \omega = \langle v, w \rangle_G \end{aligned}$$

$\square$

<u>(31.4) DEFINITION</u>: A representation $\rho : G \to GL(V)$ of the Lie group G is called *irreducible* if there is no proper

subspace $W \subset V$ with $\varrho(g)W \subset W$.

(31.5) THEOREM: A representation $\varrho : G \to GL(V)$ of a compact Lie group decomposes into irreducible components, that is there exists a decomposition $V = V_1 \oplus V_2 \oplus \dots \oplus V_m$ into invariant subspaces $V_k \subset V$ (i.e. $\varrho(g) V_k \subset V_k \;\; \forall g \in G$) such that the restriction $\varrho_k = \varrho|_{V_k} : G \to GL(V_k)$ is irreducible (and

$$\varrho = \varrho_1 \oplus \dots \oplus \varrho_m : G \to GL(V_1 \oplus \dots \oplus V_m)$$

Proof. We assume ϱ to be unitary (resp. orthogonal). If ϱ is not irreducible there is a linear subspace $W \subset V$, $0 \neq W \neq V$, such that $\varrho(g) W \subset W$ for all $g \in G$. We see that $W^\perp$ is invariant as well. Let $v \in W^\perp$. For all $w \in W$: $\langle w, v \rangle = 0$. By the invariance of the scalar product : $\langle \varrho(g) w, \varrho(g) v \rangle = 0$ for all $g \in G$. Since $\varrho(g)$ is invertable with $\varrho(g^{-1}) W \subset W$ we have $\varrho(g) W = W$. Hence $\langle w', \varrho(g) v \rangle = 0$ for all $w' \in W$, that is $\varrho(g) v \in W^\perp$. $W^\perp$ is a proper invariant subspace and we set $V_1 = W$ and apply the same procedure to

the restriction $\rho|_{W^\perp} : G \to GL(W^\perp)$, $\rho|_{W^\perp}(g) = \rho(g)|_{W^\perp}$.
By induction the result follows. $\square$

Now, the adjoint representation is induced by the inner automorphism

$$\alpha_g := \mathcal{L}_g \circ \mathcal{R}_{g^{-1}} : G \to G \ , \quad a \longmapsto g a g^{-1}, \quad g \in G$$

on a Lie group G.

(31.6) PROPOSITION - DEFINITION: 1° The tangent map $T\alpha_g : TG \to TG$ induces a a representation

$$Ad : G \to GL(\mathfrak{g}), \quad g \longmapsto T_e\alpha_g \ , \ g \in G,$$

where $\mathfrak{g} = \mathrm{Lie}\, G$. Ad is the <u>adjoint representation of</u> G.

2° The tangent map of Ad induces a representation ad of $\mathfrak{g}$

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \longmapsto T_e Ad(X) : Y \longmapsto [X,Y],$$

called the <u>adjoint representation</u> of $\mathfrak{g}$.

Note, that in the case of a matrix group $G \subset GL(n, \mathbb{K})$

$$Ad(g) : \mathfrak{g} \to \mathfrak{g} \,, \quad X \mapsto gXg^{-1}$$

Proof of the proposition:

1° Since α_g is smooth Ad is smooth as well, and since $\alpha_{gh} = \alpha_g \circ \alpha_h$ for $g, h \in G$ Ad is a group homomorphism

$$Ad(gh) = T_e \alpha_{gh} = T_e(\alpha_g \circ \alpha_h) = T_e \alpha_g \circ T_e \alpha_h = Ad(g) Ad(h) .$$

2° $ad = T_e Ad : T_e G = \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is linear and a Lie a algebra homomorphism. Moreover, for $X, Y \in \mathcal{V}(G)$ we have in general (cf. 8.8 & 8.9)

$$[X,Y](a) = \frac{d}{dt}\left(T\varphi^X_{-t}\left(Y(\varphi_t a)\right)\right)\Big|_{t=0} \quad \left(= \frac{d}{dt}\left((\varphi^X_t)^* Y\right)_{t=0}\right),$$

where $\varphi^X_t(a) = \varphi^X(t,a)$ is the flow of X. In the case of a left invariant $\tilde{X} \in \operatorname{Lie} G = \mathfrak{g}$,

$$\varphi^{\tilde{X}}(t, g) = \exp(tX) g \,, \quad X = X(e) \,.$$

$$\left(X = [\exp tX]_e \,, \quad \tilde{X}(g) = T_e \mathcal{R}_g(X) = [\mathcal{R}_g(\exp tX)] = (\exp tX) g . \right)$$

$$\varphi^{\tilde{X}}_t(g) = \mathcal{L}_{\gamma(t)} g \,, \quad \gamma(t) = \exp tX \,.$$

$$\begin{aligned} ad(X)(Y) &= TAd(X)(Y) = [Ad(\gamma)](Y) = [T(\mathcal{R}_{\gamma^{-1}} \circ \mathcal{L}_\gamma)](Y) \\ &= [T\mathcal{R}_{\gamma^{-1}} \circ T\mathcal{L}_\gamma(Y)] = \frac{d}{dt}\left(T\mathcal{R}_{\gamma^{-1}}\, Y(\gamma)\right) = [X,Y] \end{aligned}$$

$\square$

<u>(31.7) DEFINITION:</u> $\mu = \mu_G \in \mathcal{A}^1(G, \mathfrak{g})$ given by

$$\mu(X)(a) := T_a \mathcal{L}_{a^{-1}}(X(a)) \in T_e G \cong \mathfrak{g} \quad a \in G \ \& \ X \in \mathfrak{X}(G)$$

is the <u>Maurer-Cartan form</u>.

For a left invariant vector field, i.e. $X(g) = T_e \mathcal{L}_g(X(e))$ we obtain

$$\mu(X)(a) = T_a \mathcal{L}_{a^{-1}} T_e \mathcal{L}_a(X(e)) = X(e).$$

Similarly, we have ${}^R\mu \in \mathcal{A}^1(G, \mathfrak{g})$ with $\mu(X)(a) = T_a \mathcal{R}_{a^{-1}}(X(a))$, reproducing always $X(e)$ for right invariant vector fields.

<u>(31.8) PROPOSITION:</u> $\mathcal{L}_g^* \mu = \mu$ & $\mathcal{R}_g^* \mu = Ad(g^{-1}) \circ \mu$

<u>(31.9) DEFINITION:</u> A metric h on G (i.e. on TG) is called <u>left invariant</u> (resp. <u>right invariant</u>) if $\mathcal{L}_g^* h = h$ (resp. $\mathcal{R}_g^* h = h$) for all $g \in G$. h is biinvariant if h is left and right invariant.

Recall that for smooth $\varphi : G \to G$:

$$\varphi^* h(X,Y) = h(T\varphi(X), T\varphi(Y)), \quad X,Y \in \mathfrak{X}(G).$$

Given a scalar product $\langle\,,\rangle$ on $\mathfrak{g}$ a left invariant metric h is defined by

$$h(X,Y) = \langle \mu(X), \mu(Y) \rangle \quad , X,Y \in \mathfrak{X}(G),$$

$$h_a(X,Y) = \langle T\lambda_{a^{-1}}(X), T\lambda_{a^{-1}}(Y) \rangle \quad , X,Y \in T_aG.$$

And vice versa.

<u>(31.10) PROPOSITION:</u> A left invariant h on G is biinvariant iff $h_e = \langle\,,\rangle$ on T_eG is Ad-invariant:

$$\langle v, w \rangle = \langle \mathrm{Ad}\, v, \mathrm{Ad}\, w \rangle \quad \forall v, w \in T_eG$$

<u>Pf</u>: $X, Y \in \mathfrak{X}(G)$:

$$\mathcal{R}_g^* h(X,Y) = h(T\mathcal{R}_g(X), T\mathcal{R}_g(Y))$$

$$= \langle \mu(T\mathcal{R}_g(X), \mu(T\mathcal{R}_g(Y)) \rangle$$

$$= \langle \mathrm{Ad}(g^{-1})\mu(X), \mathrm{Ad}(g^{-1})\mu(Y) \rangle$$

$$= \langle \mu(X), \mu(Y) \rangle = h(X,Y) \qquad \square$$

Finally, the Killing form κ:

<u>(31.11) DEFINITION:</u> Given a fin. dim. Lie algebra $\mathfrak{g}$ the form $\kappa = \kappa_{\mathfrak{g}}$

$$\kappa(X,Y) := \mathrm{Tr}\,(\mathrm{ad}(X)\circ \mathrm{ad}(Y)) \quad , \; X,Y \in \mathfrak{g},$$

is the Killing form.

<u>(3.12) PROPOSITION:</u> For a Lie algebra isomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ one has $\kappa(\sigma(X),\sigma(Y)) = \kappa(X,Y)$. In particular,

$$\kappa(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = \kappa(X,Y) \quad , \; g \in G,$$

$$\kappa(\mathrm{ad}(X)Y, Z) + \kappa(Y, \mathrm{ad}(X)Z) = 0 \quad , \; X,Y,Z \in \mathfrak{g}$$

<u>Proof</u>: From $\sigma \circ \mathrm{ad}(X) = \mathrm{ad}(\sigma X)\circ \sigma$ we yield

$$\begin{aligned}\kappa(\sigma X, \sigma Y) &= \mathrm{Tr}\,(\mathrm{ad}(\sigma X)\circ \mathrm{ad}(\sigma Y)) \\ &= \mathrm{Tr}\,(\sigma \circ \mathrm{ad}(X)\circ \mathrm{ad}(Y)\circ \sigma^{-1}) \\ &= \mathrm{Tr}\,(\mathrm{ad}(X)\circ \mathrm{ad}(Y)) \\ &= \kappa(X,Y).\end{aligned}$$

Hence, true for $\sigma = \mathrm{Ad}(g)$. The last identity by using derivatives.