29. Lie Groups and Their Algebras

Version 1.1

Notiztitel

(29.1) DEFINITION: 1° A *Lie group* is a manifold* G together with a group structure $G \times G \to G$ such that the map

$$G \times G \to G, \quad (g,h) \longmapsto gh^{-1},$$

is a smooth map.

2° A *homomorphism of Lie groups* (a morphism in the category (Lie) of Lie groups) is a homomorphism

$$\varphi : G \to H$$

of groups which is a smooth map as well.

3° In a Lie group G with $g \in G$ the *right translation* is

$$\mathcal{R}_g : G \to G, \quad a \longmapsto ag, \quad a \in G.$$

$\mathcal{R}_g$ is smooth and this is true for the *left translations*, $\mathcal{L}_g(a) := ga$, as well.

4° A vector field $X \in \mathcal{V}(G)$ is called *right invariant* if $\mathcal{R}_g^* X = X$, for all $g \in G$, and it is called *left invariant* if $\mathcal{L}_g^* X = X$ for all $g \in G$.

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* As before we assume a manifold to be a smooth manifold.

i.e. $T_a \mathcal{R}_g^{-1}\left(X\left(\mathcal{R}_g(a)\right)\right) = X(a) \quad , \forall a \in G\,, \forall g \in G\,,$ or

$$T_a \mathcal{R}_g\left(X(a)\right) = X(ag) \quad , \forall a \in G,\ \forall g \in G, \text{ or}$$

$$T\mathcal{R}_g(X) = X \circ \mathcal{R}_g \quad , \forall g \in G.$$

Hence, $X \in \mathcal{W}(G)$ is right invariant if for all $g \in G$ the diagramm

$$\begin{array}{ccc} TG & \xrightarrow{T\mathcal{R}_g} & TG \\ X \uparrow & & \uparrow X \\ G & \xrightarrow[\mathcal{R}_g]{} & G \end{array}$$

is commutative.

<u>Examples:</u> $GL(n, \mathbb{K})$ is a Lie group, an open subset of $\mathbb{K}^{n \times n}$. By a general result, every closed subgroup $G \subset GL(n, \mathbb{K})$ is a submanifold of $\mathbb{K}^{n \times n}$. From this it is easy to deduce, that all closed subgroups of $GL(n, \mathbb{K})$ are Lie groups. These are the <u>matrix groups</u>.
Examples: $SO(3, \mathbb{R})$, $SU(2)$, $SL(n, \mathbb{K})$, ...
For many purposes the matrix groups provide enough examples of Lie groups, and one could restrict to the category of matrix groups.

[07.12.10]

Let e denote the neutral element of G. For a tangent vector $X_0 \in T_eG$ set

$$\tilde{X}_0(a) := T_e R_a(X), \quad a \in G.$$

$\tilde{X}_0$ is a right invariant vector field. Moreover, every right invariant vector field is of this form.

Note, that for a diffeomorphism $\varphi : M \to M$ the „pullback"

$$\varphi^* : \mathcal{V}(M) \to \mathcal{V}(M), \quad X \mapsto \left(a \mapsto (T_a\varphi)^{-1} X(\varphi a)\right),$$

is a Lie algebra homomorphism. This can be shown by using local formulas for $\varphi^* X$ and $[X,Y]$ with respect to coordinates q^k. Hence, for Lie groups $M = G$:

$$R_g^*([X,Y]) = [R_g^*(X), R_g^*(Y)], \quad g \in G, X, Y \in \mathcal{V}(G).$$

<u>(29.2) PROPOSITION:</u> Let $\mathrm{Lie}\, G := \{X \in \mathcal{V}(G) \mid X \text{ right invariant}\}$. Then $T_eG \to \mathcal{V}(G)$, $X_0 \mapsto \tilde{X}_0$, is an injective vector space homomorphism. Moreover, $\mathrm{Lie}\, G \subset \mathcal{V}(G)$ is a Lie subalgebra of $\mathcal{V}(G)$.

Pf. If X, Y are right invariant then $[X,Y]$ is right invariant as well according to the foregoing formula. $\square$

$\mathrm{Lie}\, G$ is called the "Lie algebra of G" and is sometimes denoted by $\mathfrak{g}$. Also: T_eG with the bracket induced by $X_0 \mapsto \tilde{X}_0$ from $\mathcal{V}(G)$ is called the Lie algebra of G.

(29.3) PROPOSITION: For a Lie group G the tangent bundle $TG \to G$ is trivial: A basis of T_eG yields a basis of $\mathrm{Lie}\, G$ as a vector space and a basis of $\mathcal{V}(M) = \Gamma(G, TG)$ over $\mathcal{E}(G)$.

(29.4) EXAMPLE: The Lie algebra of $GL(n,\mathbb{K})$ is („can be identified with") the Lie algebra $\mathfrak{gl}(n,\mathbb{K}) = \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n) \cong \mathbb{K}^{n\times n}$ with the commutator

$$[X,Y] := X\circ Y - Y\circ X \quad , \quad X, Y \in \mathbb{K}^{n\times n}$$

as Lie bracket. Indeed, the tangent space $T_eGL(n,\mathbb{K})$ at the unit matrix $e \in GL(n,\mathbb{K}) \subset \mathbb{K}^{n\times n}$ can be identified with $\mathbb{K}^{n\times n}$ ($GL(n,\mathbb{K})$ is an open subset of $\mathbb{K}^{n\times n}$). Let

$X \in T_e GL(n, \mathbb{K})$ be represented by the smooth curve $\gamma_X^e : [t_0, t_1] \to G$, $\gamma_X^e(0) = e$: $X = [\gamma_X^e]_e = \dot\gamma_X^e(0)$ The corresponding right invariant vector field $\tilde X$ fulfills

$$\tilde X(g) = [R_g \circ \gamma_X^e]_g = \frac{d}{dt}\left(\gamma_X^e(t) g\right)\Big|_0 = X \circ g .$$

For the induced Lie bracket we obtain for $X, Y \in T_e GL(n, \mathbb{K}) = \mathbb{K}^{n \times n}$:

$$[X, Y] = [\tilde X, \tilde Y](e) = \tilde X(\tilde Y)(e) - \tilde Y(\tilde X)(e)$$

$$= \frac{d}{dt} \tilde Y(\gamma_X(t))\Big|_{t=0} - \frac{d}{dt} \tilde X(\gamma_Y(t))\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\gamma_X(t) \circ Y - \gamma_Y(t) \circ X\right)_{t=0}$$

$$= X \circ Y - Y \circ X .$$

<u>(29.5) PROPOSITION:</u> Let $h : G \to H$ be a Lie group homomorphism. Then $h(e) = e$ and $T_e h : T_e G \to T_e H$ induces a Lie algebra homomorphism

$$\mathrm{Lie}\, h : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$$

Pf. $\mathrm{Lie}\, h(\tilde X) := \widetilde{T_e h(\tilde X)} : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$, $X \in T_e G$, respects the Lie bracket in $\mathcal{V}(M)$.

We thus have defined the functor

$$\mathrm{Lie} : (\mathrm{Lie}_{\mathbb{R}}) \to (\mathrm{lie}_{\mathbb{R}})$$

from the category of (real) Lie groups into the category of (real) Lie algebras. (Correspondingly for $\mathbb{K} = \mathbb{C}$)

For finite dimensional Lie algebras we have a kind of inverse (an „adjoint"):

(29.6) PROPOSITION: Given a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ there exists a unique connected and simply connected Lie group $G = \hat{\mathfrak{g}}$ such that

$$\mathrm{Lie}\, \hat{\mathfrak{g}} = \mathfrak{g}$$

and to every Lie algebra homom. $\varphi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique $\hat{\varphi} \in \mathrm{Lie}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ with $\mathrm{Lie}\, \hat{\varphi} = \varphi$.

This result can be deduced from the following four structure results:

(29.7) THEOREM (of Ado): Every finite dimensional Lie algebra over $\mathbb{R}$

is (Lie algebra) isomorphic to a matrix Lie algebra, i.e. to a Lie subalgebra of some $\mathfrak{gl}(n,\mathbb{R})$.

<u>(29.8) PROPOSITION:</u> Let $\mathfrak{g} \subset \mathfrak{h} = \mathrm{Lie}\, H$ a Lie subalgebra of the Lie algebra of a Lie group H. Then there exists a connected Lie subgroup $G \subset H$ such that $\mathfrak{g} = \mathrm{Lie}\, G$.

Note that a Lie subgroup H of a Lie group G is (in this course*) a subset $H \subset G$ which is a subgroup of the group G and a <u>closed</u> submanifold of the manifold G at the same time.

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* This concept of a Lie subgroup is not in agreement with all texts in Lie group theory. Quite often $H \subset G$ is already called a Lie subgroup of G if the inclusion map $H \hookrightarrow G$ is a Lie group homomorphism. With this more general „subgroup concept" it can occur that $H \subset G$ does not have the induced topology (or manifold structure) from G. For example, there exists an injective Lie group homomorphism $S^1 \to S^1 \times S^1$ with dense image. Such general Lie subgroups $H \subset G$ are Lie subgroups in our sense if the induced topology agrees with the given topology on H.

As a consequence of 29.7 a given Lie algebra $\mathfrak{g}$ can be regarded to be a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, and we obtain $\hat{\mathfrak{g}}$ as the universal covering of the subgroup $G \subset GL(n, \mathbb{K})$ given in 29.8.

<u>(29.9) Lemma:</u> The universal covering $\hat{G} \xrightarrow{\pi} G$ of a Lie group has a natural manifold structure such that $\hat{G}$ is a Lie group and π is a Lie group homomorphism and a local diffeomorphism.

Sketch of proof: The local inverses of π serve as charts.

If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism we obtain the $\hat{\varphi}$ of (29.6) by the following proposition.

<u>(29.10) PROPOSITION:</u> Let G and H be Lie groups and let $\varphi : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$ be a Lie algebra homomorphism. If G is simply connected (and connected) then there exists a unique Lie group homomorphism

$$\hat{\varphi} : G \to H$$

with $\mathrm{Lie}\,\hat{\varphi} = h$.

We conclude this section with some remarks on products and quotients.

<u>(29.11) REMARK:</u> Of course, for Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ there exists a natural Lie algebra structure on the direct sum (product)

$$\mathfrak{g} \oplus \mathfrak{h} \ , \quad [X \oplus Y, X' \oplus Y'] := [X, X'] \oplus [Y, Y'] .$$

$\mathfrak{g} \oplus \mathfrak{h}$ with this structure is a product in $(\mathrm{lie}_{\mathbb{K}})$.
Similarly, it is easy to check that for a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the quotient $\mathfrak{g}/\mathfrak{h}$ has a natural Lie algebra structure such that it is the quotient in $(\mathrm{lie}_{\mathbb{K}})$.

In the case Lie groups we have the following.

<u>(29.12) PROPOSITION - DEFINITION:</u> Given two Lie groups G, H the product manifold $G \times H$ with the product group structure is a Lie group. Moreover, a map $h : K \to G \times H$ from any

Lie group K into $G \times H$ is a Lie group homomorphism if and only if the compositions $pr_G \circ h : K \to G$, $pr_H \circ h : K \to H$ are Lie group homomorphisms.

Concerning quotients: For a Lie subgroup $H \subset G$ (in particular H is a closed submanifold) the quotient G/H exists as a manifold quotient (i.e. G/H has a unique manifold structure such that the projection $G \to G/H$ is smooth and a map $f : G/H \to M$ into any manifold M is smooth if and only if $f \circ \pi : G \to M$ is smooth.) G/H is a Lie group if H is in addition a normal subgroup, i.e. $Hg = gH$ for all $g \in G$. In that case, G/H is a group (quotient group) and the group operations are smooth on G/H.

(29.13) FINAL REMARK: We have treated the case $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ in the same way. In the case $\mathbb{K} = \mathbb{C}$, however, there exists also a refined theory of complex Lie groups where one requires the manifold structure to be the structure of a complex manifold, i.e. given by an atlas, $\varphi_i : U_i \to Q_i \subset \mathbb{C}^n$, $i \in I$, where the change of coordinates $\varphi_i \circ \varphi_j^{-1}$ is always a holomorphic map.