28. Metric and Orientation

Two further geometric structures on vector bundles are of interest: Orientation and metrics.

(28.1) DEFINITION: A vector bundle $E \to M$ on a manifold M is orientable if there exists an atlas $(\psi_i)_{i \in I}$ of bundle charts such that the transition functions ψ_{ij} have positive determinant.

Recall: On $U_{ij} = U_i \cap U_j \neq \emptyset$:

$$U_{ij} \times \mathbb{K}^r \xleftarrow{\psi_j} E|_{U_{ij}} \xrightarrow{\psi_i} U_{ij} \times \mathbb{K}^r$$

$$(a, v) \longmapsto (a, \psi_{ij}(a)), \ a \in U_{ij}.$$

Any choice of such an atlas fixes an orientation*.

(28.2) PROPOSITION: For a real vector bundle $E \to M$ („real" means $\mathbb{K} = \mathbb{R}$) the following assertions are equivalent:

1° $E \to M$ is orientable.

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* An orientation on E is an equivalence class of such atlasses.

2° $\det E$ is orientable.

3° $\det E$ is trivial.

(Proof left to the reader.)

EXAMPLES: Every trivial vector bundle is orientable.
$TS^2 \to S^2$ is orientable.
The möbius band $M \to S^1$ is not orientable.

(28.3) DEFINITION: Let $E \to M$ be a real vector bundle.
A metric on E is a tensor $g \in \mathcal{T}_2^0(M,E)$ which is symmetric and non-degenerate.

Hence for $\xi, \eta \in E_a$, $a \in M$, we obtain

$$g_a(\xi,\eta) = \langle \xi,\eta \rangle_a = \langle \xi,\eta \rangle$$

as a symmetric and non-degenerate bilinear form, such that for local sections $s,t \in \Gamma(W,E)$ the induced

$$a \longmapsto \langle s(a), t(a) \rangle \quad , a \in \mathcal{U},$$

is smooth.

g is a <u>Riemannian</u> metric if g_a is positive definite for all $a \in M$.

<u>Remark:</u> There is a "Riemannian geometry" on vector bundles with a Riemannian metric which is in many aspects similar to the Riemannian geometry of Riemannian spaces which are the manifolds M together with a Riemannian metric g on the tangent bundle.

The Metric can be used to obtain results about the existence of complements to subbundles of E, which we used in section 21 (21.3 and 21.2).

<u>(28.4) PROPOSITION:</u> Let $E \to M$ be a vector bundle with a Riemannian metric.

1° Any subbundle $F \subset E$ has an orthogonal complement, i.e. a subbundle $H \subset E$ such that $F \oplus H = E$ and $F_a \perp H_a$ for all $a \in M$. H is unique.

2° To every surjective vector bundle homomorphism $\Phi : E \to G$ there exist an orthogonal decomposition

$$E = \ker \Phi \oplus H$$

with $H \cong G$ isomorphic.

(28.5) PROPOSITION: Every real vector bundle $E \to M$ has a Riemannian metric, if M is of countable topology.

Proof. Use smooth partition of unity.

In particular, if M is of countable topology and $E \to M$ is an arbitrary vector bundle we conclude (cf. section 21): Any subbundle $F \subset E$ has a complement H: $F \oplus H = E$ and every morphism $\Phi : E \to G$ induces an isomorphism $E \cong \ker \Phi \oplus G$.

Complex vector bundles ($\mathbb{K} = \mathbb{C}$):

Are always orientable.

Hermitian metrics on complex vector bundles: Analogous to Riemannian metrics but sesquilinear in the fibres.